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PRODUCTS OF SIMPLY CONTINUOUS AND QUASICONTINUOUS FUNCTIONS[1]

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ABSTRACT. Functions which are products of simply continuous and quasicontinuous functions are characterized here.

In [5], T. Natkaniec proved that a function $h\colon \mathbb{R} \to \mathbb{R}$ is a product of quasicontinuous functions if and only if h is cliquish, and each of the sets $h^{-1}(0)$, $h^{-1}\big((-\infty,0)\big)$, $h^{-1}\big((0,\infty)\big)$ is the union of an open set and a nowhere dense set. More precisely, he proved that such function is a product of 8 quasicontinuous functions. We shall show that 3 quasicontinuous functions are sufficient. Moreover, we shall generalize this theorem for functions defined on a T_3 second countable topological space.

In what follows, X denotes a topological space. For a subset A of a topological space denote by $\operatorname{Cl} A$ and $\operatorname{Int} A$ the closure and the interior of A, respectively. The letters $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$ stand for the set of natural, rational and real numbers, respectively.

We recall that a function $f\colon X \to \mathbb{R}$ is *quasicontinuous* (*cliquish*) *at a point* $x \in X$ if for each $\varepsilon > 0$ and each neighbourhood U of x there is a nonempty open set $G \subset U$ such that $|f(y) - f(x)| < \varepsilon$ for each $y \in G$ ($|f(y) - f(z)| < \varepsilon$ for each $y, z \in G$). A function $f\colon X \to \mathbb{R}$ is said to be *quasicontinuous* (*cliquish*) if it is quasicontinuous (cliquish) at each point $x \in X$ (see [6]).

A function $f\colon X \to \mathbb{R}$ is *simply continuous* if $f^{-1}(V)$ is a simply open set in X for each open set V in $\mathbb{R}$. A set A is *simply open* if it is the union of an open set and a nowhere dense set (see [1]).

By [1], the union and the intersection of two simply open sets is a simply open set; the complement of a simply open set is a simply open set.

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If $\mathcal{F} \subset \mathbb{R}^X$ is a class of functions defined on X, we denote by $P(\mathcal{F})$ the collection of all functions which can be factored into a (finite) product of functions from $\mathcal{F}$. Further, denote by $\mathcal{Q}$, $\mathcal{S}$ and $\mathcal{K}$ the set of all functions which are quasicontinuous, simply continuous and cliquish, respectively. Now, let

$$\mathcal{H} = \{f\colon X \to \mathbb{R};\ f \text{ is cliquish and the sets } f^{-1}\big((0,\infty)\big) \text{ and } f^{-1}\big((-\infty,0)\big) \text{ are simply open}\}.$$

It is easy to see that $\mathcal{Q} \subset \mathcal{S}$ and $\mathcal{Q} \subset \mathcal{H}$. In [7], it is shown that if X is a Baire space, then every simply continuous function $f\colon X \to \mathbb{R}$ is cliquish. [4; Example 1] shows that the assumption "X is a Baire space" cannot be omitted. Thus, if X is a Baire space, then $\mathcal{S} \subset \mathcal{H}$. It is easy to see that $P(\mathcal{K}) = \mathcal{K}$.

LEMMA 1. *For an arbitrary topological space X we have $P(\mathcal{H}) = \mathcal{H}$.*

Proof. Let $f_1, f_2 \in \mathcal{H}$ and $f = f_1 \cdot f_2$. Then f is cliquish because $P(\mathcal{K}) = \mathcal{K}$. Further, the sets $f_1^{-1}\big((-\infty,0)\big)$, $f_2^{-1}\big((-\infty,0)\big)$, $f_1^{-1}\big((0,\infty)\big)$ and $f_2^{-1}\big((0,\infty)\big)$ are simply open, and hence $f^{-1}\big((-\infty,0)\big) = \big(f_1^{-1}\big((-\infty,0)\big) \cap f_2^{-1}\big((0,\infty)\big)\big) \cup \big(f_1^{-1}\big((0,\infty)\big) \cap f_2^{-1}\big((-\infty,0)\big)\big)$ is simply open. Similarly for $f^{-1}\big((0,\infty)\big)$. □

Therefore $P(\mathcal{Q}) \subset \mathcal{H}$, and if X is a Baire space, then also $P(\mathcal{S}) \subset \mathcal{H}$. We recall that a π-base for X is a family $\mathcal{A}$ of open subsets of X such that every nonempty open subset of X contains some nonempty $A \in \mathcal{A}$ (see [8]).

