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Dedicated to Professor Tibor Šalát
on the occasion of his 70th birthday

ON CONVERGENCE PRESERVING TRANSFORMATIONS OF INFINITE SERIES[1]

Pavel Kostyrko

(*Communicated by Ľubica Holá*)

ABSTRACT. The paper generalizes and expands some of known results concerning convergence preserving transformations of infinite series of elements of a finite dimensional Banach space.

The paper deals with infinite series of elements of a Banach space. The aim of the present paper is to give a characterization of the convergence preserving transformations. Moreover, we shall show that known methods of proofs can be used in more general setting.

Let $\mathbb{R}$ be the set of all real numbers, and let N be a natural number. Then the product $\mathbb{R}^N = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ endowed with a norm is a Banach space over $\mathbb{R}$. It is well know that, in the finite dimensional space $\mathbb{R}^N$, any two norms are equivalent ([Sh; p. 56]). We shall suppose that $\mathbb{R}^N$ is endowed with the norm $\|(\xi_1, \dots, \xi_N)\| = \sum\limits_{i=1}^{N} |\xi_i|$. Let $(a_1, \dots, a_N)$ be an N-tuple such that, for each $i = 1, \dots, N$, $a_i = +1$ or $a_i = -1$. Every subset of $\mathbb{R}^N$ of the form $O(a_1, \dots, a_N) = \{(\xi_1, \dots, \xi_N) : \text{ for each } i = 1, \dots, N, \ \operatorname{sgn} \xi_i = \operatorname{sgn} a_i \text{ or } \xi_i = 0\}$ will be called the *closed orthant* on $\mathbb{R}^N$. Obviously, $\mathbb{R}^N$ is the union of 2^N its nonoverlappig closed orthants, and $(\xi_1, \dots, \xi_N) \in \mathbb{R}^N$ belongs to two or more closed orthants if and only if some of its coordinates is zero. Note that if $x = (\xi_1, \dots, \xi_N)$

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and $y = (\eta_1, \ldots, \eta_N)$ belong to the same closed orthant, then $\|x + y\| = \|(\xi_1 + \eta_1, \ldots, \xi_N + \eta_N)\| = \sum_{j=1}^{N} |\xi_j + \eta_j| = \sum_{j=1}^{N} |\xi_j| + \sum_{j=1}^{N} |\eta_j| = \|x\| + \|y\|$.

In the following definitions, E stands for a linear space, and $\{a_n\}_n$ $\left(\sum_{n=1}^{\infty} a_n\right)$ is a sequence (a series) of elements of E.

DEFINITION 1. Let $\sum_{n=1}^{\infty} a_n$ be a convergent series, and let $f\colon E \to E$. Then the series $\sum_{n=1}^{\infty} f(a_n)$ is called the *f-transformation* of the series $\sum_{n=1}^{\infty} a_n$.

DEFINITION 2. A function $f\colon E \to E$ is said to be a *convergence preserving transformation* if for each convergent series $\sum_{n=1}^{\infty} a_n$ its f-transformation $\sum_{n=1}^{\infty} f(a_n)$ is convergent.

The following theorem deals with Banach space $(\mathbb{R}^N, \|\cdot\|)$.

THEOREM. *Let $f\colon \mathbb{R}^N \to \mathbb{R}^N$. Then the following statements are equivalent:*

(a) *f is a convergence preserving transformation;*

(b) *if the series $\sum_{n=1}^{\infty} a_n$ is convergent, then the sequence of partial sums of its f-transformation $\sum_{n=1}^{\infty} f(a_n)$ is bounded;*

(c) (i) *$f(0) = 0$ and f is continuous at 0, and*
(ii) *there is a $\delta > 0$ such that $f(x + y) = f(x) + f(y)$ holds whenever $\|x\| < \delta$, $\|y\| < \delta$ and $\|x + y\| < \delta$.*

Proof.

(a) $\Longrightarrow$ (b): Obvious.

