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Dedicated to Professor Tibor Šalát
on the occasion of his 70th birthday

ON ESTIMATES OF FUNCTIONALS IN SOME CLASSES OF FUNCTIONS WITH POSITIVE REAL PART[1]

JAROSLAV FUKA* — Z. J. JAKUBOWSKI**

(*Communicated by Michal Zajac*)

ABSTRACT. Let $\mathcal{P}$ denote the well-known class of functions of the form $p(z) = 1 + q_1 z + \dots + q_n z^n + \dots$ holomorphic in the unit disc $\mathbb{D}$ with $\operatorname{Re} p(z) > 0$ in $\mathbb{D}$. The subclasses $\mathcal{P}(B, b, \alpha; F)$ and $\mathcal{P}(B, b, \alpha)$ of the class $\mathcal{P}$ will be studied (see Definition 1). For $\mathcal{P}(B, b, \alpha; F)$ the set of values of the kth coefficient, $k = 1, 2, \dots$, will be described in the case $F = F_n = \bigcup_{k=1}^{n} F_n^k$, $n = 1, 2, \dots$, where $F_n^k = \{z \in \mathbb{T};\ z = \mathrm{e}^{\frac{2k\pi\mathrm{i}}{n}} \mathrm{e}^{\rho\mathrm{i}},\ -\frac{\alpha\pi}{n} \le \rho \le \frac{\alpha\pi}{n}\}$, $\mathbb{T}$ – unit circle. In $\mathcal{P}(B, b, \alpha)$, the set of values of the kth coefficient will be described, the sharp twosided estimates of $\operatorname{Re} p(z)$ and $\operatorname{Im} p(z)$ in a given point $z \in \mathbb{D}$ are found, and the non-compactness of $\mathcal{P}(B, b, \alpha)$ in the topology given by uniform convergence on compact subsets of $\mathbb{D}$ is proved. The article belongs to the series of papers [1]–[4], where different classes of functions defined by conditions on the unit circle $\mathbb{T}$ were studied.

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The geometric function theory originated around the turn of the century in connection with the work of P. Koebe on the uniformization of Riemann surfaces. It developed in an independent branch with proper methods, combining

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geometric and analytic considerations, and with proper problems highly stimulated by the famous Bieberbach's conjecture from 1916 about the estimates of Taylor coefficients of normalized univalent functions in the unit disc $\mathbb{D}$. This problem, solved almost seventy years later in 1984 by L. de Branges, suggested in a natural way the study of other functionals and other subclasses of functions holomorphic in $\mathbb{D}$.

Our aim in this article is to study a subclass $\mathcal{P}(B, b, \alpha)$ (see Definition 1) of the well-known Carathéodory class $\mathcal{P}$. The motivation for introducing this class can roughly be described as follows. Several authors investigated new classes lying in some sense, which will not be specified here, between two known classes and joining them "homotopically". These between-classes depending on a parameter were defined by analytic conditions imposed in the whole unit disc. The question arises, if it is possible to join homotopically two classes by conditions imposed only near the boundary of $\mathbb{D}$. The classes $\mathcal{P}(B, b, \alpha)$, $0 \le b < B < 1$, realize this idea in the dependence on the parameter α: for $\alpha = 0$ and $\alpha = 1$, $\mathcal{P}(b, B, \alpha)$ reduces to the class of Carathéodory functions of the order b and B, respectively.

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As usual, we shall denote by $\mathbb{C}$ the complex plane, by $\mathbb{D} = \{z \in \mathbb{C};\ |z| < 1\}$ the unit disc, by $\mathbb{T} = \{z \in \mathbb{C};\ |z| = 1\}$ the unit circle, by $K(S; r) = \{z \in \mathbb{C};\ |z - S| \le r\}$ the closed disc centered in S and with radius $r \ge 0$, by $K(S; r_1, r_2) = \{z \in \mathbb{C};\ r_1 \le |z - S| \le r_2\}$ the closed ring centered in S and with radii $0 < r_1 \le r_2$.

Let $\mathcal{P}$ denote the class of functions of the form

$$p(z) = 1 + q_1 z + \dots + q_n z^n + \dots \tag{1}$$

holomorphic in $\mathbb{D}$ with $\operatorname{Re} p(z) > 0$ for $z \in \mathbb{D}$, and, for a given set $F \subset \mathbb{T}$, let $F_\tau = \{\xi \in \mathbb{T};\ \mathrm{e}^{-\mathrm{i}\tau}\xi \in F\}$ be the set arising by rotation of F through the angle τ. We will frequently use the convention of identifying the set $F \subset \mathbb{T}$ with the corresponding subset of $\mathbb{R}$.

Let us recall the following well-known facts. Every function $\operatorname{Re} p(z)$, $p \in \mathcal{P}$, has the Poisson representation by means of a unique positive measure μ of total mass 1 ([6; pp. 11–12]). Moreover, $\operatorname{Re} p(z)$ has nontangential limits a.e. on $\langle -\pi, \pi \rangle$ (to be denoted $\operatorname{Re} p(\cdot)$), which are equal to $f(\mathrm{e}^{\mathrm{i}\theta})$ a.e. on $\langle -\pi, \pi \rangle$, where $f(t)$ is the density of the absolutely continuous part of the Lebesgue decomposition of the representing measure μ with respect to the normalized Lebesgue measure $\dfrac{\mathrm{d}t}{2\pi}$ on $\langle -\pi, \pi \rangle$ ([6; Chapter 1, Theorem 5.3]).

DEFINITION 1. (see [4]) Let $0 \le b < 1$, $b < B$, $0 < \alpha < 1$ be fixed real numbers.

a) Let $F \subset \mathbb{T}$ be a set of Lebesgue measure $2\pi\alpha$. By $\mathcal{P}(B,b,\alpha;F)$, we denote the class of functions $p \in \mathcal{P}$ satisfying the following conditions: there exists $\tau = \tau(p) \in \langle -\pi, \pi)$ such that

$$\operatorname{Re} p(\mathrm{e}^{\mathrm{i}\theta}) \ge B \qquad \text{a.e. on} \quad F_\tau\,,$$

and

$$\operatorname{Re} p(\mathrm{e}^{\mathrm{i}\theta}) \ge b \qquad \text{a.e. on} \quad \mathbb{T} \setminus F_\tau\,.$$

b) By $\mathcal{P}(B,b,\alpha)$, we denote the class of functions $p \in \mathcal{P}$ such that there exists a measurable set $F = F(p)$, $F \subset \mathbb{T}$, of Lebesgue measure $2\pi\alpha$ such that

$$\operatorname{Re} p(\mathrm{e}^{\mathrm{i}\theta}) \ge B \qquad \text{a.e. on} \quad F\,, \tag{2}$$

and

$$\operatorname{Re} p(\mathrm{e}^{\mathrm{i}\theta}) \ge b \qquad \text{a.e. on} \quad \mathbb{T} \setminus F\,. \tag{3}$$

c) By $\tilde{\mathcal{P}}(B,b,\alpha)$, we denote the class of functions $p \in \mathcal{P}$ such that there exists an open arc $I = I(p) \subset \mathbb{T}$ of Lebesgue measure $2\pi\alpha$ such that (2) and (3) are fulfilled for $F = I$.

d) Let $F \subset \mathbb{T}$ be a fixed set of Lebesgue measure $2\pi\alpha$. By $\check{\mathcal{P}}(B,b,\alpha;F)$, we denote the class of functions $p \in \mathcal{P}$ fulfilling (2) and (3).

