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Dedicated to Professor Tibor Šalát on the occasion of his 70th birthday

GENERALIZED ALMOST CONVERGENCE AND KNOPP'S CORE THEOREM

Z. U. AHMAD-MURSALEEN — Q. A. KHAN

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ABSTRACT. In [J. Math. Anal. Appl. **132** (1988), 226–233], Choudhary has extended the well-known Knopp's core theorem. The purpose of this paper is to generalize the results due to Choudhary by using the concept of $F_{\mathscr{B}}$-convergence [Math. Japon. **18** (1973), 53–70].

1. Introduction

We list the following functionals defined on m, the space of bounded real sequences $x = (x_k)$

$$\ell(x) = \liminf_k x_k\,, \qquad L(x) = \limsup_k x_k\,,$$
$$q(x) = \liminf_k |x_k|\,, \qquad Q(x) = \limsup_k |x_k|\,,$$
$$\|x\| = \sup |x_k|\,, \qquad \omega(x) = \inf\{L(x+z):\ z \in bs\}\,,$$

where bs denotes the space of all bounded sequences $x = (x_k)$ such that

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$\sup_n \left| \sum_{k=0}^{n} x_k \right| < +\infty$. Further

$$\ell^*(x) = \liminf_n \sup_i \frac{1}{n+1} \sum_{r=i}^{i+n} x_r ,$$

$$L^*(x) = \limsup_n \sup_i \frac{1}{n+1} \sum_{r=i}^{i+n} x_r ,$$

$$w^*(x) = \inf\{L^*(x+z) : \ z \in bs\} .$$

In [3], [4] and [6], we have the following inequalities

$$\ell \le \omega \le L \le \|\cdot\| ; \qquad \omega \le Q \le \|\cdot\| ;$$
$$L \le Q ; \qquad \ell \le q \le Q ;$$
$$\ell \le \ell^* \le L^* \le L ; \qquad w^* \le L^* .$$

Before giving some other functionals, we recall the following definition of $F_{\mathscr{B}}$-convergence ([7]).

Let $\mathscr{B} = (B_i)$ be a sequence of infinite matrices with $B_i = \big(b_{nk}(i)\big)$. A sequence $x = (x_k) \in m$ is said to be *$F_{\mathscr{B}}$-convergent to the value* $\operatorname{Lim} \mathscr{B}x$ if

$$\lim_n (B_i x)_n = \lim_n \sum_k b_{nk}(i)\, x_k = \operatorname{Lim} \mathscr{B}x ,$$

uniformly for $i = 0, 1, 2, \dots$. By $F_{\mathscr{B}}$, we mean the space of all $F_{\mathscr{B}}$-convergent sequences, and $\operatorname{Lim} \mathscr{B}x$ denotes the generalized limit. The space $F_{\mathscr{B}}$ depends on the fixed chosen sequence $\mathscr{B} = (B_i)$ of matrices. In case $\mathscr{B} = \mathscr{B}_0 = (I)$, the unit matrix, $F_{\mathscr{B}} = c$. For $\mathscr{B} = \mathscr{B}_1 = \big(B_i^{(1)}\big)$, $F_{\mathscr{B}} = f$, the space of almost convergent sequences ([5]); where $B_i^{(1)} = \big(b_{nk}^{(1)}(i)\big)$ and

$$b_{nk}^{(1)}(i) = \begin{cases} \frac{1}{n+1} , & i \le k \le i+n , \\ 0 , & \text{otherwise.} \end{cases}$$

