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INDEXED ANNIHILATORS IN ORDERED SETS

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ABSTRACT. The concept of lattice annihilator is modified and generalized for ordered sets as an indexed annihilator. The set of all indexed annihilators $\mathrm{IA}(S)$ forms a complete lattice. Some properties of $\mathrm{IA}(S)$ in connection with the lattice of all ideals of S are studied .

M. Mandelker [5] introduced the concept of annihilator in a lattice. He proved that a lattice L is distributive if and only if every of its annihilators is an ideal of L. Annihilators in lattices were intensively studied by B. Davey and J. Nieminen, see [2], [3]. Recently, this concept has been generalized also for ordered sets, see [4]. Let us recall some basic concepts.

Let $(S,\leq)$ be an ordered set and X be a subset of S.

Denote

$$L(X)=\{y\in S;\ y\leq x \text{ for each } x\in X\},$$
$$U(X)=\{y\in S;\ x\leq y \text{ for each } x\in X\}.$$

If $X=\{a,b\}$ or $X=A\cup B$ or $X=A\cup\{b\}$, we will write briefly $L(a,b)$ or $L(A,B)$ or $L(A,b)$, respectively and, analogously, $U(a,b)$ or $U(A,B)$ or $U(A,b)$. We will also use the notation $UL(X)$ instead of $U\big(L(X)\big)$ and $LU(X)$ instead of $L\big(U(X)\big)$.

An ordered set $(S,\leq)$ is called *distributive* (see [1], [6]) if

$$L\big(U(a,b),c\big)=LU\big(L(a,c),L(b,c)\big) \qquad \text{for each} \quad a,b,c\in S.$$

DEFINITION 1. (see [4]) Let $(S,\leq)$ be an ordered set. A subset $I\subseteq S$ is called an *ideal* of S if $x,y\in I$ implies $LU(x,y)\subseteq I$. An ideal I of $(S,\leq)$ is called *strong* if for every non-void finite subset $F\subseteq I$ also $LU(F)\subseteq I$. Let $a,b\in S$. By an *annihilator* $\langle a,b\rangle$ is meant the set

$$\langle a,b\rangle=\big\{x\in S;\ UL(a,x)\supseteq U(b)\big\}.$$

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Let us note that, if $(S, \leq)$ is a lattice, the concepts of ideal and strong ideal coincide with the lattice ideal, and the concept of annihilator coincides with that of [5] or [2], [3].

As it was shown in [4], annihilators are important tools for some investigations of ordered sets. Unfortunately, there is an essential difference, compared with the set of all ideals of an ordered set, namely, the set of all annihilators of S does not form a lattice in a general case:

Example 1. Let $S = \{a, b, c, d, 1\}$ and the ordered set $(S, \leq)$ have the diagram as shown in Fig. 1.

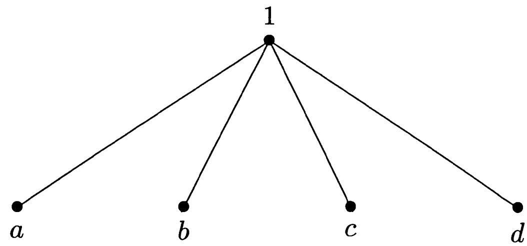

Figure 1.

Then we have

$$\langle a, c\rangle = \{b, c, d\} \quad \text{and} \quad \langle b, c\rangle = \{a, c, d\},$$

but for no $x, y \in S$ we have

$$\langle x, y\rangle = \{c, d\} = \langle a, c\rangle \cap \langle b, c\rangle .$$

To avoid this disadvantage, we can introduce the following new concept:

DEFINITION 2. Let $(S, \leq)$ be an ordered set, and $a_\gamma, b_\gamma \in S$ for $\gamma \in \Gamma \neq \emptyset$. By an *indexed annihilator* determined by a_γ, b_γ $(\gamma \in \Gamma)$ is meant the set $\{z \in S;\ UL(z, a_\gamma) \supseteq U(b_\gamma),\ \gamma \in \Gamma\}$.

LEMMA. *Let $(S, \leq)$ be an ordered set.*

(i) *If $A = \{z \in S;\ UL(z, a_\gamma) \supseteq U(b_\gamma),\ \gamma \in \Gamma\}$ is an indexed annihilator of $(S, \leq)$, then*

$$A = \bigcap\{\langle a_\gamma, b_\gamma\rangle;\ \gamma \in \Gamma\}.$$

(ii) *Let $B \subseteq S$ and $a \in S$. Denote $\langle a, B\rangle = \{x \in S;\ UL(a, x) \supseteq U(B)\}$. If $U(B) = \emptyset$, then $\langle a, B\rangle = S$. If $U(B) \neq \emptyset$, then*

$$\langle a, B\rangle = \bigcap\{\langle a, b_\gamma\rangle;\ b_\gamma \in U(B)\}.$$

Proof.