In this section, the set of values of the nth coefficient in the classes $\mathcal{P}(B,b,\alpha;F)$ and $\mathcal{P}(B,b,\alpha)$ will be examined. Recall that the set of values of the nth coefficient in some class C of functions holomorphic in $\mathbb{D}$ is the set $C^{(n)} = \left\{c_n(f)\,;\ f \in C\,,\ f(z) = \sum\limits_{k=0}^{\infty} c_k(f) z^k\right\}$, and that the condition $\eta \le 1$, where $\eta = \alpha B + (1-\alpha)b$, is necessary and sufficient for $\check{\mathcal{P}}(B,b,\alpha;F) \ne \emptyset$ (see [4; Corollary 1]). For $\eta = 1$ the class $\check{\mathcal{P}}(B,b,\alpha;F)$ contains only one function $p_F(z) = b + \dfrac{B-b}{2\pi} \displaystyle\int_F \frac{\mathrm{e}^{\mathrm{i}t} + z}{\mathrm{e}^{\mathrm{i}t} - z}\,\mathrm{d}t$, $z \in \mathbb{D}$ ([4; Corollary 1]). The triple (B,b,α) fulfilling the conditions of Definition 1 and $\alpha B + (1-\alpha)b \le 1$ will be called *admissible*, and we further tacitly assume, that all the triples (B,b,α) are admissible.

We begin with two simple but important lemmas.

LEMMA 1. *The set* $\check{\mathcal{P}}^{(k)}(B,b,\alpha;F)$, $k = 1,2,\dots$, *is the disc*

$$K\big(S_F; 2(1-\eta)\big)\,,$$

where $S_F = \dfrac{B-b}{\pi} \displaystyle\int_F \mathrm{e}^{-\mathrm{i}kt}\,\mathrm{d}t\,.$

P r o o f. By [4; Proposition 1], $\check{\mathcal{P}}(B,b,\alpha;F)=\{p_F+(1-\eta)p;\ p\in\mathcal{P}\}$, and the correspondence $p\rightarrow p_F+(1-\eta)p$ between the classes $\mathcal{P}$ and $\check{\mathcal{P}}(B,b,\alpha;F)$ is one-to-one. Here as above $p_F(z)=b+\frac{B-b}{2\pi}\int\limits_F\frac{e^{it}+z}{e^{it}-z}\,dt$. So, for every $\check{p}\in\check{\mathcal{P}}(B,b,\alpha;F)$, $\check{p}=p_F+(1-\eta)p$, we obtain by elementary calculation

$$\check{p}(z)=1+\sum_{k=1}^{\infty}\left[\frac{B-b}{\pi}\int\limits_F e^{-ikt}\,dt+(1-\eta)q_k\right]z^k,$$

where $p(z)=1+\sum\limits_{k=1}^{\infty}q_kz^k$. Hence, writing $\check{p}(z)=1+\sum\limits_{k=1}^{\infty}\check{q}_kz^k$, we obtain

$$\check{q}_k=\frac{B-b}{\pi}\int\limits_F e^{-ikt}\,dt+(1-\eta)q_k\,,\qquad k=1,2,\dots\,, \tag{4}$$

and

$$\left|\,\check{q}_k-\frac{B-b}{\pi}\int\limits_F e^{-ikt}\,dt\,\right|=(1-\eta)|q_k|\leq 2(1-\eta)\,,$$

because, for every $p\in\mathcal{P}$, $|q_k|\leq 2$ (see, e.g., [5; vol. I, pp. 80–81]).

On the other hand, let the point $w\in\mathbb{C}$ fulfil

$$|w-S_F|\leq 2(1-\eta)\,,\qquad k=1,2,\dots\,,\ \ \eta\leq 1\,.$$

If $\eta=1$, put $\check{p}=p_F$. If $\eta<1$, put

$$u=\frac{w-S_F}{1-\eta}\,.$$

Then $|u|\leq 2$. Let $\sqrt[k]{\frac{u}{2}}$ be a fixed kth root from $\frac{u}{2}$. The function

$$p(z)=\frac{1+\sqrt[k]{\frac{u}{2}}z}{1-\sqrt[k]{\frac{u}{2}}z}\,,\qquad z\in\mathbb{D},$$

belongs to $\mathcal{P}$, its kth coefficient is exactly u, and, from (4), it follows that w is the kth coefficient of the function $\check{p}=p_F+(1-\eta)p$ lying in $\check{P}(B,b,\alpha;F)$. The lemma is proved. □

LEMMA 2. *The set* $\mathcal{P}^{(k)}(B,b,\alpha;F)$, $k=1,2,\dots$, *is the sum* $\bigcup\limits_{\tau\in(-\pi,\pi)}K\big(S_\tau;2(1-\eta)\big)$, *where the centers* S_τ *fill the whole circle*

$$|w|=\frac{B-b}{\pi}\left|\int\limits_F e^{-ikt}\,dt\right|=|S_F|\,.$$

Proof. Follows at once from Lemma 1, if we put $F = F_\tau$, $\tau \in \langle -\pi, \pi)$, and realize that $\int\limits_{F_\tau} \mathrm{e}^{-ikt}\, \mathrm{d}t = \mathrm{e}^{-ik\tau} \int\limits_{F} \mathrm{e}^{-ikt}\, \mathrm{d}t$.

From Lemma 2, it follows that the sets of values of the kth coefficient, $k = 1, 2, \dots$, in $\mathcal{P}(B, b, \alpha; F)$ are discs or rings. However, it is not clear that the second case, which would be interesting from the point of view of the geometric function theory, can actually take place. Therefore we will analyse now a concrete but characteristic special case $F = F_n$, $n = 1, 2, \dots$, where

$$F_n = \bigcup_{k=1}^{n} F_n^k, \qquad F_n^k = \left\{ z \in \mathbb{T};\ z = \mathrm{e}^{\frac{2k\pi\mathrm{i}}{n}} \cdot \mathrm{e}^{\mathrm{i}\rho},\ -\frac{\alpha\pi}{n} \le \rho \le \frac{\alpha\pi}{n} \right\}. \tag{5}$$

□

In this case, we have the following

THEOREM 1. *Let n be a given positive integer, k an arbitrary positive integer, and (B, b, α) and admissible triple. Then the set $\mathcal{P}^{(k)}(B, b, \alpha; F_n)$ of values of the kth coefficient in the class $\mathcal{P}(B, b, \alpha; F_n)$ is*

(i) *the disc*

$$K\big(0; 2(1-\eta)\big)$$

if n does not divide k,

(ii) *the disc*

$$K\Big(0;\ 2\Big[(B-b)\tfrac{|\sin \alpha\pi r|}{\pi r} + 1 - \eta\Big]\Big)$$

if $k = rn$ and $\alpha + \frac{|\sin \alpha\pi r|}{\pi r} \le \frac{1-b}{B-b}$, $r = 1, 2, \dots$,

(iii) *the ring*

$$K\Big(0;\ 2\Big[(B-b)\tfrac{|\sin \alpha\pi r|}{\pi r} - (1-\eta)\Big],\ 2\Big[(B-b)\tfrac{|\sin \alpha\pi r|}{\pi r} + 1 - \eta\Big]\Big)$$

if $k = rn$, $\frac{1-b}{B-b} < \alpha + \frac{|\sin \alpha\pi r|}{\pi r}$ and $B > 1$.