We further give some new functionals defined in [6] for $\mathscr{B} = (B_i)$ with

$$\|\mathscr{B}\| = \sup_{n,i} \sum_k |b_{nk}(i)| ,$$

$$Q_{\mathscr{B}}(x) = \limsup_n \sup_i \sum_k b_{nk}(i)\, x_k ,$$

and

$$q_{\mathscr{B}}(x) = \liminf_n \sup_i \sum_k b_{nk}(i)\, x_k ,$$

obviously,

$$q_{\mathscr{B}} \le Q_{\mathscr{B}} \le \|\mathscr{B}\| \,. \tag{1.1}$$

If $Q_{\mathscr{B}}(x) = q_{\mathscr{B}}(x)$, we say that $\lim\limits_n \sum\limits_k b_{nk}(i)x_k$ exists uniformly in i. For an infinite matrix $A = (a_{nk})$, we write

$$Q_{\mathscr{B}}(Ax) = \limsup_n \sup_i \sum_\ell b_{n\ell}(i) \sum_k a_{\ell k} x_k \,,$$

$$q_{\mathscr{B}}(Ax) = \liminf_n \sup_i \sum_\ell b_{n\ell}(i) \sum_k a_{\ell k} x_k \,.$$

The object of this paper is to generalize Theorem 1 and 2 due to Choudhary [1].

Let X and Y be any two sequence spaces, and $A = (a_{nk})$ be an infinite matrix. We write $Ax = \big(A_n(x)\big)$, where $A_n(x) = \sum\limits_k a_{nk}x_k$, provided the series converges for each n. If $x = (x_k) \in X$ implies $Ax \in Y$, we say that A defines a matrix transformation from X into Y, and we denote it by $A \in (X, Y)$; and (X, Y) denotes the class of all such matrices. $A \in (X, Y)_{\text{reg}}$, we mean $A \in (X, Y)$ and $\lim x = \lim Ax$.

In order to prove our results, we need the following lemmas.

2. Lemmas

Lemma 2.1. ([7]) *Let $\mathscr{B} = (B_i)$ be a sequence of infinite matrices with*

$$\sup_n \sum_k |b_{nk}(i)| < \infty \qquad \text{for each} \quad i \,.$$

Then $A \in (c, F_{\mathscr{B}})_{\text{reg}}$ if and only if

(i) $\|A\| < \infty$,

(ii) *for $r \ge 0$*

$$\sup_{\substack{0 \le i < \infty \\ r \le n < \infty}} \sum_k \Big| \sum_\ell b_{n\ell}(i) a_{\ell k} \Big| < \infty \,,$$

(iii) $\lim\limits_n \sum\limits_\ell b_{n\ell}(i)\, a_{\ell k} = 0$ *uniformly in i for each k,*

(iv) $\lim\limits_n \sum\limits_\ell b_{n\ell}(i) \sum\limits_k a_{\ell k} = 1$ *uniformly in i.*

For $A \in (c, F_{\mathscr{B}})_{\text{reg}}$

$$\operatorname{Lim} \mathscr{B}(Ax) = \lim x \,, \qquad x \in c \,.$$

LEMMA 2.2. ([7]) *Let* $\mathscr{B} = (B_i)$ *with* $\|\mathscr{B}\| < \infty$. *Then* $A \in (f, F_{\mathscr{B}})_{\text{reg}}$ *if and only if*

(i) *Conditions* (i), (iii) *and* (iv) *of Lemma* 2.1 *hold, and*
(ii) $\lim_n \sum_k \left|\sum_\ell b_{n\ell}(i)(a_{\ell k} - a_{\ell,k+1})\right| = 0$ *uniformly in* i.

For $A \in (f, F_{\mathscr{B}})_{\text{reg}}$

$$\operatorname{Lim} \mathscr{B}(Ax) = f - \lim x\,, \qquad x \in f\,.$$

3. Main results

THEOREM 3.1. *Let* $\mathscr{B} = (B_i)$ *be a sequence of infinite matrices with* $\|\mathscr{B}\| < \infty$. *Let* $A = (a_{nk})$ *be a normal regular matrix with* $A^{-1} = (a_{nk}^{-1})$ *its reciprocal. Let*

$$\sum_{k=0}^{J+1} \left| \sum_{\ell=J+1}^{\infty} b_{n\ell}(i)\, a_{\ell k}^{-1} \right| \to 0 \qquad \text{as} \quad J \to \infty\,, \tag{3.1.1}$$

uniformly in i, *for any fixed* n.

