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THE SUM OF ALL DISTANCES IN BIPARTITE GRAPHS[1]

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ABSTRACT. The sum W of the distances between all pairs of vertices in a connected bipartite graph may be any positive integer, except $2, 3, 5, 6, 7, 11, 12, 13, 15, 17, 19, 33, 37$ and 39. We also identify the graph having the second maximal value of W among connected graphs with a given number of vertices.

1. Introduction and the main result

The sum of the distances between all pairs of vertices of a graph attracts the attention of mathematicians for quite a long time (see, for example, [2] – [4], [6] – [8], [10]). In this paper, we determine which values this sum can assume in the case of bipartite graphs.

Let G be a finite connected graph with undirected edges and without loops. The vertex and the edge sets of G are denoted by $V(G)$ and $E(G)$, respectively. The distance between the vertices u and v of G is denoted by $d(u, v \mid G)$; it is equal to the number of edges in the shortest path that connects u and v ([2]). The *vertex distance* $d(u \mid G)$ of the vertex u of G, and the *graph distance* $W(G)$ of the graph G are defined as

$$d(u \mid G) := \sum_{v \in V(G)} d(u, v \mid G),$$

$$W(G) := \frac{1}{2} \sum_{u \in V(G)} d(u \mid G).$$

The quantity $W(G)$ is sometimes called the Wiener number of the graph G, because it was first studied by Harold Wiener in connection with certain

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chemical applications [9]; more details on usages of $W(G)$ in chemistry can be found in [5].

Let $\mathbb{N}$ be the set of all non-negative integers. Then the following elementary result holds ([6]), which we restate here because of completeness.

THEOREM 1. *Let $\mathcal{G}$ be the set of all connected graphs. Then*

$$\mathbb{N} \setminus \{W(G) \mid G \in \mathcal{G}\} = \{2, 5\}.$$

Sketch of the proof. It is sufficient to consider graphs with diameter less than three. For these graphs, $W(G) = |V(G)|\big(|V(G)| - 1\big) - |E(G)|$, and therefore $|E(G)|$ can assume any integer value between $|V(G)| - 1$ and $(1/2)|V(G)|\big(|V(G)| - 1\big)$. □

The main result of this paper is

THEOREM 2. *Let $\mathscr{B}$ be the set of all finite, connected bipartite graphs. Then*

$$\mathbb{N} \setminus \{W(G) \mid G \in \mathscr{B}\} = \{2, 3, 5, 6, 7, 11, 12, 13, 15, 17, 19, 33, 37, 39\}.$$

In order to prove Theorem 2, we need some preparations.

2. Some auxiliary results

The results formulated here as Lemmas 1–5 and 8 are either immediate consequences of the definitions of the quantities $W(G)$ and $d(u \mid G)$, or have been proven elsewhere.

LEMMA 1. *If v is a vertex of the graph G having degree one and being adjacent to the vertex u, then*

$$W(G) = W(G - v) + d(u \mid G - v) + |V(G - v)|.$$

LEMMA 2. *If e is an edge of the graph G, and both G and $G - e$ are connected, then*

$$W(G) < W(G - e).$$

LEMMA 3. *If G is a connected graph and T its spanning tree, then*

$$W(G) < W(T),$$

with equality only if G itself is a tree.

LEMMA 4. ([3]) *If C_n is the circuit with n vertices, then*

$$W(C_n) = \begin{cases} n^3/8 & \textit{if } n \textit{ is even,} \\ (n^3 - n)/8 & \textit{if } n \textit{ is odd.} \end{cases}$$

LEMMA 5. ([3]) *Among connected graphs with n vertices, $n \geq 1$, the path P_n (see Fig. 1) has maximal graph distance. Furthermore,*

$$W(P_n) = \binom{n+1}{3}.$$

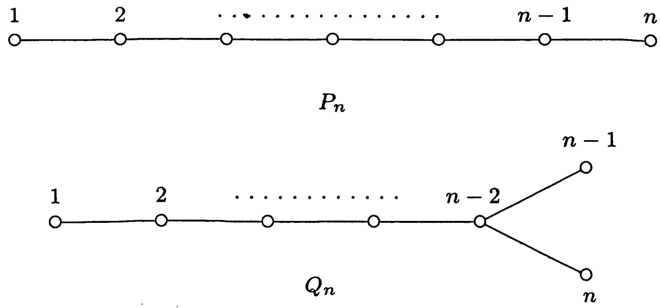


Figure 1.