Proof. First we have to find the coefficients in the class $\check{P}(B, b, \alpha; F_n)$. By the definition of p_F and by (5), we have to determine the coefficients of the function

$$h_{F_n}(z) = \frac{1}{2\pi} \sum_{j=1}^{n} \int\limits_{(-\alpha+2j)\frac{\pi}{n}}^{(\alpha+2j)\frac{\pi}{n}} \frac{\mathrm{e}^{\mathrm{i}t} + z}{\mathrm{e}^{\mathrm{i}t} - z}\, \mathrm{d}t\,.$$

By elementary calculations and using that $\sum\limits_{j=1}^{n} \big(\mathrm{e}^{-2\mathrm{i}\pi\frac{k}{n}}\big)^j$ is n if n divides k and is zero if n does not divide k, we obtain

$$h_{F_n}(z) = \alpha + 2 \sum_{r=1}^{\infty} \frac{\sin \alpha\pi r}{\pi r} z^{rn}\,. \tag{6}$$

If p is of form (1), we finally obtain (using $b+(B-b)\alpha=\eta$)

$$\check{p}(z)=1+2(B-b)\sum_{r=1}^{\infty}\frac{\sin\alpha\pi r}{\pi r}z^{rn}+(1-\eta)\sum_{k=1}^{\infty}q_k z^k\,,\qquad z\in\mathbb{D}, \tag{7}$$

for every $\check{p}\in\check{P}(B,b,\alpha;F)$. Writing $\check{p}(z)=1+\sum_{k=1}^{\infty}\check{q}_k z^k$, we have by (7) $\check{q}_k=(1-\eta)q_k$ if n does not divide k, and so, by Lemma 1, we obtain (i). Let n divide k, i.e, $k=rn$ for some $r=1,2,\dots$. Then $\check{q}_k=2(B-b)\frac{\sin\alpha\pi r}{\pi r}+(1-\eta)q_k$. By Lemma 2, we have to compare the quantities $2(B-b)\frac{|\sin\alpha\pi r|}{\pi r}$ and $2(1-\eta)$. By definition of η, we obtain

$$(B-b)\frac{|\sin\alpha\pi r|}{\pi r}-(1-\eta)=(B-b)\left[\alpha+\frac{|\sin\alpha\pi r|}{\pi r}-\frac{1-b}{B-b}\right]. \tag{8}$$

From (8), we see that, if $\alpha+\frac{|\sin\alpha\pi r|}{\pi r}\le\frac{1-b}{B-b}$, $r=1,2,\dots$, we obtain (ii), and if $\frac{1-b}{B-b}<\alpha+\frac{|\sin\alpha\pi r|}{\pi r}$, we obtain (iii). □

Remark 1. If $B\le 1$ and $0\le b<B$, the inequality $\frac{1-b}{B-b}<\alpha+\frac{|\sin\alpha\pi r|}{\pi r}$, $r=1,2,\dots$, cannot be satisfied for $\alpha\in(0,1)$. This follows from the following considerations. It is easy to show, that the function $\Phi_r(\alpha)=\alpha+\frac{|\sin\alpha\pi r|}{\pi r}$ is increasing in the interval $\langle 0,1\rangle$. So $\Phi_r(\alpha)\le 1$ for $\alpha\in\langle 0,1\rangle$. This gives, together with the inequality $1\le\frac{1-b}{B-b}$ valid for $B\le 1$, the inequality $\alpha+\frac{|\sin\alpha\pi r|}{\pi r}\le\frac{1-b}{B-b}$.

Notice also that, in case (iii) of Theorem 1, the possibility $\sin\alpha\pi r=0$ is automatically excluded, because we supposed that the triple (B,b,α) is admissible, and the inequality $\eta\le 1$ is equivalent to $\alpha\le\frac{1-b}{B-b}$.

COROLLARY 1. *The set $\mathcal{P}^{(n)}(B,b,\alpha;F_n)$, $n=1,2,\dots$, is*

(i) *the disc*

$$K\Big(0;\,2\big[(B-b)\tfrac{\sin\alpha\pi}{\pi}+1-\eta\big]\Big)$$

if $\alpha+\frac{\sin\alpha\pi}{\pi}\le\frac{1-b}{B-b}$,

(ii) *the ring*

$$K\Big(0;\,2\big[(B-b)\tfrac{\sin\alpha\pi}{\pi}-(1-\eta)\big],\,2\big[(B-b)\tfrac{\sin\alpha\pi}{\pi}+1-\eta\big]\Big)$$

if $\frac{1-b}{B-b}<\alpha+\frac{\sin\alpha\pi}{\pi}$ and $B>1$.

By Definition 1 c) and Theorem 1, we obtain

COROLLARY 2. *The set* $\tilde{\mathcal{P}}^{(k)}(B,b,\alpha)$, $k=1,2,\dots$, *is*

(i) *the disc*

$$K\Big(0;\ 2\big[(B-b)\tfrac{|\sin\alpha k\pi|}{k\pi}+1-\eta\big]\Big)$$

if $\alpha+\frac{|\sin\alpha k\pi|}{k\pi}\le\frac{1-b}{B-b}$,

(ii) *the ring*

$$K\Big(0;\ 2\big[(B-b)\tfrac{|\sin\alpha k\pi|}{k\pi}-(1-\eta)\big],\ 2\big[(B-b)\tfrac{|\sin\alpha k\pi|}{k\pi}+1-\eta\big]\Big)$$

if $\frac{1-b}{B-b}<\alpha+\frac{|\sin\alpha k\pi|}{k\pi}$ *and* $B>1$.

Remark 2. Let us point out that there exists an admissible triple (B,b,α) and a positive integer p, $p\neq 1$, so that the set of values of the pth coefficient in the class $\mathcal{P}(B,b,\alpha;F_1)\cup\mathcal{P}(B,b,\alpha;F_p)$, which is clearly equal to $\mathcal{P}^{(p)}(B,b,\alpha;F_1)\cup\mathcal{P}^{(p)}(B,b,\alpha;F_p)$, consists of two disjoint rings.

We will not discuss in detail various possibilities offering themselves in this direction and only sketch the above mentioned case. Let $B>1$ and $0<b<1$. Then $\dfrac{1-b}{B-b}<1$, and so for every α such that $0<\alpha<\dfrac{1-b}{B-b}$ the triple (B,b,a) is admissible. For every $r=1,2,\dots$ the function $\Psi_r(\alpha)=\alpha+\dfrac{|\sin\alpha\pi r|}{\pi r}-\dfrac{1-b}{B-b}$ is continuous and increasing in $\langle 0,1\rangle$. Since $\Psi_r(0)=-\dfrac{1-b}{B-b}<0$ and $\Psi_r\big(\frac{1-b}{B-b}\big)\ge 0$, there exists exactly one root α_r of the equation $\Psi_r(\alpha)=0$ in the interval $\big(0,\frac{1-b}{B-b}\big\rangle$. If $\alpha_r=\dfrac{1-b}{B-b}$, then $\sin\alpha_r\pi r=0$, and so $\alpha_r=\dfrac{\ell}{r}$, $\ell=1,2,\dots,r-1$. But, for any given $B>1$, one can easily choose $b\in(0,1)$ so that $\dfrac{1-b}{B-b}$ is an irrational number lying in $(0,1)$. Then, for each $r=1,2,\dots$, $\alpha_r\in\big(0,\frac{1-b}{B-b}\big)$, and for every $\alpha\in\big(\alpha_r,\frac{1-b}{B-b}\big)$ one has $\alpha+\dfrac{|\sin\alpha\pi r|}{\pi r}>\dfrac{1-b}{B-b}$, since $\Psi_r(\alpha)$ is increasing and $\Psi_r(\alpha_r)=0$. Summarizing, we have the following. To every $B>1$ there exists $b\in(0,1)$ with the following property: for each $r=1,2,\dots$ there is a unique $\alpha_r\in\big(0,\frac{1-b}{B-b}\big)$ so that for every $\alpha\in\big(\alpha_r,\frac{1-b}{B-b}\big)$ the triple (B,b,α) is admissible, and, at the same time, $\alpha+\dfrac{|\sin\alpha\pi r|}{\pi r}>\dfrac{1-b}{B-b}$. Hence, in this situation, it is possible to use Corollary 1 (ii) and Corollary 2 (ii), and we obtain

$$\mathcal{P}^{(p)}(B,b,\alpha;F_p) = K\Big(0;\ 2\big[(B-b)\tfrac{\sin\alpha\pi}{\pi} - (1-\eta)\big],\ 2\big[(B-b)\tfrac{\sin\alpha\pi}{\pi} + 1 - \eta\big]\Big)$$

and

$$\mathcal{P}^{(p)}(B,b,\alpha;F_1) = K\Big(0;\ 2\big[(B-b)\tfrac{|\sin\alpha p\pi|}{p\pi} - (1-\eta)\big],\ 2\big[(B-b)\tfrac{|\sin\alpha p\pi|}{p\pi} + 1 - \eta\big]\Big),$$

respectively. So we have to show that the inequality

$$2\left[(B-b)\frac{|\sin\alpha p\pi|}{p\pi} + 1 - \eta\right] < 2\left[(B-b)\frac{|\sin\alpha p\pi|}{p\pi} - (1-\eta)\right]$$

holds for some α, p, $\max(\alpha_1,\alpha_p) < \alpha < \dfrac{1-b}{B-b}$. This inequality is equivalent to

$$2\pi\Big(\frac{1-b}{B-b} - \alpha\Big) < \sin\alpha\pi - \frac{|\sin\alpha p\pi|}{p\pi}\,. \tag{9}$$

