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SYMMETRIC DIFFERENCE IN ORTHOMODULAR LATTICES

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(*Communicated by Tibor Katriňák*)

ABSTRACT. Symmetric differences and a binary operation very close to symmetric differences on orthomodular lattices and relations between them are studied. In addition, associativity, invertibility and regularity of symmetric differences are investigated and the connection with the Boolean structure of the corresponding orthomodular lattice is presented.

1. Introduction

The symmetric difference $A \triangle B$ of two subsets A and B of a non-void base set Ω of a measurable space $(\Omega, \mathcal{S})$ is defined in different equivalent forms, e.g.,

$$A \triangle B := (A \cup B) \setminus (A \cap B)$$
$$A \triangle B := (A \setminus B) \cup (B \setminus A)$$

and it plays an important rôle in classical measure theory. The congruence $\sim_\mu$ on $\mathcal{S}$ induced by the symmetric difference $\triangle$ and a measure μ on $\mathcal{S}$ via $A \sim_\mu B$ if and only if $\mu(A \triangle B) = 0$ converts $\mathcal{S}$ into a complete metric space $\mathcal{S}/\sim_\mu$ ([Hal; §40], [DuSc; Part III.7]), and it enables to use methods of functional analysis to obtain such important results of measure theory as the Vitali-Hahn-Saks theorem ([DuSc; Theorem III.7.2]), Nikodým's convergence theorem ([DuSc; Corollary III.7.4]), Nikodým's boundedness theorem ([DuSc; Theorem IV.9.8]), etc. Original proofs of these results are based on the Baire category theorem

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([DuSc; Theorem I.6.9]) and on the fact that any σ-algebra $\mathcal{S}$ is distributive. The Baire category theorem, however, has an intimate connection to a weaker form of the axiom of choice ([Bla]), but nowadays there exist elementary methods not using the Baire result [AnSw] to obtain all of the above and other results of measure theory.

In connection with mathematical foundations of quantum mechanics ([Kal], [Var]), orthomodular lattices (OMLs for short) are studied. In general, OMLs are not distributive, and states (= probability measures) on OMLs need not be subadditive. Therefore, any attempt to generalize classical results of measure theory to orthomodular lattices meets serious problems. However, today there are some methods which enable us to do this in special cases, where the use of other methods, e.g., topological methods ([SAKC], [Rie]), is important.

In the present note, we shall study different types of symmetric differences and a binary operation between them, and we prove some inequalities closely related to them. Moreover, associativity, invertibility and regularity of the operation of symmetric difference and connections to the Boolean structure of the corresponding orthomodular lattice are investigated.

2. Operations closely related to symmetric differences

An *orthomodular lattice* is an algebra $\langle L, \vee, \wedge, ', 0, 1\rangle$ of type $(2, 2, 1, 0, 0)$ such that $\langle L, \vee, \wedge, 0, 1\rangle$ is a bounded lattice (which induces a partial ordering $\le$ on L via $a \le b$ if and only if $a = a \wedge b$, $a, b \in L$) and $'$ is a unary operation on L such that for all $a, b \in L$

(i) $a'' = a$;
(ii) if $a \le b$, then $b' \le a'$;
(iii) $a \vee a' = 1$;
(iv) if $a \le b$, then $b = a \vee b \wedge a'$ (orthomodular law).

(Here and in the following, we assume that the operation $\wedge$ has higher priority than the operations $\vee$, $\triangle$, $\bigtriangledown$ and $+$.) For more details on OMLs, see, e.g., [Kal].

In the sequel, let L denote an arbitrary, fixed OML.

Let $a, b \in L$. a, b are said to be *orthogonal* to each other, in signs $a \perp b$, if $a \le b'$. a, b are said to be *compatible* with each other, in signs $a\, C\, b$, if there exist three mutually orthogonal elements $a_1, b_1, c \in L$ such that $a = a_1 \vee c$ and $b = b_1 \vee c$. It is possible to show that for $a, b \in L$ the following conditions are equivalent:

(i) $a\, C\, b$;
(ii) $a = a \wedge b \vee a \wedge b'$;
(iii) $b = b \wedge a \vee b \wedge a'$

(cf. [Var; Lemma 6.7]). Moreover, it is well known that $a, b \in L$ are compatible if and only if there exists a Boolean subalgebra B of L with $a, b \in B$.