The assertion (i) is evident.

Prove (ii). If $U(B) = \emptyset$, then, trivially, $\langle a, B\rangle = S$ for each $a \in S$.

Suppose $U(B) \neq \emptyset$. If $x \in \langle a, B\rangle$, then

$$UL(a,x) \supseteq U(B),$$

which gives

$$L(a,x) = LUL(a,x) \subseteq LU(B) \subseteq L(b_\gamma)$$

for each $b_\gamma \in U(B)$, thus $UL(a,x) \supseteq U(b_\gamma)$ and $x \in \langle a, b_\gamma\rangle$ for each $b_\gamma \in U(B)$.

Conversely, if $x \in \bigcap\{\langle a, b_\gamma\rangle;\ b_\gamma \in U(B)\}$, then $x \in \langle a, b_\gamma\rangle$ for each $b_\gamma \in U(B)$, whence $UL(a,x) \supseteq U(b_\gamma)$. This yields $L(a,x) \subseteq L(b_\gamma)$ for each $\gamma \in \Gamma$, i.e., $L(a,x) \subseteq LU(B)$. Hence, $UL(a,x) \supseteq ULU(B) = U(B)$ proving $x \in \langle a, B\rangle$. □

THEOREM 1. *The set* $\mathrm{IA}(S)$ *of all indexed annihilators of* $(S, \leq)$ *forms a complete lattice with respect to set inclusion. The greatest element of* $\mathrm{IA}(S)$ *is equal to* S *and the operation meet coincides with set intersection.*

Proof. Let $A_\lambda \in \mathrm{IA}(S)$ for $\lambda \in \Lambda$, $A_\lambda = \bigcap\{\langle a^\gamma_\lambda, b^\gamma_\lambda\rangle;\ \gamma \in \Gamma_\lambda\}$. Then, by (i) of Lemma,

$$\bigcap\{A_\lambda;\ \lambda \in \Lambda\} = \bigcap\{\langle a^\gamma_\lambda, b^\gamma_\lambda\rangle;\ \gamma \in \Gamma_\lambda,\ \lambda \in \Lambda\} \in \mathrm{IA}(S),$$

thus $(\mathrm{IA}(S), \cap)$ is a closure system. Since $\langle a, a\rangle = S$ for each $a \in S$, S is the greatest element of $(\mathrm{IA}(S), \subseteq)$, thus it is a complete lattice where meet coincides with set intersection. □

COROLLARY 1. *Let* $(S, \leq)$ *be an ordered set and* $X \subseteq S$. *Then there exists the least indexed annihilator of* $(S, \leq)$ *containing* X, *the so called generated by* X.

We are able to give an explicit construction of the indexed annihilator $\mathscr{A}(X)$ generated by the set X:

Construction. Let $X \subseteq S$. For each $a \in S$ let $B_a = \{b_{\gamma a};\ \gamma_a \in \Gamma_a\}$, the so called *polar* of a, i.e., the set of all elements $b_{\gamma a} \in S$ satisfying the condition

$$UL(a,x) \subseteq U(b_{\gamma a}) \qquad \text{for each} \quad x \in X$$

($B_a \neq \emptyset$ since $a \in B_a$). Put

$$A_a = \bigcap\{\langle a, b_{\gamma a}\rangle;\ \gamma_a \in \Gamma_a\}.$$

Then $\mathscr{A}(X) = \bigcap\{A_a;\ a \in S\}$.

The proof of this Construction is a consequence of (ii) of Lemma.

Example 2. Let the diagram of $(S, \leq)$ be given in Fig. 2.

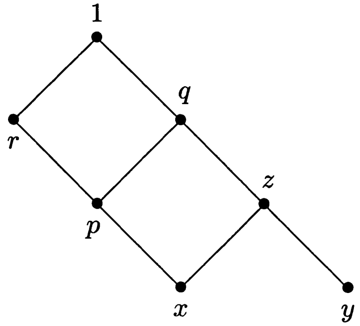

Figure 2.