The limit of the left hand side of (9) for $\alpha \to \dfrac{1-b}{B-b}$ is zero, and the limit of the right hand side of (9) for $\alpha \to \dfrac{1-b}{B-b}$ is not less than $\sin\dfrac{1-b}{B-b}\pi - \dfrac{1}{p}$, and this expression can be made positive by choosing p sufficiently large.

Remark 3. Letting $B \to b$ or $\alpha \to 0$ or $\alpha \to 1$ we obtain easily from Theorem 1 the well-known result

$$\mathcal{P}_b^{(k)} = K\big(0; 2(1-b)\big)$$

for the class $\mathcal{P}_b$ of Carathéodory functions of the order b, $0 \le b < 1$.

It seems to be an interesting problem to characterize the set $\mathcal{P}^{(n)}(B,b,\alpha)$, $n = 1,2,\dots$, for all admissible (B,b,α). We have the following

THEOREM 2. *The set $\mathcal{P}^{(n)}(B,b,\alpha)$, $n = 1,2,\dots$, of values of the nth coefficient in the class $\mathcal{P}(B,b,\alpha)$ is the disc $K(0,R)$, where $R = 2\big[(B-b)\frac{\sin\alpha\pi}{\pi} + 1 - \eta\big]$.*

Proof. Let $p \in \mathcal{P}(B,b,\alpha)$ be of form (1). By [4; Theorem 8], we have $|q_n| \le R$, hence $\mathcal{P}^{(n)}(B,b,\alpha) \subset K(0,R)$.

If $p \in \mathcal{P}(B,b,\alpha)$, then $p_\varepsilon(z) =: p(\varepsilon z)$, $|\varepsilon| = 1$, belongs also to $\mathcal{P}(B,b,\alpha)$. By (1),

$$p_\varepsilon(z) = 1 + q_1\varepsilon z + \dots + q_n\varepsilon^n z^n + \dots\,, \qquad z \in \mathbb{D}.$$

So, if $0 < w_0 \le R$ and $w_0 \in \mathcal{P}^{(n)}(B,b,\alpha)$, the whole circle $|w| = w_0$ is contained in $\mathcal{P}^{(n)}(B,b,\alpha)$. Consequently, it suffices to prove

$$\langle 0, R\rangle \subset \mathcal{P}^{(n)}(B,b,\alpha)\,. \tag{10}$$

Fix $n \geq 1$. We will first describe the idea of the proof. Consider the set $F(0)$, arising by rotation of F_{2n} (see (5)) through the angle $\frac{\pi}{2n}$. The intervals $F_k(0)$ constituting the set $F(0)$ are concentrated around the roots $\tau_k = (2k+1)\frac{\pi}{2n}$, $k = -n, -n+1, \dots, n-1$, of $\cos nt$ in the interval $(-\pi, \pi)$. Translating every $F_k(0)$ in the direction of the growth of $\cos nt$, we obtain a family $F(x)$ of sets consisting of $2n$ intervals for $0 \leq x < \frac{1-\alpha}{2}$. For $x \to \frac{1-\alpha}{2}$ these intervals glue to n intervals forming the set $F\left(\frac{1-\alpha}{2}\right) = F_n$ from (5). We show that the Taylor coefficients

$$q_n\left(h_{F(x)}\right) = \frac{1}{\pi} \int\limits_{F(x)} \mathrm{e}^{-\,int}\, \mathrm{d}t \tag{6'}$$

of functions $h_{F(x)}(z) = \frac{1}{2\pi} \int\limits_{F(x)} \frac{\mathrm{e}^{it} + z}{\mathrm{e}^{it} - z}\, \mathrm{d}t$, $0 \leq x \leq \frac{1-\alpha}{2}$, fill the whole interval $\left\langle 0, \frac{2 \sin \alpha\pi}{\pi} \right\rangle$. A second homotopy (see (15)), gives then the result.

So define $F(0) = \sum\limits_{k=-n}^{n-1} F_k(0)$, where $F_k(0) = \left\langle \tau_k - \frac{\alpha\pi}{2n}, \tau_k + \frac{\alpha\pi}{2n} \right\rangle$. Clearly, $F_i(0) \cap F_j(0) = \emptyset$ for $i \neq j$, and the Lebesgue measure of $F(0)$ is $2\pi\alpha$. By Theorem 1 (i) (see also (6')), $q_n\left(h_{F_{2n}}\right) = 0$, $q_n\left(h_{F(0)}\right) = \left(\mathrm{e}^{\mathrm{i}\frac{\pi}{2n}}\right)^n q_n\left(h_{F_{2n}}\right) = 0$.

First, let n be even, $n = 2m$, $m = 1, 2, \dots$. Define

$$F_k(x) = \left\langle \tau_k - \tfrac{\alpha\pi}{2n} - \tfrac{x\pi}{n}, \tau_k + \tfrac{\alpha\pi}{2n} - \tfrac{x\pi}{n} \right\rangle \quad \text{if} \quad k = -n, -n+2, \dots, n-2, \tag{11}$$

$$F_k(x) = \left\langle \tau_k - \tfrac{\alpha\pi}{2n} + \tfrac{x\pi}{n}, \tau_k + \tfrac{\alpha\pi}{2n} + \tfrac{x\pi}{n} \right\rangle \quad \text{if} \quad k = -n+1, -n+3, \dots, n-1, \tag{12}$$

$$F(x) = \bigcup_{k=-n}^{n-1} F_k(x), \tag{13}$$

for $0 \leq x < \frac{1-\alpha}{2}$. $F(x)$ is the sum of $2n$ nonoverlapping closed intervals of length $\frac{\alpha\pi}{n}$, so the Lebesgue measure of $F(x)$ is $2\pi\alpha$. For $x = \frac{1-\alpha}{2}$, taking into account the special role played by $-\pi$ and π, $F\left(\frac{1-\alpha}{2}\right)$ is the sum of n nonoverlapping intervals of length $\frac{2\alpha\pi}{n}$ and $F\left(\frac{1-\alpha}{2}\right) = F_n$ (see (5)). Now, we calculate $q_n\left(h_{F(x)}\right)$. We have, by (11), (12), (13),

$$\begin{aligned}
\pi q_n\big(h_{F(x)}\big) &= \int\limits_{F(x)} \mathrm{e}^{-int}\,\mathrm{d}t \\
&= \sum_{\ell=-m}^{m-1} \int\limits_{F_{2\ell}(x)} \mathrm{e}^{-int}\,\mathrm{d}t + \sum_{\ell=-m}^{m-1} \int\limits_{F_{2\ell+1}(x)} \mathrm{e}^{-int}\,\mathrm{d}t \\
&= \sum_{\ell=-m}^{m-1} \Big(\frac{1}{\mathrm{i}n}\Big)\, \mathrm{e}^{-\mathrm{i}\frac{\pi}{2}(4\ell+1-2x)} \big(\mathrm{e}^{-\mathrm{i}\frac{\alpha\pi}{2}} - \mathrm{e}^{\mathrm{i}\frac{\alpha\pi}{2}}\big) \\
&\qquad + \sum_{\ell=-m}^{m-1} \Big(\frac{1}{-\mathrm{i}n}\Big)\, \mathrm{e}^{-\mathrm{i}\frac{\pi}{2}(4\ell+1+2x)} \big(\mathrm{e}^{-\mathrm{i}\frac{\alpha\pi}{2}} - \mathrm{e}^{\mathrm{i}\frac{\alpha\pi}{2}}\big) \\
&= 4\sin\frac{\alpha\pi}{2}\sin\pi x \cdot \frac{1}{n} \sum_{\ell=-m}^{m-1} \mathrm{e}^{-2\mathrm{i}\pi\ell} = 4\sin\frac{\alpha\pi}{2}\cdot\sin\pi x
\end{aligned}$$

with regard to $n = 2m$.