Then

$$Q_{\mathscr{B}}(x) \le L(Ax) \tag{3.1.2}$$

if and only if

$$A^{-1} \in (c, F_{\mathscr{B}})_{\text{reg}}\,, \tag{3.1.3}$$

$$\lim_n \sup_i \sum_\ell \left| b_{n\ell}(i) \sum_k a_{\ell k}^{-1} \right| = 1\,. \tag{3.1.4}$$

Proof.

Necessity. Let $x \in c$. Then

$$\ell(x) = L(x) = \lim x\,. \tag{3.1.5}$$

It is obvious that

$$\ell(x) \le q_{\mathscr{B}}(x)\,,$$

since A is regular. Therefore $\ell(x) = \ell(Ax)$. Hence

$$\ell(Ax) \le q_{\mathscr{B}}(x)\,,$$

since A is normal. Therefore

$$\ell(x) \le q_{\mathscr{B}}(A^{-1}x)\,.$$

Now, by (1.1), (3.1.2), and (3.1.5), we have

$$\lim x = \ell(x) \le q_{\mathscr{B}}(A^{-1}x) \le Q_{\mathscr{B}}(A^{-1}x) \le L(x) = \lim x\,,$$

that is,

$$q_{\mathscr{B}}(A^{-1}x) = Q_{\mathscr{B}}(A^{-1}x) = \lim x\,.$$

Therefore

$$\operatorname{Lim} \mathscr{B}(A^{-1}x) = \lim x \qquad \text{for all} \quad x \in c\,.$$

Hence (3.1.3) holds.

Now, by [2; Lemma 2], there exists $y \in m$ such that $\|y\| \le 1$ and

$$Q_{\mathscr{B}}(A^{-1}y) = \limsup_n \sup_i \sum_\ell \left| b_{n\ell}(i) \sum_k a_{\ell k}^{-1} \right|. \tag{3.1.6}$$

Hence

$$\begin{aligned}
1 = q_{\mathscr{B}}(A^{-1}e) &\le \liminf_n \sup_i \sum_\ell \left| b_{n\ell}(i) \sum_k a_{\ell k}^{-1} \right| \\
&\le \limsup_n \sup_i \sum_\ell \left| b_{n\ell}(i) \sum_k a_{\ell k}^{-1} \right| \\
&= Q_{\mathscr{B}}(A^{-1}y) \qquad \text{(by (3.1.6))} \\
&\le L(y) \le \|y\| \le 1\,,
\end{aligned}$$

which proves the necessity of (3.1.4).

Sufficiency. We define for any real λ

$$\lambda^+ = \max(\lambda, 0) \quad \text{and} \quad \lambda^- = \max(-\lambda, 0)\,,$$

then

$$|\lambda| = \lambda^+ + \lambda^- \quad \text{and} \quad \lambda = \lambda^+ - \lambda^-\,.$$

Therefore, for any positive integer k_0

$$\begin{aligned}
&\sum_\ell b_{n\ell}(i) \sum_k a_{\ell k}^{-1} y_k \\
&= \sum_\ell b_{n\ell}(i) \sum_{k<k_0} a_{\ell k}^{-1} y_k + \sum_\ell b_{n\ell}(i) \sum_{k\ge k_0} (a_{\ell k}^{-1})^+ y_k - \sum_\ell b_{n\ell}(i) \sum_{k\ge k_0} (a_{\ell k}^{-1})^- y_k \\
&\le \|y\| \sum_\ell b_{n\ell}(i) \sum_{k<k_0} |a_{\ell k}^{-1}| + \Big(\sup_{k\ge k_0} y_k\Big) \sum_\ell b_{n\ell}(i) \sum_{k\ge k_0} |a_{\ell k}^{-1}| \\
&\qquad\qquad + \|y\| \sum_\ell b_{n\ell}(i) \sum_{k\ge k_0} \left(|a_{\ell k}^{-1}| - a_{\ell k}^{-1}\right) \\
&\le \|y\| \sum\nolimits_1 + \Big(\sup_k y_k\Big) \sum\nolimits_2 + \|y\| \sum\nolimits_3
\end{aligned}$$