For $X = \{x, y\}$ we have the polars

$$\begin{aligned} B_1 &= \{1, q, z\} = B_q = B_z \,, \\ B_r &= \{1, r, q, p, z, x\} = B_p = B_x \,, \\ B_y &= \{1, q, z, y\} \,. \end{aligned}$$

Hence,

$$\begin{aligned} &A_1 = \langle 1, 1\rangle \cap \langle 1, q\rangle \cap \langle 1, z\rangle = S \cap \{q, p, x, y, z\} \cap \{x, y, z\} = \{x, y, z\} \,, \\ &A_p = \{x, y, z\} = A_q = A_p \,, \qquad A_x = A_y = A_z = S \,, \end{aligned}$$

thus $\mathscr{A}(X) = \{x, y, z\}$.

COROLLARY 2. *Denote by* $\vee$ *the operation join in the lattice* $\mathrm{IA}(S)$. *For* $B, C \in \mathrm{IA}(S)$ *we have*

$$B \vee C = \mathscr{A}(B \cup C) \,.$$

THEOREM 2. *An ordered set* $(S, \leq)$ *is distributive if and only if every indexed annihilator of* $(S, \leq)$ *is an ideal.*

Proof. Let $A = \bigcap \{ \langle a_\gamma, b_\gamma \rangle \,;\ \gamma \in \Gamma \}$ be an indexed annihilator of $(S, \leq)$, let $x, y \in A$ and $z \in LU(x, y)$. Then $L(z) \subseteq LU(x, y)$ and $U(z) \supseteq ULU(x, y) = U(x, y)$, thus for each $\gamma \in \Gamma$ we have

$$\begin{aligned} UL(a_\gamma, z) = UL\big(a_\gamma, U(z)\big) \supseteq UL\big(a_\gamma, U(x, y)\big) &= ULU\big(L(a_\gamma, x), L(a_\gamma, y)\big) \\ = U\big(L(a_\gamma, x), L(a_\gamma, y)\big) = UL(a_\gamma, x) \cap UL(a_\gamma, y) &\supseteq U(b_\gamma) \end{aligned}$$

using distributivity of S. Hence, $z \in \bigcap \{ \langle a_\gamma, b_\gamma \rangle \,;\ \gamma \in \Gamma \}$, i.e., A is an ideal of $(S, \leq)$.

Conversely, suppose that every indexed annihilator of S is an ideal. Thus also every annihilator $\langle a, b \rangle$ is an ideal of S. It is clear that set intersection of every set of ideals of S is an ideal of S again. Hence, by Lemma, also $\langle a, B\rangle$ is an ideal of S for each $a \in A$ and every $B \subseteq S$.

Let $a, b, x \in S$. Then

$$UL(a,x) \supseteq UL(a,x) \cap UL(b,x) = U\big(L(a,x), L(b,x)\big)\,,$$
$$UL(b,x) \supseteq U\big(L(a,x), L(b,x)\big)\,.$$

Hence, for $B = L(a,x) \cup L(b,x)$ we have $a, b \in \langle x, B\rangle$. But $\langle x, B\rangle$ is an ideal of S, thus $LU(a,b) \subseteq \langle x, B\rangle$.

Let $z \in L\big(U(a,b), x\big)$. Then $z \in LU(a,b) \cap L(x)$, and thus $z \in \langle x, B\rangle$. Therefore, $UL(z,x) \supseteq U\big(L(a,x), L(b,x)\big)$ and $z \in L(x)$ implies $L(z,x) = L(z)$. Finally, we obtain

$$U(z) \supseteq U\big(L(a,x), L(b,x)\big)\,, \quad L(z) \subseteq LU\big(L(a,x), L(b,x)\big)\,, \qquad \text{i.e.,}$$
$$z \in LU\big(L(a,x), L(b,x)\big) \quad \text{and} \quad L\big(U(a,b), x\big) \subseteq LU\big(L(a,x), L(b,x)\big)\,.$$

However, the opposite inclusion is valid for all elements of S, hence S is distributive. □

Example 3. The ordered set $S = \{a, b, c\}$ visualized in Fig. 3 is not distributive (see e.g. [1]).

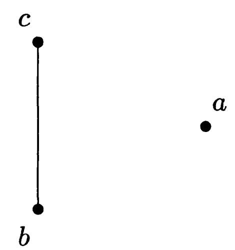

Figure 3.