The function $f(x) = 4\sin\frac{\alpha\pi}{2}\sin\pi x$ is continuous and strictly increasing in $\langle 0, \frac{1-\alpha}{2}\rangle$, $f(0) = 0$, $f\big(\frac{1-\alpha}{2}\big) = 2\sin\alpha\pi$, so the numbers $q_n\big(F(x)\big)$, $0 \le x \le \dfrac{1-\alpha}{2}$, fill the whole interval $\langle 0, \frac{2\sin\alpha\pi}{\pi}\rangle$. Hence we see, since $p_{F(x)}(z) = b + (B-b)h_{F(x)}(z) + 1 - \eta$ belongs to $\mathcal{P}(B,b,\alpha)$,

$$\big\langle 0, 2(B-b)\tfrac{\sin\alpha\pi}{\pi}\big\rangle \subset \mathcal{P}^{(n)}(B,b,\alpha)\,. \tag{14}$$

Finally, for $0 \le y \le 1$, the function

$$p_y(z) = b + (B-b)h_{F(\frac{1-\alpha}{2})}(z) + (1-\eta)\frac{1+yz^n}{1-yz^n} \tag{15}$$

belongs to $\mathcal{P}(B,b,\alpha)$, and the coefficient $q_n(p_y) = 2\Big[(B-b)\dfrac{\sin\alpha\pi}{\pi} + (1-\eta)y\Big]$, so

$$\big\langle 2(B-b)\tfrac{\sin\alpha\pi}{\pi},\ R\big\rangle \subset \mathcal{P}^{(n)}(B,b,\alpha)\,. \tag{14'}$$

By (14), (14′), we have (10), and so the proof of the Theorem 2, for n even, is finished. □

If n is odd, we define $F_k(x)$ by (11) for $k = -n+1, -n+3, \dots, n-1$, and, by (12), for $k = -n, -n+2, \dots, n-2$ respectively, and proceeded as in the former case. The details are omitted.

2

Our aim in this section is to find sharp twosided estimates of $\operatorname{Re} p(z)$ and $\operatorname{Im} p(z)$ in a given point $0 \ne z \in \mathbb{D}$ if p runs over the whole class $\mathcal{P}(B,b,\alpha)$. As usual, $\operatorname{arctg} t \in \big(0, \frac{\pi}{2}\big)$ for $t > 0$ and $\arccos t \in \big(0, \frac{\pi}{2}\big)$ for $0 < t < 1$.

THEOREM 3. *Let* $p \in \mathcal{P}(B,b,\alpha)$, $0 \neq z \in \mathbb{D}$, $z = r\,\mathrm{e}^{\mathrm{i}\phi}$. *Then*

$$\begin{aligned} & b + \frac{2(B-b)}{\pi}\operatorname{arctg}\Big(\frac{1-r}{1+r}\operatorname{tg}\frac{\alpha\pi}{2}\Big) + (1-\eta)\frac{1-r}{1+r} \\ & \leq \operatorname{Re} p(z) \\ & \leq b + \frac{2(B-b)}{\pi}\operatorname{arctg}\Big(\frac{1+r}{1-r}\operatorname{tg}\frac{\alpha\pi}{2}\Big) + (1-\eta)\frac{1+r}{1-r}, \end{aligned} \tag{16}$$

and

$$\begin{aligned} & \frac{B-b}{2\pi}\log\frac{P_r(\gamma_0 - 2\pi x_0 + 2\pi\alpha)}{P_r(\gamma_0 - 2\pi x_0)} - (1-\eta)\frac{2r}{1-r^2} \\ & \leq \operatorname{Im} p(z) \\ & \leq \frac{B-b}{2\pi}\log\frac{P_r(\gamma_0 - 2\pi x_0)}{P_r(\gamma_0 - 2\pi x_0 + 2\pi\alpha)} + (1-\eta)\frac{2r}{1-r^2}, \end{aligned} \tag{17}$$

where x_0 *is the only root of the equation* $\cos(\gamma_0 - 2\pi x_0 + \pi\alpha) = \cos\gamma_0 \cdot \cos\alpha\pi$ *lying in the interval* $\left(0, \frac{\gamma_0}{\pi}\right)$, $\cos\gamma_0 = \dfrac{2r}{1+r^2}$, $\gamma_0 \in \left(0, \frac{\pi}{2}\right)$ *and* $P_r(t) = \dfrac{1-r^2}{1-2r\cos t + r^2}$.

The estimates (16) *and* (17) *are sharp and are attained by the functions* $p_0(\varepsilon z)$, $|\varepsilon| = 1$, *where*

$$p_0(z) = b + \frac{B-b}{2\pi}\int_F \frac{\mathrm{e}^{\mathrm{i}t} + z}{\mathrm{e}^{\mathrm{i}t} - z}\,\mathrm{d}t + (1-\eta)\frac{1+z}{1-z}, \qquad z \in \mathbb{D}, \tag{18}$$

and

$$\begin{aligned} F = F_1 &= \{\mathrm{e}^{\mathrm{i}t};\ t \in \langle -\pi, -\pi(1-\alpha)\rangle \cup \langle \pi(1-\alpha), \pi)\}, \\ F = F_2 &= \{\mathrm{e}^{\mathrm{i}t};\ t \in \langle -\alpha\pi, \alpha\pi\rangle\} \end{aligned} \tag{18$'$}$$

for estimates (16), *and*

$$\begin{aligned} F = F_3 &= \{\mathrm{e}^{\mathrm{i}t};\ t \in \langle \gamma_0 - 2\pi x_0, \gamma_0 - 2\pi x_0 + 2\pi\alpha\rangle\}, \\ F = F_4 &= \{\mathrm{e}^{\mathrm{i}t};\ t \in \langle 2\pi x_0 - \gamma_0 - 2\pi\alpha, 2\pi x_0 - \gamma_0\rangle\} \end{aligned} \tag{18$''$}$$

for estimates (17).