By virtue of condition (3.1.1),

$$\sum\nolimits_1 = \sum_{\ell} b_{n\ell}(i) \sum_{k<k_0} |a_{\ell k}^{-1}| \to 0 \qquad \text{uniformly in } i\,, \text{ for fixed } n\,;$$

and condition (3.1.4) gives that

$$\sum\nolimits_2 = \sup_i \sum_{\ell} b_{n\ell}(i) \sum_{k\geq k_0} |a_{\ell k}^{-1}| \to 1 \qquad \text{as} \quad n\to\infty\,.$$

Also, condition (3.1.4) alongwith condition (iv) of Lemma 2.1 gives

$$\sum\nolimits_3 = \sup_i \sum_{\ell} b_{n\ell}(i) \sum_{k\geq k_0} \left(|a_{\ell k}^{-1}| - a_{\ell k}^{-1}\right) \to 0 \qquad \text{as} \quad n\to\infty\,.$$

Finally, we have

$$\limsup_n \sup_i \sum_{\ell} b_{n\ell}(i) \sum_k a_{\ell k}^{-1} y_k \leq \limsup_k y_k\,,$$

that is,

$$Q_{\mathscr{B}}(A^{-1}y) \leq L(y)\,.$$

Since A is normal, we have

$$Q_{\mathscr{B}}(x) \leq L(Ax)\,,$$

where $x = A^{-1}y = \left(\sum_k a_{\ell k}^{-1} x_k\right)$.

This completes the proof of the theorem. □

THEOREM 3.2. *Let A and $\mathscr{B}$ be matrices as in Theorem 3.1. Let (3.1.1) hold. Then*

$$Q_{\mathscr{B}}(x) \leq L^*(Ax) \tag{3.2.1}$$

if and only if

$$A^{-1} \in (f, F_{\mathscr{B}})_{\text{reg}} \tag{3.2.2}$$

and (3.1.4) hold.

Proof.

Necessity. Let $x\in f$. Then

$$\ell^*(x) = L^*(x) = f - \lim x\,.$$

By (3.2.1), we have

$$\ell^*(x) \leq q_{\mathscr{B}}(A^{-1}x) \leq Q_{\mathscr{B}}(A^{-1}x) \leq L^*(x)\,.$$

Therefore, for all $x \in f$

$$\operatorname{Lim} \mathscr{B}(A^{-1}x) = f - \lim x \,,$$

and hence (3.2.2) holds.

Since $(f, F_{\mathscr{B}})_{\mathrm{reg}} \subset (c, F_{\mathscr{B}})_{\mathrm{reg}}$, condition (3.1.4) follows from Theorem 3.1.

Sufficiency. Given $\varepsilon > 0$, we can find a positive integer p such that for $y \in m$ and for all $k \geq 0$

$$\frac{1}{p+1} \sum_{r=k}^{k+p} y_r < L^*(y) + \varepsilon \,, \tag{3.2.3}$$

holds for fixed p whose choice depends on $y \in m$.

We can proceed as in the proof of Theorem 3 due to Orhan [6]. It is easy to write that

$$\sum_k a_{\ell k}^{-1} y_k = \sum_k a_{\ell k}^{-1} \frac{1}{p+1} \sum_{r=k}^{k+p} y_r - \sum_{k=p}^{\infty} \left(\frac{a_{\ell k}^{-1} + \cdots + a_{\ell, k-p}^{-1}}{p+1} - a_{\ell k}^{-1} \right) y_k$$
$$+ \sum_{k=0}^{p-1} a_{\ell k}^{-1} y_k + \sum_{k=0}^{p-1} \left(\frac{a_{\ell k}^{-1} + \cdots + a_{\ell, k-p+1}^{-1}}{p+1} \right) y_k \,.$$