LEMMA 6. *Among trees with n vertices, $n \geq 4$, that are different from P_n, the vertex 1 of the graph Q_n (see Fig. 1) has maximal vertex distance.*

Proof. Let $\delta_i = \delta_i(v \mid G)$ be the number of vertices of the graph G that are at distance i from the vertex v, $i \geq 0$. Then

$$d(v \mid G) = \sum_{i \geq 0} i\delta_i(v \mid G). \tag{1}$$

It is evident that the choice $\delta_i = 1$ for $0 \leq i \leq n-1$ gives the maximum value for the right-hand side of (1). This occurs in the case of the vertices 1 and n of P_n (cf. Fig. 1). The second largest value for the right-hand side of (1) is obtained when $\delta_i = 1$ for $0 \leq i \leq n-3$ and $\delta_i = 2$ for $i = n-2$. These conditions are satisfied by the vertex 1 of Q_n and only by it. □

LEMMA 7. *Among trees with n vertices, $n \geq 4$, that are different from P_n, the graph Q_n has maximal graph distance.*

Proof proceeds by induction on the number n of vertices. For $n = 4$ and $n = 5$ the validity of Lemma 7 is checked by direct calculation.

Suppose now that $W(Q_m)$ is maximal for all trees with m vertices, different from P_m. Let T be any tree with $m+1$ vertices, other than P_{m+1} or Q_{m+1}, and let v be its vertex of degree one adjacent to the vertex u. If $m \geq 5$, we can always choose v so that $T - v$ is different from P_m. Then, from Lemma 1,

$$W(T) = W(T-v) + d(u \mid T-v) + m, \tag{2}$$

$$W(Q_{m+1}) = W(Q_m) + d(1 \mid Q_m) + m. \tag{3}$$

$T-v$ is a tree with m vertices. Because of Lemma 6, $d(1 \mid Q_m) \geq d(u \mid T-v)$, whereas, by the induction hypothesis, $W(Q_m) \geq W(T-v)$. These inequalities, combined with (2) and (3) imply $W(Q_{m+1}) \geq W(T)$. □

LEMMA 8.

$$W(Q_n) = \binom{n+1}{3} - (n-3)\,.$$

LEMMA 9. *Among connected graphs with n vertices, $n \geq 4$, that are different from P_n, the graph Q_n has maximal graph distance.*

Proof. From Lemmas 4 and 8, we immediately conclude that for $n \geq 4$, $W(C_n) \leq W(Q_n)$. Therefore the circuit C_n can be excluded from the consideration.

All connected graphs with n vertices, except P_n and C_n, have a spanning tree that is different from P_n. Then, in view of Lemma 3, the graph with the second maximal W must be a tree. From Lemma 7, we known that this is Q_n. □

3. Proof of Theorem 2

The vertex set of a bipartite graph G can be partitioned into two disjoint subsets $V_a(G)$ and $V_b(G)$, such that $u,v \in V_a(G) \implies (u,v) \notin E(G)$ and $u,v \in V_b(G) \implies (u,v) \notin E(G)$. We call the vertices from $V_a(G)$ white and the vertices from $V_b(G)$ black. Further, let $|V_a(G)| = \underline{a}$ and $|V_b(G)| = \underline{b}$. Then we say that G is a bipartite graph on $\underline{a} + \underline{b}$ vertices.

Denote by $K_{a,b}$ the complete bipartite graph on $\underline{a} + \underline{b}$ vertices.

LEMMA 10. *If G is a connected bipartite graph on $\underline{a} + \underline{b}$ vertices, then*

$$W(G) \geq (a+b)(a+b-1) - ab \tag{4}$$

with equality only if $G = K_{a,b}$.

Proof. Observe that the right-hand side of (4) is just $W(K_{a,b})$. Every other bipartite graph on $\underline{a} + \underline{b}$ vertices is obtained by deleting some edges from $K_{a,b}$. Lemma 10 follows from Lemma 2. □

LEMMA 11. *If G is a connected bipartite graph on n vertices, $n \geq 1$, then*

$$W(G) \geq \begin{cases} n(n-1) - n^2/4 & \text{if } n \text{ is even,} \\ n(n-1) - (n^2-1)/4 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Lemma 11 is a corollary of Lemma 10 because $n = \underline{a} + \underline{b}$. □

For the first few values of n, the minimal and maximal values of W of connected bipartite graphs on n vertices are given as follows:

Table. Minimal and maximal distances of n-vertex bipartite graphs.

n	1	2	3	4	5	6	7	8	9	10
$W_{\min}$	0	1	4	8	14	21	30	40	52	65
$W_{\max}$	0	1	4	10	20	35	56	84	120	165

□

The subsequent lemma is a previously known result [1]. For the sake of completeness, we, nevertheless, sketch its proof.