Proof. Let us recall first (see [4; Corollary 2]) that the extreme points in the class $\check{P}(B,b,\alpha;F)$ are of the form

$$p(z;\gamma,F) = b + \frac{B-b}{2\pi}\int_F \frac{\mathrm{e}^{\mathrm{i}t} + z}{\mathrm{e}^{\mathrm{i}t} - z}\,\mathrm{d}t + (1-\eta)\frac{\mathrm{e}^{\mathrm{i}\gamma} + z}{\mathrm{e}^{\mathrm{i}\gamma} - z}, \qquad \gamma \text{ real}, \ z \in \mathbb{D}. \tag{19}$$

Since, by Definition 1 b), d),

$$\mathcal{P}(B,b,\alpha) = \bigcup_F \check{\mathcal{P}}(B,b,\alpha;F),$$

where the sum is taken over all subsets $F \subset \mathbb{T}$ of Lebesgue measure $2\pi\alpha$, it is clear, that, for $z \in \mathbb{D}$ fixed,

$$\sup_{p\in\mathcal{P}(B,b,\alpha)} \operatorname{Re} p(z) = \sup_{\gamma,F}\left\{\operatorname{Re} p(z;\gamma,F);\ \gamma\in\langle-\pi,\pi),\ F\subset\mathbb{T},\ m(F)=2\pi\alpha\right\},$$

where, by $m(E)$, we denote the Lebesgue measure of a measurable set $E \subset \mathbb{T}$. Corresponding formulae hold also for

$$\inf_{p\in\mathcal{P}(B,b,\alpha)} \operatorname{Re} p(z), \qquad \sup_{p\in\mathcal{P}(B,b,\alpha)} \operatorname{Im} p(z), \qquad \inf_{p\in\mathcal{P}(B,b,\alpha)} \operatorname{Im} p(z).$$

Since with every function $p \in \mathcal{P}(B,b,\alpha)$ the function $q(z) = p(\varepsilon z)$, $|\varepsilon| = 1$, $z \in \mathbb{D}$, is also contained in $\mathcal{P}(B,b,\alpha)$, we can reformulate our problem in the following way:

Let $r \in (0,1)$. Find

$$\sup_{\gamma,F} \operatorname{Re} p(r;\gamma,F), \quad \inf_{\gamma,F} \operatorname{Re} p(r;\gamma,F), \quad \sup_{\gamma,F} \operatorname{Im} p(r;\gamma F), \quad \inf_{\gamma,F} \operatorname{Im} p(r;\gamma,F), \tag{20}$$

where $p(z;\gamma,F)$ is of form (19).

Further, $\dfrac{e^{i\gamma}+r}{e^{i\gamma}-r} = \dfrac{1-r^2}{1-2r\cos\gamma+r^2} + i\dfrac{-2r\sin\gamma}{1-2r\cos\gamma+r^2}$, and so we have

$$\frac{1-r}{1+r} \le \operatorname{Re}\frac{e^{i\gamma}+z}{e^{i\gamma}-z} \le \frac{1+r}{1-r}, \tag{21}$$

and, by elementary calculus,

$$-\frac{2r}{1-r^2} \le \operatorname{Im}\frac{e^{i\gamma}+z}{e^{i\gamma}-z} \le \frac{2r}{1-r^2}. \tag{22}$$

Equalities are attained for $\gamma = 0$ and $\gamma = \pi$ in (21) and for $\gamma = -\gamma_0$, $\gamma = \gamma_0$ in (22), respectively.

Hence, our problem reduces to the following. Find

$$\begin{gathered}\sup_F\left\{\frac{1}{2\pi}\int_F P_r(t)\,dt;\ F\subset\mathbb{T},\ m(F)=2\pi\alpha\right\},\\ \inf_F\left\{\frac{1}{2\pi}\int_F P_r(t)\,dt;\ F\subset\mathbb{T},\ m(F)=2\pi\alpha\right\},\end{gathered} \tag{23}$$

and

$$\begin{gathered}\sup_F\left\{\frac{1}{2\pi}\int_F Q_r(t)\,dt;\ F\subset\mathbb{T},\ m(F)=2\pi\alpha\right\},\\ \inf_F\left\{\frac{1}{2\pi}\int_F Q_r(t)\,dt;\ F\subset\mathbb{T},\ m(F)=2\pi\alpha\right\},\end{gathered} \tag{24}$$

where $Q_r(t) = \dfrac{-2r\sin t}{1-2r\cos t + r^2}$.

Recall now the following:

LEMMA. *Let $a, b \in \mathbb{R}$, and let $E \subset \langle a, b\rangle$ be a Lebesgue measurable set, and f a bounded nondecreasing function on $\langle a, b\rangle$. Then*

$$\int\limits_a^{a+m(E)} f(t)\ \mathrm{dt} \le \int\limits_E f(t)\ \mathrm{dt} \le \int\limits_{b-m(E)}^{b} f(t)\ \mathrm{dt} \tag{25}$$

(for the proof, see [4; Lemma 4]).

First, we prove the first inequality in (16) (see (23)). The proof of the second follows the same lines, and is therefore omitted. Since $P_r(t)$, $t \in \langle -\pi, \pi)$ is increasing in $\langle -\pi, 0\rangle$, we obtain by (25)

$$\int\limits_{F_1} P_r(t) \le \int\limits_{-2a\pi}^{0} P_r(t)\ \mathrm{dt},$$

where $F_1 = F \cap \langle -\pi, 0)$, $m(F_1) = 2a\pi \in \langle 0, \pi\rangle$. Similarly, using (25) to the function $-P_r(t)$, we obtain

$$\int\limits_{F_2} P_r(t)\ \mathrm{dt} \le \int\limits_{0}^{2\pi b} P_r(t)\ \mathrm{dt},$$

where $F_2 = F \cap \langle 0, \pi)$, $m(F_2) = 2\pi b \in \langle 0, \pi\rangle$. So

$$\int\limits_{F} P_r(t)\ \mathrm{dt} \le \int\limits_{-2\pi a}^{2\pi b} P_r(t)\ \mathrm{dt} =: \lambda(a,b) \tag{26}$$

and $a + b = \alpha$, since $m(F) = 2\pi\alpha$.

Hence we have to find

$$\max_{(a,b)\in L_\alpha} \lambda(a,b) = \max_{(a,b)\in L_\alpha} \int\limits_{-2\pi a}^{2\pi b} P_r(t)\ \mathrm{dt}, \tag{27}$$

where

$$L_\alpha = \Big\{(a,b) \in \mathbb{R}^2\,;\ 0 \le a \le \frac{1}{2},\ 0 \le b \le \frac{1}{2},\ a + b = \alpha\Big\}.$$

So, for a given α, L_α is the segment $L_{\alpha,1} = \{(a, \alpha-a)\,;\ 0 \le a \le \alpha\}$ if $0 < \alpha \le \frac{1}{2}$, and the segment $L_{\alpha,2} = \{(a, \alpha - a)\,;\ \alpha - \frac{1}{2} \le a \le \frac{1}{2}\}$ if

$\frac{1}{2} \le \alpha < 1$. Let $u(a) = \lambda(a, \alpha - a)$, $a \in L_\alpha$. By (26), we have $u'(a) = 2\pi\big(P_r(2\pi a) - P_r(2\pi(\alpha - a))\big)$. Hence, in both cases, $u'(a) = 0$ for $a = \frac{\alpha}{2}$ (note that $\frac{\alpha}{2} \in (0,\alpha)$ and $\frac{\alpha}{2} \in \big(\alpha - \frac{1}{2}, \frac{1}{2}\big)$ since $\alpha \in (0,1)$). Since $u''(a) = 2\pi\big(P_r'(2\pi a) + P_r'(2\pi(\alpha - a))\big)$, we have $u''\big(\frac{\alpha}{2}\big) = 8\pi^2 P_r'(\alpha\pi) < 0$, because $\alpha\pi \in (0,\pi)$, and the signum of $P_r'(t)$ is the same as the signum of $-\sin t$. Since $\frac{\alpha}{2}$ is clearly the only root of the function $u'(a)$ on $\langle 0, \alpha\rangle$, the maximum of $u(a)$ in (27) is attained for $a = \frac{\alpha}{2}$. By calculating the integral $u(\frac{\alpha}{2}) = \lambda(\frac{\alpha}{2}, \frac{\alpha}{2}) = \int\limits_{-\alpha\pi}^{\alpha\pi} P_r(t)\, \mathrm{d}t$ and using (19), (20), (21), we obtain the right hand side in (16).