For $a \in L$, put $a^1 := a$ and $a^{-1} := a'$. Let $n > 1$ be an integer and put $N := \{1, \dots, n\}$. For $a_1, \dots, a_n \in L$ let

$$\operatorname{com}(a_1, \dots, a_n) := \bigvee_{j_1, \dots, j_n \in \{-1,1\}} \bigwedge_{i \in N} a_i^{j_i}$$

denote the commutator of $a_1, \dots, a_n$. It is well known that the elements $a_1, \dots \dots, a_n$ mutually commute if and only if their commutator equals 1.

We define two types of symmetric differences:

$$a \bigtriangleup b := (a \vee b) \wedge (a \wedge b)' ,$$
$$a \bigtriangledown b := a \wedge b' \vee b \wedge a'$$

($a, b \in L$). Then $\bigtriangleup$ and $\bigtriangledown$ are commutative binary operations on L which are closely related to each other via

$$a \bigtriangleup b = (a \bigtriangledown b')' ,$$
$$a \bigtriangledown b = (a \bigtriangleup b')'$$

($a, b \in L$). Moreover, it holds

$$(a \bigtriangleup b) \bigtriangleup c = \big((a \bigtriangledown b') \bigtriangledown c\big)' ,$$
$$a \bigtriangleup (b \bigtriangleup c) = \big(a \bigtriangledown (b' \bigtriangledown c)\big)'$$

for all $a, b, c \in L$. By [SAKC; p. 19, Proposition 3], we have

$$a \bigtriangleup b = a \wedge (a \wedge b)' \vee b \wedge (a \wedge b)' ,$$
$$a \bigtriangleup c \le (a \bigtriangleup b) \vee (b \bigtriangleup c) ,$$
$$(a \vee b) \bigtriangleup (c \vee d) \le (a \bigtriangleup c) \vee (a \bigtriangleup d) \vee (b \bigtriangleup c) \vee (b \bigtriangleup d)$$

for every $a, b, c, d \in L$. It is simple to see that

$$a \bigtriangledown b \le a \bigtriangleup b$$

for any $a, b \in L$, and it is easy to give an example that this inequality may be proper. For this purpose, consider the OML $L(\mathbb{R}^2)$ of all (closed) linear subspaces of the real plane $\mathbb{R}^2$. Let a, b be any two one-dimensional subspaces of $\mathbb{R}^2$ which are not compatible. Then $a \bigtriangleup b = \mathbb{R}^2$ and $a \bigtriangledown b = \{(0,0)\}$.

For $a = (a_1, \dots, a_n) \in L^n$ put

$$\bigtriangledown(a) := \bigvee_{\substack{j_1, \dots, j_n \in \{-1,1\} \\ |\{k \in N \mid j_k = 1\}| \text{ odd}}} \bigwedge_{i \in N} a_i^{j_i}$$

and

$$\triangle(a) := \bigwedge_{\substack{j_1,\dots,j_n\in\{-1,1\}\\ |\{k\in N \mid j_k=-1\}| \text{ even}}} \bigvee_{i\in N} a_i^{j_i} .$$

Then $\bigtriangledown(b,c) = b \bigtriangledown c$ and $\triangle(b,c) = b \triangle c$ for all $b, c \in L$. For every binary operation $\circ$ on L, every positive integer k and arbitrary elements $a_1, \dots, a_k \in L$ let $a_1 \circ \dots \circ a_k$ denote the element $\big(\dots(a_1 \circ a_2)\dots\big) \circ a_k$.

THEOREM 2.1. *Let $+$ be a binary operation on L such that*

$$a \bigtriangledown b \le a + b \le a \triangle b$$

for all $a, b \in L$. Then for arbitrary $c = (c_1, \dots, c_n) \in L^n$ we have

$$\bigtriangledown(c) \le S(c) \le \triangle(c) , \tag{1}$$

where $S(c) := c_1 + \dots + c_n$.