LEMMA 2. (see [3; Theorem 1]) *Let X be a Baire second countable T_3-space such that the family of all open connected sets is a π-base for X. Then every cliquish function $f\colon X \to \mathbb{R}$ is the sum of two simply continuous functions.*

LEMMA 3. *Let X be as in Lemma 2. If $f\colon X \to \mathbb{R}$ is a positive (negative) cliquish function, then f is the product of two simply continuous functions.*

Proof. Put $g = |f|$. Then $\ln g$ is cliquish, and, by Lemma 2, there are simply continuous functions $g_1, g_2\colon X \to \mathbb{R}$ such that $\ln g = g_1 + g_2$. The functions $f_1 = \operatorname{sign} f \cdot \exp g_1$ and $f_2 = \exp g_2$ are simply continuous and $f = f_1 \cdot f_2$. □

THEOREM 1. *Let X be a Baire T_3 second countable space such that the family of all open connected sets is a π-base for X. Then $P(\mathcal{S}) = \mathcal{H}$. Further, every function from $\mathcal{H}$ is the product of two simply continuous functions.*

Proof. Let $f \in \mathcal{H}$. Put $A = f^{-1}\big((0,\infty)\big)$, $B = f^{-1}\big((-\infty,0)\big)$, $C = f^{-1}(0)$. According to Lemma 3, there are simply continuous functions $g_1, g_2\colon \operatorname{Int} A \to \mathbb{R}$, $h_1, h_2\colon \operatorname{Int} B \to \mathbb{R}$ such that $f|_{\operatorname{Int} A} = g_1 \cdot g_2$ and $f|_{\operatorname{Int} B} = h_1 \cdot h_2$.

Now define functions $f_1, f_2\colon X \to \mathbb{R}$ as follows:

$$f_1(x) = \begin{cases} g_1(x) & \text{for } x \in \operatorname{Int} A\,, \\ h_1(x) & \text{for } x \in \operatorname{Int} B\,, \\ f(x) & \text{otherwise;} \end{cases}$$

$$f_2(x) = \begin{cases} g_2(x) & \text{for } x \in \operatorname{Int} A\,, \\ h_2(x) & \text{for } x \in \operatorname{Int} B\,, \\ 1 & \text{otherwise.} \end{cases}$$

Then $f = f_1 \cdot f_2$. Let V be an open set in $\mathbb{R}$. Since $f_1^{-1}(V) \cap \operatorname{Int} A = g_1^{-1}(V)$ is simply open and A, B and C are simply open, the set $f_1^{-1}(V) = \big(f_1^{-1}(V) \cap \operatorname{Int} A\big) \cup \big(f_1^{-1}(V) \cap \operatorname{Int} B\big) \cup \big(f_1^{-1}(V) \cap \operatorname{Int} C\big) \cup \big(f_1^{-1}(V) \cap \big((A \setminus \operatorname{Int} A) \cup (B \setminus \operatorname{Int} B) \cup (C \setminus \operatorname{Int} C)\big)\big)$ is simply open. Similarly for $f_2^{-1}(V)$. □

LEMMA 4. (see [2; Theorem]) *Let X be a T_3 second countable space. Then every cliquish $f\colon X \to \mathbb{R}$ is the sum of three quasicontinuous functions.*

LEMMA 5. *Let X be as in Lemma 4. If $f\colon X \to \mathbb{R}$ is a positive (negative) cliquish function, then f is the product of three quasicontinuous functions.*

Proof. Similar as in Lemma 3. □

LEMMA 6. (see [9; Lemma 1]) *Let X be a separable metrizable space without isolated points. If A is a nowhere dense nonempty set in X, and $B \subset X$ is an open set such that $\operatorname{Cl} A \subset \operatorname{Cl} B$, then there exists a family $(K_{n,m})_{n\in\mathbb{N},\, m\leqq n}$ of nonempty open sets satisfying the following conditions:*