LEMMA 12. *The distance of a bipartite graph on $\underline{a}+\underline{b}$ vertices is odd if and only if both $\underline{a}$ and $\underline{b}$ are odd numbers. In particular, the distance of a bipartite graph with odd number of vertices is even.*

P r o o f. The distances between two white or two black vertices are necessarily even. The distance between a white and a black vertex is odd. Hence, the parity of $W(G)$ is equal to the parity of the number of pairs of oppositely colored vertices, i.e., to the parity of the product $\underline{a}\underline{b}$. □

Construct now the bipartite graphs $G_{a2}(i)$, $G_{a3}(i,j)$, $G_{a4}(i,j,k)$ and $G_{a5}(i,j,k,\ell)$ with $\underline{a}$ white and $\underline{b}=2$, $\underline{b}=3$, $\underline{b}=4$ and $\underline{b}=5$ black vertices, respectively. Assuming that $\underline{a}\geq 1$ and $1\leq \ell\leq k\leq j\leq i\leq \underline{a}$, the construction proceeds in the following manner. Denote the white vertices by $v_1,\dots,v_a$ and the black vertices by $w_1,\dots,w_b$. Then $G_{a2}(i)$ is obtained by connecting w_1 with $v_1,\dots,v_a$ and w_2 with $v_1,\dots,v_i$. The graph $G_{a3}(i,j)$ is obtained from $G_{a2}(i)$ by introducing a new black vertex (w_3) and connecting it with $v_1,\dots,v_j$. The graph $G_{a4}(i,j,k)$ is obtained from $G_{a3}(i,j)$ by introducing a new black vertex (w_4) and connecting it with $v_i,\dots,v_k$. The graph $G_{a5}(i,j,k,\ell)$ is obtained from $G_{a4}(i,j,k)$ by introducing another black vertex (w_5) and connecting it with $v_1,\dots,v_\ell$.

By an elementary, yet somewhat tedious calculation we arrive at

$$W\big(G_{a2}(i)\big)=a^2+3a+2-2i\,, \tag{5}$$

$$W\big(G_{a3}(i,j)\big)=a^2+6a+6-2(i+j)\,, \tag{6}$$

$$W\big(G_{a4}(i,j,k)\big)=a^2+9a+12-2(i+j+k)\,, \tag{7}$$

$$W\big(G_{a5}(i,j,k,\ell)\big)=a^2+12a+20-2(i+j+k+\ell)\,. \tag{8}$$

It is now easy to verify that for $\underline{a}=1,2,3,\dots$ and $i+j+k+\ell=4,5,6,\dots,4\underline{a}$, the right-hand side of (8) takes all possible (non-negative) integer values, except the 44 values listed below.

We summarize this observation as:

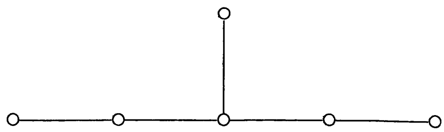
LEMMA 13. *Let $\mathcal{G}_5$ be the set of all graphs $G_{a5}(i,j,k,\ell)$, $\underline{a} \geq 1$, $4 \leq i+j+k+\ell \leq 4\underline{a}$. Then*

$$\mathbb{N} \setminus \{W(G) \mid G \in \mathcal{G}_5\} = \{0,1,2,\ldots,24,26,27,28,29,30,31,33,35,\\ 37,39,42,44,46,48,50,59,61,63,78\}.$$

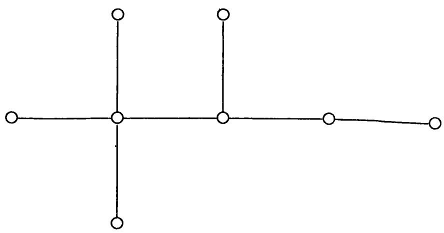
Let, further, $\mathcal{G}_2$, $\mathcal{G}_3$ and $\mathcal{G}_4$ be the sets of all graphs $G_{a2}(i)$, $G_{a3}(i,j)$ and $G_{a4}(i,j,k)$ respectively, $\underline{a} \geq 1$, $1 \leq i \leq \underline{a}$, $2 \leq i+j \leq 2\underline{a}$ and $3 \leq i+j+k \leq 3\underline{a}$. Then a direct calculation, based on (5), (6) and (7), shows that $4,8,10,14,18,22,24,26,28,44,46,48,50,78 \in \mathcal{G}_2$, $9,21,23,27,29,30,42 \in \mathcal{G}_3$ and $16 \in \mathcal{G}_4$. These observations, combined with Lemma 12, result in:

$$\begin{aligned} &\mathbb{N}\setminus\{W(G) \mid G \in (\mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_4 \cup \mathcal{G}_5)\} \\ &\quad = \{0,1,2,3,5,6,7,11,12,13,15,17,19,20,31,33,35,37,39,59,61,63\}. \end{aligned} \tag{9}$$

From the formula given in Lemma 5, we calculate that $W(P_1) = 0$, $W(P_2) = 1$, $W(P_5) = 20$ and $W(P_6) = 35$. In Fig. 2, there are depicted four bipartite graphs whose distances are $31, 59, 61$ and 63. These, together with the examples that lead to (9), demonstrate that there exist bipartite graphs with distances being equal to any positive integer, except those 14 numbers listed in Theorem 2.



$W = 31$



$W = 63$

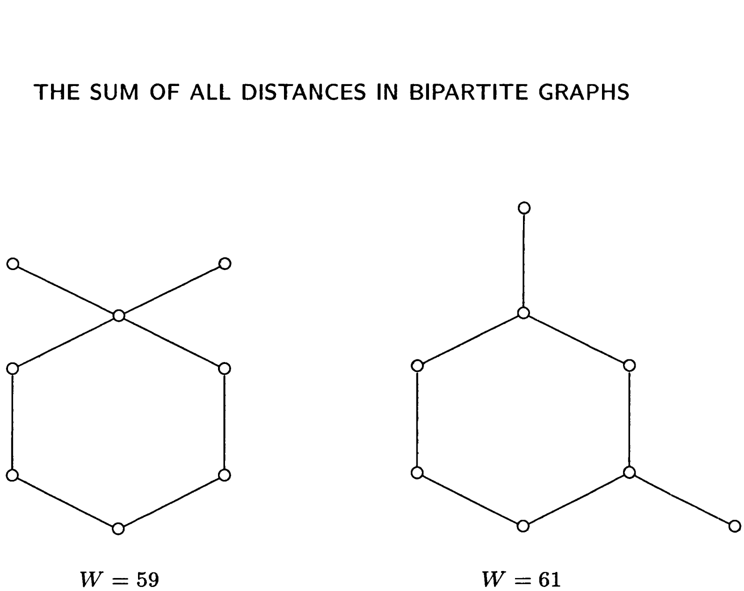


Figure 2.

In order to complete the proof of Theorem 2, it remains to verify that the respective 14 numbers cannot be the distances of any bipartite graphs. This will be done by examining the following five cases.

Case 1: $W = 2, 3, 5, 6, 7$. Because of Lemma 11, bipartite graphs for which $W < 8$ must have less than four vertices. There are exactly three connected bipartite graphs with less than four vertices, and, by direct checking, we conclude that their distances are not equal to $2, 3, 5, 6$ or 7.

Case 2: $W = 11, 12, 13$. Because of Lemma 11, the number of vertices must be less than five. On the other hand, Lemma 5 implies that the maximal possible distance of such graphs is $W(P_4) = 10$. Consequently, W cannot have the values $11, 12, 13$.

Case 3: $W = 15, 17, 19$. From Lemma 11, it follows that the number of vertices must be less than six. The number of vertices cannot be five because, by Lemma 12, if the number of vertices is odd, then W is even. However, the distance of a graph with less than five vertices does not exceed 10.

Case 4: $W = 37, 39$. Because of Lemma 11, the respective graphs cannot have more than seven vertices, and because of Lemma 12, they cannot have seven vertices. The maximal distance of a six-vertex graph is $W(P_6) = 35$.

Case 5: $W = 33$. As in the previous cases, from Lemmas 5, 11 and 12 it follows that the number of vertices must be less than seven and greater than five, i.e., it must be six. Using Lemmas 5 and 8 we obtain $W(P_6) = 35$ and $W(Q_6) = 32$. Then, as a consequence of Lemma 9, no six-vertex graph can have a W value between 32 and 35.

By this, we showed that none of the numbers $2, 3, 5, 6, 7, 11, 12, 13, 15, 17, 19, 33, 37$ and 39 is the distance of a bipartite graph.

This completes the proof of Theorem 2.

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