Its indexed annihilators are:

$$\begin{array}{ll} \langle a,b\rangle = \langle a,c\rangle = \{b,c\}\,, & \langle b,a\rangle = \langle c,a\rangle = \{a\}\,, \\ \langle c,b\rangle = \{a,b\}\,, & \langle b,c\rangle = S = \langle x,x\rangle \quad \text{for each } x \in S\,, \\ \langle a,b\rangle \cap \langle b,a\rangle = \emptyset\,, & \langle a,b\rangle \cap \langle c,b\rangle = \{b\}\,, \end{array}$$

thus $\mathrm{IA}(S)$ has the diagram as shown in Fig. 4 (a).

On the other hand, the ideal lattice is different, see Fig. 4 (b), since the indexed annihilator $\{a, b\}$ is not an ideal of S. However, every ideal $J \in \mathrm{Id}(S)$ is an indexed annihilator of S, thus, especially, $\mathscr{A}(J) = J$. Hence, it is a natural problem if every ideal of an ordered set S is an indexed annihilator at least in the case of distributive $(S, \leq)$.

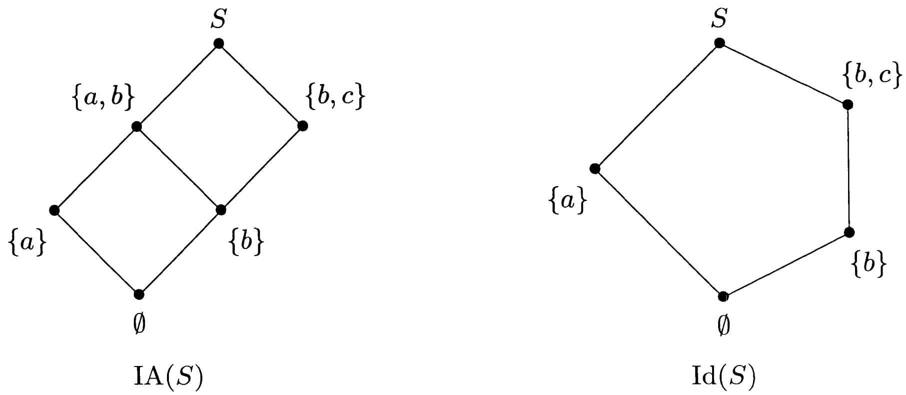

Figure 4.

Especially, if $(S, \leq)$ is a finite distributive lattice, then $\mathrm{Id}(S) = \mathrm{IA}(S)$ since every ideal J of S is a principal ideal, i.e., $J = L(x)$ for some $x \in S$ and $L(x) = \langle 1, x\rangle$, where 1 is the greatest element of S. We proceed to show that the answer to our problem is negative (in infinite case).

E x a m p l e 4. Let M be an infinite set. Consider the set $A = \mathrm{Exp}\, M$ of all subsets of M ordered by set inclusion (trivially, $(\mathrm{Exp}\, M, \subseteq)$ is a distributive lattice). The set J of all finite subsets of M forms an ideal of $(A, \subseteq)$. By using Construction of $\mathscr{A}(X)$, we obtain

$$\mathscr{A}(J) = A \neq J\,.$$

Hence J is not an indexed annihilator of A.

It motivates our investigation for which ideal J of $(S, \leq)$ we have $\mathscr{A}(J) = J$.

THEOREM 3. *Let $(S, \leq)$ be an ordered set and J be a principal ideal of S, i.e., $J = L(c)$ for some $c \in S$. Then $\mathscr{A}(J) = J$.*

P r o o f . For each $a \in S$ we have $B_a = \{x \in S;\ L(a, c) \subseteq L(x)\}$ for the polar B_a, see Construction. Trivially, $c \in B_a$ for each $a \in S$. If $b \in B_a$ and $z \in \langle a, b\rangle$, then $UL(z, a) \supseteq U(b)$. For $a = z$ and $b = c$ we obtain $U(z) = UL(z) = UL(a, z) \supseteq U(c)$ since $c \in B_a$. Hence $z \leq c$ proving $z \in J$. By Construction of $\mathscr{A}(J)$, we have $\mathscr{A}(J) \subseteq J$. The converse inclusion is trivial. □

The foregoing result can be generalized for strong ideals in the case of finite sets:

THEOREM 4. *Let $(S,\leq)$ be a finite ordered set. Then $\mathscr{A}(J)=J$ for every strong ideal J of $(S,\leq)$.*

P r o o f . Let J be a strong ideal of S. If $U(J)=\emptyset$, then $LU(J)=S$. Since S is finite, also J is finite, thus $LU(J)\subseteq J$. Hence $J=S$, but $\mathscr{A}(S)=S$ holds trivially.