P r o o f . We use induction on n. According to the assumption, (1) holds for $n = 2$. Now assume $n \ge 2$ and suppose that (1) holds. Let $a_1, \dots, a_{n+1} \in L$ and $j_1, \dots, j_{n+1} \in \{-1, 1\}$. First assume $\big|\{k \in \{1, \dots, n+1\} \mid j_k = 1\}\big|$ to be odd. If $j_{n+1} = 1$, then

$$\begin{aligned} a_1 + \dots + a_{n+1} &= (a_1 + \dots + a_n) + a_{n+1} \ge (a_1 + \dots + a_n)' \wedge a_{n+1} \\ &\ge \Big(\bigvee_{i\in N} a_i^{-j_i}\Big)' \wedge a_{n+1} = \bigwedge_{i=1}^{n+1} a_i^{j_i} . \end{aligned}$$

If $j_{n+1} = -1$, then

$$\begin{aligned} a_1 + \dots + a_{n+1} &= (a_1 + \dots + a_n) + a_{n+1} \ge (a_1 + \dots + a_n) \wedge a'_{n+1} \\ &\ge \bigwedge_{i\in N} a_i^{j_i} \wedge a'_{n+1} = \bigwedge_{i=1}^{n+1} a_i^{j_i} . \end{aligned}$$

Now assume $\big|\{k \in \{1, \dots, n+1\} \mid j_k = -1\}\big|$ to be even. If $j_{n+1} = 1$, then

$$\begin{aligned} a_1 + \dots + a_{n+1} &= (a_1 + \dots + a_n) + a_{n+1} \le (a_1 + \dots + a_n) \vee a_{n+1} \\ &\le \bigvee_{i\in N} a_i^{j_i} \vee a_{n+1} = \bigvee_{i=1}^{n+1} a_i^{j_i} . \end{aligned}$$

If $j_{n+1} = -1$, then

$$\begin{aligned} a_1 + \dots + a_{n+1} &= (a_1 + \dots + a_n) + a_{n+1} \le (a_1 + \dots + a_n)' \vee a'_{n+1} \\ &\le \Big(\bigwedge_{i\in N} a_i^{-j_i}\Big)' \vee a'_{n+1} = \bigvee_{i=1}^{n+1} a_i^{j_i} . \end{aligned}$$

The rest of the proof is trivial. □

Remark 2.2.

(i) Since the lower and upper bound occurring in Theorem 2.1 are symmetric in $a_1, \dots, a_n$, this theorem remains valid if $a_1 + \dots + a_n$ is replaced by $a_{\pi(1)} + \dots + a_{\pi(n)}$, where π is an arbitrary permutation of N.

(ii) For $a \in L^n$ and $b \in L$ we have

1. $\mathrm{com}(a) = \nabla(a) \vee (\triangle(a))'$,
2. $\nabla(a) = \triangle(a) \wedge \mathrm{com}(a)$,
3. $\triangle(a) = \nabla(a) \vee (\mathrm{com}(a))'$,
4. $\nabla(a) = \triangle(a)$ if and only if $\mathrm{com}(a) = 1$,
5. $b + 0 = b$, $b + 1 = b'$, $b + b' = 1$ and $b + b = 0$.

THEOREM 2.3. *Assume that the conditions of Theorem* 2.1 *are satisfied, and let* $a = (a_1, \dots, a_n) \in L^n$. *Then* $\mathrm{com}(a)\, C\, S(a)$ *and*

$$\begin{aligned} S(a) \wedge \mathrm{com}(a) &= a_1 \wedge \mathrm{com}(a) + \dots + a_n \wedge \mathrm{com}(a) \\ &= a_{\pi(1)} \wedge \mathrm{com}(a) + \dots + a_{\pi(n)} \wedge \mathrm{com}(a) \\ &= \triangle(a) \wedge \mathrm{com}(a), \end{aligned}$$

where π *is an arbitrary permutation of* N.

Proof. Because of $\nabla(a) \le S(a) \le \triangle(a)$, we have $S(a)\, C\, \nabla(a)$ and $S(a)\, C\, \triangle(a)$, and hence $S(a)\, C\, \big(\nabla(a) \vee (\triangle(a))'\big)$, which means $S(a)\, C\, \mathrm{com}(a)$ (cf. 1. of Remark 2.2).