Next, we prove inequalities (17). Because of $Q_r(-t) = -Q_r(t)$, only the infimum in (24) is to be determined. Since Q_r has the period 2π, we can suppose $F \subset \big\{\mathrm{e}^{\mathrm{i}t};\ t \in \langle -\gamma_0, -\gamma_0 + 2\pi)\big\}$. Denote $F_3 = F \cap \langle -\gamma_0, \gamma_0\rangle$, $F_4 = F \cap \langle \gamma_0, 2\pi - \gamma_0)$ and write $\int\limits_F Q_r(t)\, \mathrm{d}t = \int\limits_{F_3} Q_r(t)\, \mathrm{d}t + \int\limits_{F_4} Q_r(t)\, \mathrm{d}t$. Since the function $Q_r(t)$ is increasing in $\langle \gamma_0, 2\pi - \gamma_0)$ and decreasing in $\langle -\gamma_0, \gamma_0\rangle$, so applying the lemma to the intervals $\langle \gamma_0, 2\pi - \gamma_0)$, $\langle -\gamma_0, \gamma_0\rangle$ and to the sets F_4, F_3 and functions $Q_r(t)$, $-Q_r(t)$, respectively, we obtain, with regard to $m(F_3) + m(F_4) = 2\pi\alpha$, $\int\limits_F Q_r(t)\, \mathrm{d}t \ge \int\limits_{\gamma_0 - m(F_3)}^{\gamma_0 + m(F_4)} Q_r(t)\, \mathrm{d}t$. Here $\gamma_0 + m(F_4) \le 2\pi - \gamma_0$, $\gamma_0 - m(F_3) \ge -\gamma_0$, so $0 \le m(F_3) \le 2\gamma_0$, $0 \le m(F_4) \le 2(\pi - \gamma_0)$. Hence, denoting $m(F_3) = 2\pi x$, $m(F_4) = 2\pi y$, $\mu(x,y) = \int\limits_{\gamma_0 - 2\pi x}^{\gamma_0 + 2\pi y} Q_r(t)\, \mathrm{d}t$, we realize, that we have to determine

$$\min_{(x,y)\in M_\alpha} \mu(x,y) = \min_{(x,y)\in M_\alpha} \int\limits_{\gamma_0 - 2\pi x}^{\gamma_0 + 2\pi y} Q_r(t)\, \mathrm{d}t\,, \tag{28}$$

where

$$M_\alpha = \big\{(x,y) \in \mathbb{R}^2;\ 0 \le x \le \tfrac{\gamma_0}{\pi},\ 0 \le y \le 1 - \tfrac{\gamma_0}{\pi},\ x + y = \alpha\big\}.$$

Because $r \in (0,1)$, and therefore $\gamma_0 \in \big(0, \frac{\pi}{2}\big)$ is fixed, we easily see, that the segment M_α is given by

$$\begin{aligned}
M_\alpha &= \big\{(x, \alpha - x);\ 0 \le x \le \alpha\big\} && \text{if } 0 < \alpha < \tfrac{\gamma_0}{\pi},\\
M_\alpha &= \big\{(x, \alpha - x);\ 0 \le x \le \tfrac{\gamma_0}{\pi}\big\} && \text{if } \tfrac{\gamma_0}{\pi} \le \alpha \le 1 - \tfrac{\gamma_0}{\pi},\\
M_\alpha &= \big\{(x, \alpha - x);\ \alpha - 1 + \tfrac{\gamma_0}{\pi} \le x \le \tfrac{\gamma_0}{\pi}\big\} && \text{if } 1 - \tfrac{\gamma_0}{\pi} < \alpha < 1.
\end{aligned} \tag{29}$$

Let

$$v(x) = \mu(x, \alpha - x)\,. \tag{30}$$

By the definition of $\mu(\cdot,\cdot)$ and (30),

$$v'(x) = -2\pi\big[Q_r(\gamma_0 - 2\pi x + 2\pi\alpha) - Q_r(\gamma_0 - 2\pi x)\big]\,,$$
$$v''(x) = 4\pi^2\big[Q_r'(\gamma_0 - 2\pi x + 2\pi\alpha) - Q_r'(\gamma_0 - 2\pi x)\big]\,.$$

Now, we easily calculate $Q_r'(t) = -\dfrac{2r[(1+r^2)\cos t - 2r]}{(1-2r\cos t + r^2)^2}$. For every $x \in \langle 0, \frac{\gamma_0}{\pi}\rangle$ we have, by (29), $-\gamma_0 \le \gamma_0 - 2\pi x \le \gamma_0$ and $0 \le 2\pi(\alpha - x) \le 2\pi - 2\gamma_0$, so $\gamma_0 \le \gamma_0 - 2\pi x + 2\pi\alpha \le 2\pi - \gamma_0$, and, with regard to $\gamma_0 \in \left(0, \frac{\pi}{2}\right)$, we have $\cos t \ge \cos\gamma_0$ for $-\gamma_0 \le t \le \gamma_0$, and $\cos t < \cos\gamma_0$ for $\gamma_0 < t < 2\pi - \gamma_0$. Hence $\cos(\gamma_0 - 2\pi x + 2\pi\alpha) < \cos(\gamma_0 - 2\pi x)$ for $\gamma_0 \in \left(0, \frac{\pi}{2}\right)$, $\alpha \in (0,1)$ and $x \in \left\langle 0, \frac{\gamma_0}{\pi}\right\rangle$. From this, we conclude that $v''(x) > 0$, and so $v'(x)$ is increasing on $\left\langle 0, \frac{\gamma_0}{\pi}\right\rangle$. But $Q_r(\gamma_0) \le Q_r(t) \le -Q_r(\gamma_0)$ by (22), and so

$$\begin{aligned}
v'(0) &= -2\pi\big[Q_r(\gamma_0 + 2\pi\alpha) - Q_r(\gamma_0)\big] < 0\,,\\
v'(\alpha) &= -2\pi\big[Q_r(\gamma_0) - Q_r(\gamma_0 - 2\pi\alpha)\big] > 0\,,\\
v'\Big(\frac{\gamma_0}{\pi}\Big) &= -2\pi\big[Q_r(2\pi\alpha - \gamma_0) - Q_r(-\gamma_0)\big] > 0\,,\\
v'\Big(\alpha - 1 + \frac{\gamma_0}{\pi}\Big) &= -2\pi\big[Q_r(2\pi - \gamma_0) - Q_r(2\pi - \gamma_0 - 2\pi\alpha)\big] < 0\,.
\end{aligned}$$

From this, we see by (30) that for any $\alpha \in (0,1)$, $\gamma \in \left(0, \frac{\pi}{2}\right)$ there exists a unique root x_0 of the equation $v'(x) = 0$ with

$$\begin{aligned}
&x_0 \in (0,\alpha) && \text{if} \quad \alpha \in \left(0, \tfrac{\gamma_0}{\pi}\right),\\
&x_0 \in \left(0, \tfrac{\gamma_0}{\pi}\right) && \text{if} \quad \alpha \in \left\langle \tfrac{\gamma_0}{\pi}, 1 - \tfrac{\gamma_0}{\pi}\right\rangle,\\
&x_0 \in \left(\alpha - 1 + \tfrac{\gamma_0}{\pi}, \tfrac{\gamma_0}{\pi}\right) && \text{if} \quad \alpha \in \left(1 - \tfrac{\gamma_0}{\pi}, 1\right).
\end{aligned} \tag{31}$$

So x_0 is given by the equation

$$Q_r(\gamma_0 - 2\pi x_0 + 2\pi\alpha) = Q_r(\gamma_0 - 2\pi x_0)\,, \tag{32}$$

from which we infer by elementary calculations an equivalent equation for x_0

$$\cos(\gamma_0 - 2\pi x_0 + \pi\alpha) = \cos\gamma_0 \cos\pi\alpha\,.$$

So the minimum in (28) is attained in the point x_0, given by equation (32). At the same time, the point x_0 is lying in the corresponding intervals (31), and hence in the interval $\left(0, \frac{\gamma_0}{\pi}\right)$. Now, since

$$\big(\log P_r(t)\big)' = Q_r(t)\,,$$

we have $\mu(x_0, \alpha - x_0) = v(x_0) = \log \dfrac{P_r(\gamma_0 - 2\pi x_0 + 2\pi\alpha)}{P_r(\gamma_0 - 2\pi x_0)}$, and, by (19), (20), (22), the first inequality in (17) is proved.