(b) $\Longrightarrow$ (c): Proof of (i): Since the series $0+0+\cdots+0+\ldots$ is convergent, the series $f(0) + f(0) + \cdots + f(0) + \ldots$ has bounded partial sums, and there is $K > 0$ such that for each $m = 1, 2, \ldots$ we have $\left\| \sum_{n=1}^{m} f(0) \right\| = m\|f(0)\| \le K$, consequently, $f(0) = 0$. We prove continuity of f at 0. In the contrary, suppose that there is a sequence $\{a_n\}_n$, $a_n \to 0$, and $\varepsilon > 0$ such that $\|f(a_{n_k})\| \ge \varepsilon$ holds for some sequence $\{n_k\}_k$. Without loss of generality, we can suppose $\|a_{n_k}\| \le 2^{-k}$. Terms of the sequence $\{f(a_{n_k})\}_k$ are contained in 2^N closed orthants of $\mathbb{R}^N$, hence there is a closed orthant P which contains infinitely many terms of $\{f(a_{n_k})\}_k$. Let $\{n_{k_r}\}_r$ be such a subsequence of $\{n_k\}_k$ that $f(a_{n_{k_r}}) \in P$ holds for each r. The series $\sum_{r=1}^{\infty} a_{n_{k_r}}$ absolutely converges, and

$\left\| \sum\limits_{r=1}^{s} f(a_{n_{k_r}}) \right\| = \sum\limits_{r=1}^{s} \|f(a_{n_{k_r}})\| \geq s\varepsilon$ holds for each $s = 1, 2, \ldots$. Hence the sequence of partial sums of $\sum\limits_{r=1}^{\infty} f(a_{n_{k_r}})$ is not bounded – a contradiction.

Proof of (ii): There are two following cases:

1. For every $k = 1, 2, \ldots$ there are three N-tuples u_k, v_k, w_k such that

$$\|u_k\| < 2^{-k}, \qquad \|v_k\| < 2^{-k}, \qquad \|w_k\| < 2^{-k}, \tag{1}$$

$$u_k + v_k + w_k = 0, \tag{2}$$

$$f(u_k) + f(v_k) + f(w_k) \neq 0. \tag{3}$$

2. The case opposite to 1.

1. Put $n_1 = 1$. By induction for each $k = 2, 3, \ldots$, we can choose a positive integer n_k such that

$$n_k \|f(u_k) + f(v_k) + f(w_k)\| \geq k + \left\| \sum_{j=1}^{k-1} n_j \big(f(u_j) + f(v_j) + f(w_j)\big) \right\|. \tag{4}$$

It is possible by virtue of (3). Put

$$\begin{aligned} \sum_{n=1}^{\infty} a_n = u_1 + v_1 + w_1 + u_2 + v_2 + w_2 + \cdots + u_2 + v_2 + w_2 + \ldots \\ \cdots + u_k + v_k + w_k + \cdots + u_k + v_k + w_k + \ldots \end{aligned} \tag{5}$$

(the sum $u_k + v_k + w_k$ appears in (5) n_k-times). The convergence of the series (5) follows from properties (1), (2), and from Cauchy's criterion of convergence. On the other hand, the property (4) implies that the sequence of partial sums of the series $\sum\limits_{n=1}^{\infty} f(a_n)$ is not bounded. Consequently, this case 1 does not occur.

2. There exists a positive integer k such that, for $\delta = 2^{-k}$, $f(u) + f(v) + f(w) = 0$ holds whenever $\|u\| < \delta$, $\|v\| < \delta$, $\|w\| < \delta$, and $u + v + w = 0$. Then for $\|x\| < \delta$, $\|y\| < \delta$ and $\|x + y\| < \delta$ we have $f(x) + f(-x) = f(x) + f(-x) + f(0) = 0$, hence $f(-x) = -f(x)$. The equality $(x + y) + (-x) + (-y) = 0$ implies $f(x + y) = f(x) + f(y)$.

(c) $\Longrightarrow$ (a): Let $\sum\limits_{n=1}^{\infty} a_n$ be a convergent series. We show that the series $\sum\limits_{n=1}^{\infty} f(a_n)$ fulfils Cauchy's criterion of convergence. Since the series $\sum\limits_{n=1}^{\infty} a_n$ is convergent, it follows that there is a positive integer m_0 such that $\|a_n\| < \delta$ holds for each $n > m_0$. Using the mathematical induction and (c) (ii) we can show for each natural number s

$$f\left(\sum_{j=1}^{s} x_j\right) = \sum_{j=1}^{s} f(x_j) \tag{6}$$

whenever $\|x_1\| < \delta$, $\|x_2\| < \delta$, $\dots$, $\|x_s\| < \delta$, $\|x_1 + x_2\| < \delta$, $\dots$, $\|x_1 + x_2 + \dots + x_s\| < \delta$. Let $\varepsilon > 0$. It follows from the continuity of f at 0 that there is an η, $0 < \eta < \delta$, such that