(1) $\operatorname{Cl} K_{n,m} \subset B \setminus \operatorname{Cl} A$ $(n \in \mathbb{N},\ m \leqq n)$,
(2) $\operatorname{Cl} K_{r,s} \cap \operatorname{Cl} K_{i,j} = \emptyset$ *whenever* $(r,s) \neq (i,j)$ $(r, i \in \mathbb{N},\ s \leqq r,\ j \leqq i)$,
(3) *for each* $x \in \operatorname{Cl} A$, *each neighbourhood* U *of* x *and an arbitrary* m *there exists an* $n \geqq m$ *such that* $\operatorname{Cl} K_{n,m} \subset U$,
(4) *for each* $x \in X \setminus \operatorname{Cl} A$ *there exists a neighbourhood* U *of* x *such that the set* $\{(n,m) : U \cap \operatorname{Cl} K_{n,m} \neq \emptyset\}$ *has at most one element.*

LEMMA 7. *Let G be an open subset of X and let $f\colon X \to \mathbb{R}$ be a cliquish function. Then the restrictions $f|_G$ and $f|_{\operatorname{Cl} G}$ are cliquish functions.*

We omit the easy proof. Remark that the restriction of a cliquish function to an arbitrary closed set need not be cliquish. (Let C be the Cantor set and $C = A \cup B$, where A and B are dense disjoint in C. Then $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = 1$ for $x \in A$ and $f(x) = 0$ otherwise, is cliquish, but $f|_C$ is not cliquish.) The following lemma is obvious.

LEMMA 8. *Let G be an open subset of X, let $f\colon X \to \mathbb{R}$ be a function, and let $x \in \operatorname{Cl} G$ $(x \in G)$. If $f|_{\operatorname{Cl} G}$ $(f|_G)$ is quasicontinuous at x, then f is quasicontinuous at x.*

THEOREM 2. *Let X be a T_3 second countable (=separable metrizable) space. Then $P(\mathcal{Q}) = \mathcal{H}$. More precisely, every function from $\mathcal{H}$ is the product of three quasicontinuous functions.*

Proof. Let $f \in \mathcal{H}$. Denote by D the set of all isolated points of X. Put $B = X \setminus \mathrm{Cl}\, D$. Now denote by

$$G_1 = B \cap \mathrm{Int}\, f^{-1}\big((0,\infty)\big)\,,$$
$$G_2 = B \cap \mathrm{Int}\, f^{-1}\big((-\infty,0)\big)\,,$$
$$G_3 = B \cap \mathrm{Int}\, f^{-1}(0)\,.$$

Then the set

$$A = B \setminus (G_1 \cup G_2 \cup G_3) = B \cap \big((\mathrm{Cl}\, G_1 \setminus G_1) \cup (\mathrm{Cl}\, G_2 \setminus G_2) \cup (\mathrm{Cl}\, G_3 \setminus G_3)\big)$$

is nowhere dense and $\mathrm{Cl}\, A \subset \mathrm{Cl}\, B$. Hence, by Lemma 6, there is a family $(K_{n,m})_{n\in\mathbb{N},\, m\leqq n}$ of nonempty open sets satisfying (1), (2), (3) and (4). Put

$$C = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{n} \mathrm{Cl}\, K_{n,m}\,.$$

Let $j \in \{1,2\}$.
Let $x \in G_j \setminus C$. Then $x \notin \mathrm{Cl}\, A$, and hence, by (4), there is a neighbourhood U of x such that $\{(n,m):\ U \cap \mathrm{Cl}\, K_{n,m} \neq \emptyset\}$ has at most one element. Thus there is (r,s), $r \geqq s$ such that $U \cap \mathrm{Cl}\, K_{n,m} = \emptyset$ for each $(n,m) \neq (r,s)$. Then $G_j \cap U \setminus \mathrm{Cl}\, K_{r,s} \subset G_j \setminus C$ is a neighbourhood of x, and hence $G_j \setminus C$ is an open set.

By Lemma 7, the function $f|_{G_j \setminus C}$ is cliquish, and hence, by Lemma 5, there are quasicontinuous functions $t_1^j,\, t_2^j,\, t_3^j \colon G_j \setminus C \to \mathbb{R}$ such that

$$f|_{G_j \setminus C} = t_1^j \cdot t_2^j \cdot t_3^j\,.$$

Now let $j \in \{1,2\}$, $n \in \mathbb{N}$ and $m \leqq n$.
By Lemma 7, the function $f|_{\mathrm{Cl}\, K_{n,m} \cap G_j}$ is cliquish, and hence, by Lemma 5 there are quasicontinuous functions $g_{n,m,1}^j,\, g_{n,m,2}^j,\, g_{n,m,3}^j \colon \mathrm{Cl}\, K_{n,m} \cap G_j \to \mathbb{R}$ such that

$$f|_{\mathrm{Cl}\, K_{n,m} \cap G_j} = g_{n,m,1}^j \cdot g_{n,m,2}^j \cdot g_{n,m,3}^j\,.$$