On the other hand, we have $a_j\, C \bigwedge_{i\in N} a_i^{k_i}$ for any $(k_1, \dots, k_n) \in \{-1, 1\}^n$ so that, by [Var; Lemma 6.10], $a_j\, C\, \mathrm{com}(a)$ for every $j \in N$. Consequently, $b\, C\, \mathrm{com}(a)$ for any element b of the orthomodular sublattice of L generated by $a_1, \dots, a_n$. Put $\hat{a}_j := a_j \wedge \mathrm{com}(a)$ for all $j \in N$, and $\hat{a} := (\hat{a}_1, \dots, \hat{a}_n)$. It is easy to see that $\mathrm{com}(\hat{a}) = 1$. Hence, in view of 4. of Remark 2.2, $\nabla(\hat{a}) = S(\hat{a}) = \triangle(\hat{a})$.

Without loss of generality, we can verify

$$\begin{aligned} \bigwedge_{j=1}^{m} \hat{a}_j' \wedge \bigwedge_{j=m+1}^{n} \hat{a}_j &= \bigwedge_{j=1}^{m} (a_j \wedge \mathrm{com}(a))' \wedge \bigwedge_{j=m+1}^{n} (a_j \wedge \mathrm{com}(a)) \\ &= \Big(\bigwedge_{j=1}^{m} (a_j' \vee (\mathrm{com}(a))') \Big) \wedge \Big(\bigwedge_{j=m+1}^{n} a_j \Big) \wedge \mathrm{com}(a) \\ &= \Big(\bigwedge_{j=1}^{m} a_j' \wedge \bigwedge_{j=m+1}^{n} a_j \Big) \wedge \mathrm{com}(a) \end{aligned}$$

for all $m = 0, \dots, n$. Hence $\bigtriangledown(\hat{a}) = \bigtriangledown(a) \wedge \mathrm{com}(a)$. Since $\bigtriangledown(a) \le S(a) \le \triangle(a)$ (by Theorem 2.1) and $\bigtriangledown(a) = \triangle(a) \wedge \mathrm{com}(a)$ (according to 2. of Remark 2.2), we have

$$\bigtriangledown(a) = \bigtriangledown(a) \wedge \mathrm{com}(a) = S(a) \wedge \mathrm{com}(a) = \triangle(a) \wedge \mathrm{com}(a)\,.$$

Therefore we obtain

$$\begin{aligned}\bigtriangledown(a) &= \bigtriangledown(\hat{a}) = S(\hat{a}) = \triangle(\hat{a}) \\ &= \bigtriangledown(a) \wedge \mathrm{com}(a) = S(a) \wedge \mathrm{com}(a) = \triangle(a) \wedge \mathrm{com}(a)\,.\end{aligned}$$

□

A mapping $m\colon L \to [0,1]$ satisfying the two conditions

$$\begin{aligned} m(1) &= 1\,, \\ m(a \vee b) &= m(a) + m(b) \ \text{ for all } a, b \in L \text{ with } a \perp b \end{aligned}$$

is said to be a *state* on L.

THEOREM 2.4. *Assume that the conditions of Theorem 2.1 are satisfied, let m be a state on L, let $a = (a_1, \dots, a_n) \in L^n$, and assume $m(\mathrm{com}(a)) = 1$. Then*

$$m(a_1 + \dots + a_n) = m\big(a_{\pi(1)} + \dots + a_{\pi(a)}\big) = m\big(\triangle(a)\big)\,,$$

where π is an arbitrary permutation of N.

P r o o f. By Theorem 2.3, $\mathrm{com}(a)\, C\, S(a)$. Therefore

$$\begin{aligned} m\big(S(a)\big) &= m\big(S(a) \wedge \mathrm{com}(a)\big) + m\big(S(a) \wedge (\mathrm{com}(a))'\big) = m\big(S(a) \wedge \mathrm{com}(a)\big) \\ &= m\big(a_1 \wedge \mathrm{com}(a) + \dots + a_n \wedge \mathrm{com}(a)\big) \\ &= m\big(\triangle(a) \wedge \mathrm{com}(a)\big) = m\big(\triangle(a)\big)\,, \end{aligned}$$

where we have used the fact that $\triangle(a)\, C\, \mathrm{com}(a)$. □

3. Associativity, invertibility and regularity of symmetric differences

In the present section, we investigate associativity, invertibility and regularity of $\triangle$, and we show that each single one of these properties forces L to be a Boolean algebra. Since we have the simple relation $a \bigtriangledown b = (a \triangle b')'$, results analogous to those obtained here also hold for $\bigtriangledown$ instead of $\triangle$.