Therefore

$$\sum_\ell b_{n\ell}(i) \sum_k a_{\ell k}^{-1} y_k = \sum{}^1 - \sum{}^2 + \sum{}^3 + \sum{}^4 \,, \tag{3.2.4}$$

where

$$\sum{}^1 = \sum_\ell b_{n\ell}(i) \sum_k a_{\ell k}^{-1} \frac{1}{p+1} \sum_{r=k}^{k+p} y_r \,,$$

$$\sum{}^2 = \sum_\ell b_{n\ell}(i) \sum_{k=p}^{\infty} \left(\frac{a_{\ell k}^{-1} + \cdots + a_{\ell, k-p}^{-1}}{p+1} - a_{\ell k}^{-1} \right) y_k \,,$$

$$\sum{}^3 = \sum_\ell b_{n\ell}(i) \sum_{k=0}^{p-1} a_{\ell k}^{-1} y_k \,,$$

$$\sum{}^4 = \sum_\ell b_{n\ell}(i) \sum_{k=0}^{p-1} \left(\frac{a_{\ell k}^{-1} + \cdots + a_{\ell, k-p+1}^{-1}}{p+1} \right) y_k \,.$$

Since $A^{-1} \in (f, F_{\mathscr{B}})_{\text{reg}}$, and, by condition (iii) of Lemma 2.1, $\sum^3$ and $\sum^4$ tend to zero as $n \to \infty$. Now

$$\begin{aligned}\left|\sum\nolimits^2\right| &\le \frac{1}{p+1}\left|\sum_\ell b_{n\ell}(i)\sum_{k=p}^{\infty}\left(a_{\ell k}^{-1} + \cdots + a_{\ell,k-p}^{-1} - (p+1)a_{\ell k}^{-1}\right)\right||y_k| \\ &\le \frac{\|x\|}{p+1}\left|\sum_\ell b_{n\ell}(i)\sum_{r=0}^{p}\sum_{k=p}^{\infty}\left(a_{\ell,k-p}^{-1} - a_{\ell k}^{-1}\right)\right| \\ &\le \frac{\|x\|}{p+1}\left|\sum_\ell b_{n\ell}(i)\sum_{r=0}^{p} r\sum_{k=0}^{\infty}\left(a_{\ell k}^{-1} - a_{\ell,k+1}\right)\right| \\ &\le \frac{p}{2}\|x\|\sum_k\left|\sum_\ell b_{n\ell}(i)\left(a_{\ell k}^{-1} - a_{\ell,k+1}\right)\right|.\end{aligned}$$

By virtue of condition (ii) of Lemma 2.2, $\left|\sum^2\right| \to 0$ as $n \to \infty$. Therefore, we have by (3.2.4)

$$\begin{aligned}Q_{\mathscr{B}}(A^{-1}y) &= \limsup_n \sup_i \sum_\ell b_{n\ell}(i)\sum_k a_{\ell k}^{-1} y_k \\ &\le \limsup_n \sup_i \sum_\ell b_{n\ell}(i)\sum_k a_{\ell k}^{-1}\left(\frac{y_k + \cdots + y_{k+p}}{p+1}\right) \\ &\le \limsup_n \sup_i \sum_\ell b_{n\ell}(i)\sum_k (a_{\ell k}^{-1})^{+}\left(\frac{y_k + \cdots + y_{k+p}}{p+1}\right) \\ &\quad - \limsup_n \sup_i \sum_\ell b_{n\ell}(i)\sum_k (a_{\ell k}^{-1})^{-}\left(\frac{y_k + \cdots + y_{k+p}}{p+1}\right).\end{aligned}$$

Using condition (3.2.3), we have

$$\begin{aligned}Q_{\mathscr{B}}(A^{-1}y) \le \left(L^*(y) + \varepsilon\right) &\limsup_n \sup_i \sum_\ell \left|b_{n\ell}(i)\sum_k a_{\ell k}^{-1}\right| \\ &+ \|y\| \limsup_n \sup_i \sum_\ell \left|b_{n\ell}(i)\sum_k a_{\ell k}^{-1}\right| \\ &- \|y\| \limsup_n \sup_i \sum_\ell b_{n\ell}(i)\sum_k a_{\ell k}^{-1}.\end{aligned}$$

Using conditions (3.1.1), (3.1.4) and condition (iv) of Lemma 2.1, we finally have

$$Q_{\mathscr{B}}(A^{-1}y) \le L^*(y).$$

Hence

$$Q_{\mathscr{B}}(x) \le L^*(Ax).$$

This completes the proof of the theorem. □

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