$$\|f(x)\| < \varepsilon \tag{7}$$

holds for each $x \in \mathbb{R}^N$ with $\|x\| < \eta$. Since the series $\sum\limits_{n=1}^{\infty} a_n$ is convergent, for $\eta > 0$ there exists a positive integer $m_1 > m_0$ such that for each $m > m_1$ and each $p \geq 1$ we have $\|a_{m+1} + \dots + a_{m+p}\| < \eta$. Consequently, (6) and (7) imply $\|f(a_{m+1} + \dots + f(a_{m+p})\| = \left\|f\Bigl(\sum\limits_{j=1}^{p} a_{m+j}\Bigr)\right\| < \varepsilon$, and, according to Cauchy's criterion, the series $\sum\limits_{n=1}^{\infty} f(a_n)$ is convergent. □

It is well known that every continuous additive function $f\colon \mathbb{R}^N \to \mathbb{R}^N$ is of the form $f(x) = xA$, where $x = (\xi_1, \dots, \xi_N) \in \mathbb{R}^N$, and A is a square matrix of the type $N \times N$ over $\mathbb{R}$. If $f(x) = (\eta_1, \dots, \eta_N)$, then $\eta_i = f_i(x)$, where $f_i\colon \mathbb{R}^N \to \mathbb{R}$ is a continuous additive function for each $i = 1, \dots, N$.

COROLLARY 1. *Let $f\colon \mathbb{R}^N \to \mathbb{R}^N$. Then f is a convergence preserving transformation if and only if there is $\delta > 0$ such that for x, $\|x\| < \delta$, $f(x) = xA$, where A is a constant square matrix of type $N \times N$ over $\mathbb{R}$.*

A proof of Corollary 1 is a consequence of the fact that every function f fulfilling conditions (c) (i) and (c) (ii) of Theorem is for $\|x\| < \delta$ the restriction of a continuous additive function $f\colon \mathbb{R}^N \to \mathbb{R}^N$ (see [K; pp. 332, 130]).

Let $\mathbb{C}$ be the set of all complex numbers. Since $\mathbb{C}$ may be considered as $\mathbb{R}\times\mathbb{R}$, Corollary 2 follows from a know representation of the continuous additive function $f\colon \mathbb{C} \to \mathbb{C}$ (see [K; p. 132]).

COROLLARY 2. *Let $f\colon \mathbb{C} \to \mathbb{C}$. Then f is a convergence preserving transformation if and only if there is $\delta > 0$ such that for z, $|z| < \delta$, $f(z) = c_1 z + c_2 \overline{z}$, where $c_1 \in \mathbb{C}$ and $c_2 \in \mathbb{C}$ are constants, and $\overline{z}$ denotes the complex conjugate of z.*

Note. The paper generalizes and expands some of known results concerning convergence preserving transformation (see [R], [Ša; p. 84], [Sm] and [W]).

PROBLEM 1. This paper was motivated by the following conjecture risen by Professor T. Šalát: Let τ be a permutation of the set of all positive integers such that for each real convergent series $\sum\limits_{n=1}^{\infty} a_n$ the series $\sum\limits_{n=1}^{\infty} a_{\tau(n)}$ has bounded its partial sums. Then for each convergent series $\sum\limits_{n=1}^{\infty} b_n$ the series $\sum\limits_{n=1}^{\infty} b_{\tau(n)}$ converges, and $\sum\limits_{n=1}^{\infty} b_{\tau(n)} = \sum\limits_{n=1}^{\infty} b_n$. This problem remains open.

PROBLEM 2. Can our Theorem be formulated and proved also for some infinite dimensional Banach spaces?

REFERENCES

[R] RADO, R.: *A theorem of infinite series*, J. London Math. Soc. **XXXV** (1960), 273–276.

[Ša] ŠALÁT, T.: *Infinite Series* (Slovak), Academia, Praha, 1974.

[Sh] SHILOV, G. E.: *Mathematical Analysis, Functions of a single variable. Part 3* (Russian), Nauka, Moskva, 1970.

[Sm] SMITH, A.: *Convergence preserving functions: an alternative discussion*, Amer. Math. Monthly **96** (1991), 831–833.

[K] KUCZMA, M.: *An Introduction to the Theory of Functional Equations and Inequalities*, PWN, Warszawa-Kraków-Katowice, 1985.

[W] WILDENBERG, G.: *Convergence preserving functions*, Amer. Math. Monthly **95** (1988), 542–544.

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