Evidently, $g_{n,m,i}^j(x) \neq 0$ for each $i \in \{1,2,3\}$ and each $x \in \mathrm{Cl}\, K_{n,m} \cap G_j$. If $\mathrm{Cl}\, K_{n,m} \cap G_j \neq \emptyset$, choose an arbitrary $a_{n,m}^j \in K_{n,m} \cap G_j$. Let $W \subset \mathrm{Cl}\, D \setminus D$ be a countable dense subset of $\mathrm{Cl}\, D \setminus D$. Then $W = \{w_i:\ i \in M\}$, where $w_r \neq w_s$ for $r \neq s$ and $M \subset \mathbb{N}$. For each $i \in M$ there is a sequence $(v_k^i)_k$ in D converging to w_i such that $v_k^i \neq v_s^r$ for $(i,k) \neq (r,s)$. Let $\mathbb{Q} \setminus \{0\} = \{q_1, q_2, q_3, \dots\}$ (one-to-one sequence of all rationals different from zero).

For each $i \in M$ let $H_i = \{v_2^i, v_4^i, v_6^i, v_8^i, \dots\}$. Now, let $\lambda_i \colon H_i \to \big(\mathbb{Q} \setminus \{0\}\big) \times \mathbb{N}$ be a bijection, and let $\pi \colon \big(\mathbb{Q} \setminus \{0\}\big) \times \mathbb{N} \to \mathbb{Q} \setminus \{0\}$, $\pi(q_r, s) = q_r$.

Put

$$L = \bigcup_{k=1}^{2} \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{[\frac{n+k}{3}]} \operatorname{Cl} K_{n,3m-k}\,.$$

Similarly as for $G_j \setminus C$, we can prove that $G_3 \setminus (C \setminus L)$ is open. Now define functions $f_1, f_2, f_3 \colon X \to \mathbb{R}$ as follows:

$$f_1(x) = \begin{cases} g^j_{n,3m-k,1}(x) \cdot g^j_{n,3m-k,k+1}(a^j_{n,3m-k}) & \text{if } x \in G_j \cap \operatorname{Cl} K_{n,3m-k} \\ & (j \in \{1,2\},\ k \in \{1,2\}, \\ & 3m-k \leqq n), \\ \dfrac{g^j_{n,3m,1}(x) \cdot q_m}{g^j_{n,3m,1}(a^j_{n,3m})} & \text{if } x \in G_j \cap \operatorname{Cl} K_{n,3m} \\ & (j \in \{1,2\},\ 3m \leqq n), \\ q_m & \text{if } x \in G_3 \cap \operatorname{Cl} K_{n,3m} \\ & (3m \leqq n), \\ \pi\big(\lambda_i(x)\big) & \text{if } x \in H_i\ (i \in M), \\ f(x) & \text{if } x \in A \cup \Big(\operatorname{Cl} D \setminus \bigcup\limits_{i \in M} H_i\Big) \\ & \cup\big(G_3 \setminus (C \setminus L)\big), \\ t_1^j(x) & \text{if } x \in G_j \setminus C\ (j \in \{1,2\}); \end{cases}$$

$$f_2(x) = \begin{cases} \dfrac{g^j_{n,3m,2}(x) \cdot g^j_{n,3m,1}(a^j_{n,3m})}{q_m} & \text{if } x \in G_j \cap \operatorname{Cl} K_{n,3m} \\ & (j \in \{1,2\},\ 3m \leqq n), \\ \dfrac{g^j_{n,3m-1,2}(x)}{g^j_{n,3m-1,2}(a^j_{n,3m-1})} & \text{if } x \in G_j \cap \operatorname{Cl} K_{n,3m-1} \\ & (j \in \{1,2\},\ 3m-1 \leqq n), \\ g^j_{n,3m-2,2}(x) & \text{if } x \in G_j \cap \operatorname{Cl} K_{n,3m-2} \\ & (j \in \{1,2\},\ 3m-2 \leqq n), \\ \dfrac{f(x)}{\pi\big(\lambda_i(x)\big)} & \text{if } x \in H_i\ (i \in M), \\ 0 & \text{if } x \in G_3 \setminus L, \\ 1 & \text{if } x \in A \cup (G_3 \cap L) \cup \\ & \Big(\operatorname{Cl} D \setminus \bigcup\limits_{i \in M} H_i\Big), \\ t_2^j(x) & \text{if } x \in G_j \setminus C\ (j \in \{1,2\}); \end{cases}$$