Suppose $U(J)\neq\emptyset$. Since S is finite, the set $\{g_1,\dots,g_n\}$ of minimal elements of $U(J)$ is finite. Further, for each $a\in S$ and each $x\in J$ we have $L(a,x)\subseteq L(g_i)$ $(i=1,\dots,n)$ since $L(a,x)\subseteq L(x)\subseteq J$ and g_i is minimal in $U(J)$. Thus $g_i\in B_a$ for the polar B_a. Hence, if $z\in\mathscr{A}(J)$, then $z\in\langle a,g_i\rangle$ for each $a\in S$ and $i=1,\dots,n$. Especially, for $a=z$ we obtain $z\in\langle z,g_i\rangle$ whence $U(z)\supseteq U(g_i)$, i.e., $z\leq g_i$ $(i=1,\dots,n)$. This gives $L(z)\subseteq\bigcap\bigl\{L(g_i);\ i=1,\dots,n\bigr\}$. However,

$$\bigcap\bigl\{L(g_i);\ i=1,\dots,n\bigr\}=LU(J)\,,$$

and since J is a strong ideal, $LU(J)\subseteq J$, i.e., $L(z)\subseteq J$, which gives $z\in J$. We have proved $\mathscr{A}(J)\subseteq J$. The converse inclusion is trivial. □

In the remaining part of the paper, we will show that the lattice $\mathrm{IA}(S)$ is pseudocomplemented.

THEOREM 5. *Let $(S,\leq)$ be an ordered set, and A, B be elements of $\mathrm{IA}(S)$ such that B is either a principal ideal, or B is equal to $L(S)$ or S. Then there exists a relative pseudocomplement $A:B$ of A relative to B in $\mathrm{IA}(S)$.*

P r o o f . Let us denote by $X=\bigl\{y\in S;\ \exists b_y{\in}B\ \forall a{\in}A:\ L(a,y)\subseteq L(b_y)\bigr\}$, and let $A:B=\mathscr{A}(X)$, an indexed annihilator generated by X. Further, for $a\in S$ let $B_a=\bigl\{b_{\gamma a}\in S;\ L(a,x)\subseteq L(b_{\gamma a})$ for all $x\in X\bigr\}$. Suppose that $y\in A\cap(A:B)$. Then, by the construction of $\mathscr{A}(X)$, we have $L(y)\subseteq L(b_{\gamma y})$ for every $b_{\gamma y}\in B_y$. Suppose that B is of the form $B=L(b^*)$. Then for every $x\in X$ and $a\in A$, $L(a,x)\subseteq L(b_{\gamma x})\subseteq L(b^*)$ holds. Thus $b^*\in B_y$, and so $L(y)\subseteq L(b^*)$, $y\leq b^*$, and, since B is an indexed annihilator, also $y\in B$. We have $A\cap(A:B)\subseteq B$.

Now, let $R\in\mathrm{IA}(S)$ and $A\cap R\subseteq B$. If $R\subseteq X$, then $\mathscr{A}(R)=R\subseteq\mathscr{A}(X)=A:B$, so $R\subseteq A:B$. Suppose that there exists an element $r\in R$, $r\notin X$. From this, we conclude that for every element $b\in B$ there exists $a_b\in A$ such that $L(a_b,r)\nsubseteq L(b)$. This means that there exists $z\in L(a_b,r)$ with $z\nleq b$. Since $a_b\in A$, $r\in R$, and $z\in L(a_b,r)$, we have $z\in A\cap R\subseteq B$. If we choose $b=b^*$, then $z\nleq b^*$ and $z\in B=L(b^*)$, which is a contradiction. This means that $A:B$ is a relative pseudocomplement of A relative to B.

If B is equal to $L(S)$, then

$$A:B=\bigl\{y\in S;\ UL(a,y)=S\ \text{for all}\ a\in A\bigr\}\,.$$

Obviously, $A \cap (A : B) = L(S)$. If $A \cap R = L(S)$ for some $R \in \mathrm{IA}(S)$ and $r \in R$, $r \notin A : B$, then there exists $a \in A$, with $UL(a, r) \neq S$, i.e., $L(a, r) \neq L(S)$. For $z \in L(a, r)$ we have $z \in A \cap R = L(S)$, a contradiction. Finally, $A : B$ is a relative pseudocomplement of A relative to B. If $B = S$, it is evident that $A : B = B$. □

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