Since function (18) belongs to the class $\mathcal{P}(B, b, \alpha)$, and the measure of sets (18′) and (18″) is $2\pi\alpha$, estimates (16) and (17) are sharp. □

Remark 4. Using the definition of γ_0, inequalities (17) can be rewritten in the form

$$\begin{aligned}&\frac{B-b}{2\pi}\log\frac{1-\cos\gamma_0\cos(\gamma_0-2\pi x_0)}{1-\cos\gamma_0\cos(\gamma_0-2\pi x_0+2\pi\alpha)}-(1-\eta)\frac{2r}{1-r^2}\\ &\le \operatorname{Im}p(z) \qquad\qquad (17')\\ &\le\frac{B-b}{2\pi}\log\frac{1-\cos\gamma_0\cos(\gamma_0-2\pi x_0+2\pi\alpha)}{1-\cos\gamma_0\cos(\gamma_0-2\pi x_0)}+(1-\eta)\frac{2r}{1-r^2}\,.\end{aligned}$$

Using (32) and $Q_r(t)=\dfrac{-2r}{1-r^2}\sin tP_r(t)$, inequalities (17) can be rewritten in the form

$$\begin{aligned}&\frac{B-b}{2\pi}\log\frac{\sin(\gamma_0-2\pi x_0)}{\sin(\gamma_0-2\pi x_0+2\pi\alpha)}-(1-\eta)\frac{2r}{1-r^2}\\ &\le \operatorname{Im}p(z) \qquad\qquad (17'')\\ &\le\frac{B-b}{2\pi}\log\frac{\sin(\gamma_0-2\pi x_0+2\pi\alpha)}{\sin(\gamma_0-2\pi x_0)}+(1-\eta)\frac{2r}{1-r^2}\qquad \text{if}\quad \alpha\ne\frac12\,,\end{aligned}$$

and

$$\begin{aligned}&\frac{B-b}{\pi}\log\frac{1-r}{1+r}-(1-\eta)\frac{2r}{1-r^2}\\ &\le \operatorname{Im}p(z) \qquad\qquad (17''')\\ &\le\frac{B-b}{\pi}\log\frac{1+r}{1-r}+(1-\eta)\frac{2r}{1-r^2}\qquad \text{if}\quad \alpha=\frac12\,.\end{aligned}$$

Remark 5. Since the functions $p_0(z)$ from (18) and also $p_0(e^{i\gamma_0}z)$ are contained in $\tilde{\mathcal{P}}(B,b,\alpha)$, the extrema of the functionals $\operatorname{Re}p(z)$ and $\operatorname{Im}p(z)$ are attained already in this class and also in the class $\mathcal{P}(B,b,\alpha)$ (see [2; p. 96, Definition] and [4; Lemma 1] or [3; Theorem 6]).

Remark 6. Passing to the limit as in Remark 3 we obtain the classical results for the class $\mathcal{P}_b$ (see, e.g., [5; Vol. I, p. 84]).

3

The estimates of the linear functionals $\operatorname{Re}p(z)$ and $\operatorname{Im}p(z)$, $z\in\mathbb{D}$ fixed, given in the preceding section, and also the estimates of the convex functionals $|q_k|$, $k=1,2,\dots$, are interesting from the following point of view: they are valid on the whole of the closed convex hull of $\mathcal{P}(B,b,\alpha)$, although $\mathcal{P}(B,b,\alpha)$ is neither convex nor compact (this will be shown in this section). Recall, that the topology on $\mathcal{P}(B,b,\alpha)$ is the restriction of the topology given by uniform convergence on compact subsets of $\mathbb{D}$ on the set of all functions holomorphic in $\mathbb{D}$, and that the class $\mathcal{P}$ is compact, and hence $\mathcal{P}(B,b,\alpha)$ is relatively compact in $\mathcal{P}$ in this topology.

THEOREM 4. *The class $\mathcal{P}(B,b,\alpha)$ is not convex.*

Proof. Take $p_1(z)=p_{F_1}(z)+(1-\eta)\frac{1+z}{1-z}$, $p_2(z)=p_{F_2}(z)+(1-\eta)\frac{1+z}{1-z}$, $z\in\mathbb{D}$, where the sets F_i, $i=1,2$, are chosen in such a manner, that $0<m(F_1\cap F_2)<\alpha$. Put $p_t=tp_1+(1-t)p_2$, $0<t<1$. Since $\operatorname{Re}\frac{1+z}{1-z}=0$ a.e. on $\mathbb{T}$, $\operatorname{Re}p_{F_i}=B$ a.e. on F_i, $\operatorname{Re}p_{F_i}=b$ a.e. on $\mathbb{T}\setminus F_i$, and $tb+(1-t)B<B$ for $0<t<1$, so $\operatorname{Re}p_t=B$ a.e. on $F_1\cap F_2$ and $\operatorname{Re}p_t<B$ a.e. on $\mathbb{T}\setminus F_1\cap F_2$. Since $m(F_1\cap F_2)<\alpha$, p_t does not satisfy (2), and so does not belong to $\mathcal{P}(B,b,\alpha)$. □

THEOREM 5. *The class $\mathcal{P}(B,b,\alpha)$ is not compact.*

Proof. It is sufficient to prove that $\mathcal{P}(B,b,\alpha)$ is not closed. Put

$$p_n(z)=b+(B-b)h_{F_n}(z)+(1-\eta)\frac{1+z}{1-z},\qquad z\in\mathbb{D},\tag{33}$$

where F_n are sets (5) and $h_{F_n}(z)$ functions (6). For $z\in\mathbb{D}$, $|z|\le\rho<1$, we have by (6)

$$|h_{F_n}(z)-\alpha|\le 2\sum_{r=1}^{\infty}\frac{|\sin\alpha\pi r|}{\pi r}\rho^{rn}\le 2\rho^n\sum_{r=0}^{\infty}(\rho^n)^r=\frac{2\rho^n}{1-\rho^n},$$

and so the sequence $\{h_{F_n}\}_{n=1}^{\infty}$ is uniformly convergent to the constant function α on every compact subset of $\mathbb{D}$. Denoting $p_0(z)=\eta+(1-\eta)\frac{1+z}{1-z}$ and using (33) we see that $p_n(z)\to p_0(z)$ uniformly on compact subsets of $\mathbb{D}$. But the function $\operatorname{Re}p_0$ is equal to η a.e. on $\mathbb{T}$, since $\operatorname{Re}\frac{1+z}{1-z}$ is zero a.e. on $\mathbb{T}$. Since $\eta=\alpha B+(1-\eta)b<\alpha B+(1-\alpha)B=B$, p_0 does not fulfill (2), and so does not belong to $\mathcal{P}(B,b,\alpha)$. □

Remark 7. The idea of the sequence $\{p_n\}$ comes from Theorem 8 in [4]: the function $p_n(z)$ realizes the maximum of the modulus of the nth coefficient in the class $\mathcal{P}(B,b,\alpha)$. The measure in the Poisson representation of p_n is the sum of two parts: the (absolutely continuous) part $\left[b+(B-b)\chi_{F_n}(t)\right]\frac{dt}{2\pi}$ and the (singular) part $(1-\eta)\varepsilon_0$, where ε_0 is the Dirac measure sitting at the point $t=0$. Now, intuitively, the measures $\chi_{F_n}\frac{dt}{2\pi}$ spread to the measure $\alpha\frac{dt}{2\pi}$, and the limit function $p_0(z)$, which is represented by the limit measure $\left[b+(B-b)\alpha\right]\frac{dt}{2\pi}+(1-\eta)\varepsilon_0=\eta\frac{dt}{2\pi}+(1-\eta)\varepsilon_0$, cannot belong to $\mathcal{P}(B,b,\alpha)$.

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