$$f_3(x) = \begin{cases} \dfrac{g^j_{n,3m-2,3}(x)}{g^j_{n,3m-2,3}(a^j_{n,3m-2})} & \text{if } x \in G_j \cap \operatorname{Cl} K_{n,3m-2} \\ & (j \in \{1,2\},\ 3m-2 \leqq n), \\ g^j_{n,3m-k,3}(x) & \text{if } x \in G_j \cap \operatorname{Cl} K_{n,3m-k} \\ & (j \in \{1,2\},\ k \in \{0,1\},\ 3m-k \leqq n), \\ 1 & \text{if } x \in A \cup \operatorname{Cl} D \cup G_3, \\ t^j_3(x) & \text{if } x \in G_j \setminus C\ (j \in \{1,2\}). \end{cases}$$

Then $f = f_1 \cdot f_2 \cdot f_3$.

We shall show that f_1, f_2, f_3 are quasicontinuous. Let $x_0 \in X$. Fix $\varepsilon > 0$ and a neighbourhood U of x_0.

a) Let $x_0 \in A$. Let $m \in \mathbb{N}$ be such that $|q_m - f(x_0)| < \frac{\varepsilon}{2}$. According to (3), there is $n \geqq 3m$ such that $\operatorname{Cl} K_{n,3m} \subset U$. By (1), we have $\operatorname{Cl} K_{n,3m} \cap (G_1 \cup G_2 \cup G_3) \neq \emptyset$.

a1) If $\operatorname{Cl} K_{n,3m} \cap G_3 \neq \emptyset$, then $G = K_{n,3m} \cap G_3$ is an open nonempty subset of U and $|f_1(y) - f_1(x_0)| = |q_m - f(x_0)| < \varepsilon$ for each $y \in G$.

a2) If $\operatorname{Cl} K_{n,3m} \cap G_j \neq \emptyset$ for $j \in \{1,2\}$, then $H = K_{n,3m} \cap G_j \subset U$ is nonempty open. Since $g^j_{n,3m,1}$ is quasicontinuous at $a^j_{n,3m}$, there is an open nonempty $G \subset H$ such that

$$|g^j_{n,3m,1}(y) - g^j_{n,3m,1}(a^j_{n,3m})| < \frac{\varepsilon}{2|q_m|} |g^j_{n,3m,1}(a^j_{n,3m})|$$

for each $y \in G$. Hence, for each $y \in G$ we have

$$|f_1(y) - f_1(a^j_{n,3m})| = \left| \frac{g^j_{n,3m,1}(y) \cdot q_m}{g^j_{n,3m,1}(a^j_{n,3m})} - \frac{g^j_{n,3m,1}(a^j_{n,3m}) \cdot q_m}{g^j_{n,3m,1}(a^j_{n,3m})} \right| < \frac{\varepsilon}{2}$$

and

$$\begin{aligned} |f_1(y) - f_1(x_0)| &\leqq |f_1(y) - f_1(a^j_{n,3m})| + |f_1(a^j_{n,3m}) - f_1(x_0)| \\ &< \frac{\varepsilon}{2} + |q_m - f(x_0)| < \varepsilon . \end{aligned}$$

Thus f_1 is quasicontinuous at $x_0 \in A$.

b) Let $x_0 \in \operatorname{Cl} D \setminus D$. Choose $w_i \in (\operatorname{Cl} D \setminus D) \cap U$ and $v^i_{2j} \in H_i \cap U$ such that

$$\left|\pi\big(\lambda_i(v^i_{2j})\big) - f(x_0)\right| < \varepsilon .$$

Then $\{v^i_{2j}\}$ is an open nonempty subset of U and $|f_1(v^i_{2j}) - f_1(x_0)| < \varepsilon$, thus f_1 is quasicontinuous at $x_0 \in \operatorname{Cl} D \setminus D$.

c) Let $x_0 \in A$. According to (3), there is $n \in \mathbb{N}$ such that $\operatorname{Cl} K_{n,2} \subset U$.

c1) If $\operatorname{Cl} K_{n,2} \cap G_3 \neq \emptyset$, then $G = K_{n,2} \cap G_3$ is an open nonempty subset of U and $|f_2(y) - f_2(x_0)| = 0$ for each $y \in G$.