In order to simplify proofs, in the following, we make frequent use of a method of Navara (cf. [Nav]) concerning the calculations within the free OML F with two free generators a and b. Put $c := \operatorname{com}(a,b)$. Then the following hold:

1. The atoms of F are $a \wedge b$, $a \wedge b'$, $a' \wedge b$, $a' \wedge b'$, $a \wedge c'$, $a' \wedge c'$, $b \wedge c'$ and $b' \wedge c'$.
2. The mappings $x \mapsto (x \wedge c, x \wedge c')$ and $(x,y) \mapsto x \vee y$ are mutually inverse isomorphisms between F and $[0,c] \times [0,c']$.
3. $[0,c]$ is the Boolean algebra with the atoms $a \wedge b$, $a \wedge b'$, $a' \wedge b$ and $a' \wedge b'$.
4. $[0,c'] = \{0,\ a \wedge c',\ a' \wedge c',\ b \wedge c',\ b' \wedge c',\ c'\} \cong \mathrm{MO2}$.
5. $a = a \wedge b \vee a \wedge b' \vee a \wedge c'$.
6. $b = a \wedge b \vee a' \wedge b \vee b \wedge c'$.

LEMMA 3.1. *Let $a, b \in L$. Then the following conditions are equivalent:*

1. $a\, C\, b$.
2. $(a \triangle b) \triangle b = a$.
3. $(a \triangle b) \triangle (a \vee b) = a \triangle \big(b \triangle (a \vee b)\big)$.

Proof. The equivalence of 1. and 2. follows from

$$(a \triangle b) \triangle b = a \wedge b \vee a \wedge b' \vee b' \wedge \big(\operatorname{com}(a,b)\big)',$$
$$a = a \wedge b \vee a \wedge b' \vee a \wedge \big(\operatorname{com}(a,b)\big)',$$

and the equivalence of 1. and 3. follows from

$$(a \triangle b) \triangle (a \vee b) = a \wedge b,$$
$$a \triangle \big(b \triangle (a \vee b)\big) = a \wedge b \vee \big(\operatorname{com}(a,b)\big)'.$$

□

LEMMA 3.2. *For $a, b \in L$ we have*

1. $(a \vee b) \triangle a \wedge b = a \triangle b$,
2. $(a \triangle b) \triangle (a \vee b) = a \wedge b$,
3. $(a \triangle b) \triangle a \wedge b = a \vee b$.

Proof.

1. is obvious. For $a, b \in L$ we have $a \vee b\ C\ a \wedge b$, and hence, by 1. and Lemma 3.1, $(a \triangle b) \triangle (a \vee b) = \big(a \wedge b \triangle (a \vee b)\big) \triangle (a \vee b) = a \wedge b$. 3. follows in an analogous way. □

Let $a, b \in L$. a, b are said to be *complemented* to each other (one is said to be a complement of the other) if $a \vee b = 1$ and $a \wedge b = 0$. This is obviously equivalent to the fact that $a \triangle b = 1$. a, b are called *perspective* to each other, in signs $a \sim b$, if they have a common complement in L, i.e., if there exists an element c of L with $a \vee c = b \vee c = 1$ and $a \wedge c = b \wedge c = 0$. Hence, perspectivity of a and b is equivalent to the existence of an element c of L satisfying $a \triangle c = b \triangle c = 1$.

PROPOSITION 3.3. *Let $a, b \in L$, and let $\langle\{a, b\}\rangle$ denote the orthomodular sublattice of L generated by a and b. Then 1.–5. hold:*

1. $a \triangle b = 0$ *if and only if* $a = b$,
2. $a \triangle b = 1$ *if and only if* $a \sim b'$ *holds in* $\langle\{a, b\}\rangle$,
3. $a \triangle b = a \vee b$ *if and only if* $a \wedge b = 0$,
4. $a \triangle b = a \wedge b$ *if and only if* $a \vee b = 0$,
5. $a \triangle b = b \wedge a'$ *if and only if* $a \leq b$.

Proof. 1. follows from 2. or 3. of Lemma 3.2, 2. follows from the fact that for $a, b \in L$ the condition that $a \sim b'$ in $\langle\{a, b\}\rangle$ is equivalent to $a \wedge b = a' \wedge b' = 0$. 3. and 4. follow from 2. and 3. of Lemma 3.2, respectively. For $a, b \in L$ it holds $a \triangle b = a \wedge b' \vee a' \wedge b \vee \big(\mathrm{com}(a, b)\big)'$, which implies 5. □

PROPOSITION 3.4. *Two elements a, b of L are perspective to each other if and only if there exists an element c of L with $a \triangle c = b \triangle c$.*

Proof. A calculation shows that $a \triangle \big((a \triangle b)' \triangle b\big) = 1$ for all $a, b \in L$. Now, let a, b be arbitrary, fixed elements of L. If there exists an element c of L with $a \triangle c = b \triangle c$, then

$$a \triangle \big((a \triangle c)' \triangle c\big) = 1 = b \triangle \big((b \triangle c)' \triangle c\big) = b \triangle \big((a \triangle c)' \triangle c\big),$$

and hence $a \sim b$. (The element $(a \triangle c)' \triangle c$ is a common complement of a and b in L.) The rest of the proof is trivial. □

LEMMA 3.5. *For $a, b \in L$ we have $a \, C \, a \triangle b \, C \, b$.*

Proof. It is clear. □

LEMMA 3.6. *Two elements a, b of L commute if and only if there exists an element c of L with $a \triangle c = b$.*

Proof. Let $a, b \in L$. If $a C b$, then $a \triangle (a \triangle b) = b$ according to Lemma 3.1. The rest follows from Lemma 3.5. □

A binary operation $\circ$ on L is called *invertible* (*regular*) if for arbitrary $a, b \in L$ each one of the equations $a \circ x = b$ and $y \circ a = b$ has at least (at most) one solution in L.

THEOREM 3.7. *The following statements are equivalent:*

1. *L is a Boolean algebra.*
2. *The binary (n-ary) operations $\triangle$ and $\bigtriangledown$ coincide.*
3. *$\triangle$ is associative.*
4. *$\triangle$ is invertible.*
5. *$\triangle$ is regular.*
6. *$\wedge$ is distributive with respect to $\triangle$.*

Proof. The equivalence of 1. and 2. follows from 4. of Remark 2.2. It is well known that 1. implies 3.–6. If 3., respectively 4., holds, then the fact that any two elements of L commute follows from Lemma 3.1, respectively Lemma 3.6, and hence L is a Boolean algebra. Proposition 3.4 together with [Kal; Proposition 1.3.7] shows that 5. implies 1. Finally, assume that 6. holds. Then for all $a, b \in L$ we have $a' \wedge (a \triangle b) = a' \wedge a \triangle a' \wedge b$, which is equivalent to $a' \wedge (a \vee b) = a' \wedge b$, hence $a\,C\,b$, and L is a Boolean algebra. □

THEOREM 3.8. *In the variety of OMLs, there does not exist a binary term inducing a regular, respectively invertible, binary operation on every OML.*

Proof. Let t be a binary term (in the variety of all OMLs), let $\{0, a, a', b, b', 1\}$ denote the OML MO2, and let t also denote the term function on MO2 induced by t. Then the transpositions $(a\ a')$ and $(b\ b')$ are automorphisms of MO2. If $t(a,b) \notin \{a, a'\}$, then $t(a,b) = (a\ a')t(a,b) = t((a\ a')a, (a\ a')b) = t(a', b)$, and if $t(a,b) \notin \{b, b'\}$, then $t(a,b) = (b\ b')t(a,b) = t((b\ b')a, (b\ b')b) = t(a, b')$. Hence, in any case, the binary operation on MO2 induced by t is neither regular nor, since MO2 is finite, invertible. □

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