c2) If $\operatorname{Cl} K_{n,2} \cap G_j \neq \emptyset$ for $j \in \{1,2\}$, then there is an open nonempty subset G of $K_{n,2} \cap G_j$ such that $|g^j_{n,2,2}(y) - g^j_{n,2,2}(a^j_{n,2})| < \varepsilon |g^j_{n,2,2}(a^j_{n,2})|$ for each $y \in G$. Therefore for each $y \in G$ we have

$$|f_2(y) - f_2(x_0)| \leqq |f_2(y) - f_2(a^j_{n,2})| + |f_2(a^j_{n,2}) - f_2(x_0)|$$
$$= \left| \frac{g^j_{n,2,2}(y)}{g^j_{n,2,2}(a^j_{n,2})} - \frac{g^j_{n,2,2}(a^j_{n,2})}{g^j_{n,2,2}(a^j_{n,2})} \right| + |1 - 1| < \varepsilon .$$

Therefore f_2 is quasicontinuous at $x_0 \in A$.

d) Let $x_0 \in \operatorname{Cl} D \setminus D$. Then there are $w_i \in (\operatorname{Cl} D \setminus D) \cap U$ and $v^i_{2j-1} \in U$. Then $\{v^i_{2j-1}\}$ is an open nonempty subset of U and $|f_2(v^i_{2j-1}) - f_2(x_0)| = 0$.

e) Let $x_0 \in A$. Then, by (3), there is $n \in \mathbb{N}$ such that $\operatorname{Cl} K_{n,1} \subset U$, and the quasicontinuity of f_3 at x_0 we can prove similarly as for f_2.

The quasicontinuity of f_1, f_2 and f_3 at other points follows from Lemma 8.

□

P r o b l e m 1. Can the assumption "the family of all open connected subsets of X is a π-base for X" in Theorem 1 be omitted?

P r o b l e m 2. Is every function f from $\mathcal{H}$ (X as in Theorem 2) the product of two quasicontinuous functions?

Evidently, a positive answer to Problem 2 implies a positive answer to Problem 1.

R e m a r k 1. The assumption "X is T_3 second countable" in Theorem 2 cannot be replaced by "X is normal (but not T_1) second countable". If $X = \mathbb{R}$ with the topology $\mathcal{T}$, where $A \in \mathcal{T}$ if and only if $A = \emptyset$ or $A = (a, \infty)$ (where $a \in \mathbb{R}$), then every quasicontinuous function on X is constant (see [2]) but there are nonconstant functions from $\mathcal{H}$ (e.g., $f(x) = 0$ for $x \leqq 0$ and $f(x) = 1$ for $x > 0$).

R e m a r k 2. If X is a Baire space, then $\mathcal{H} = \mathcal{H}^*$, where

$$\mathcal{H}^* = \{f \colon X \to \mathbb{R};\ f \text{ is cliquish and } f^{-1}(0) \text{ is simply open}\}.$$

Evidently, $\mathcal{H} \subset \mathcal{H}^*$. If there is $f \in \mathcal{H}^* \setminus \mathcal{H}$, then the set $f^{-1}\big((0,\infty)\big)$ is not simply open. Hence there is an open nonempty set E such that E is disjoint from $f^{-1}(0)$, and the sets $f^{-1}\big((0,\infty)\big)$ and $f^{-1}\big((-\infty,0)\big)$ are dense in E. Since f is cliquish, the set $\left\{x \in E:\ f(x) > \frac{1}{n}\right\}$ is nowhere dense in E for each $n \in \mathbb{N}$. Then the set

$$E = \bigcup_{n=1}^{\infty} \left\{x \in E:\ f(x) > \frac{1}{n}\right\} \cup \bigcup_{n=1}^{\infty} \left\{x \in E:\ f(x) < -\frac{1}{n}\right\}$$

is of the first category, which is a contradiction.

For an arbitrary X this equality need not hold. If $\mathbb{Q} = A \cup B$, where A and B are dense disjoint in $\mathbb{Q}$, $A = \{a_1, a_2, \dots\}$, $B = \{b_1, b_2, \dots\}$ (one-to-one sequence), then the function $f\colon \mathbb{Q} \to \mathbb{R}$, $f(a_n) = \frac{1}{n}$, $f(b_n) = -\frac{1}{n}$, belongs to $\mathcal{H}^*$, but it does not belong to $\mathcal{H}$.

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