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THE IRREGULARITY STRENGTH OF GENERALIZED PETERSEN GRAPHS

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ABSTRACT. The generalized Petersen graph $P(n,k)$, $n \geq 3$, $1 \leq k < \frac{n}{2}$, is a graph on $2n$ vertices labelled $\{a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n\}$ and edges $\{a_i b_i, a_i a_{i+1}, b_i b_{i+k} \ : \ i = 1, 2, \dots, n;$ subscripts modulo $n\}$. Assign positive integer weights to the edges of $P(n,k)$ in such a way that the graphs become irregular, i.e. the weight sums at the vertices become pairwise distinct. The minimum of the largest weights assigned over all such irregular assignments on $P(n,k)$ is determined.

1. Introduction

Let G be a simple graph having no connected components isomorphic to K_1 or K_2. A function $w\colon E(G) \to \mathbb{Z}^+$ is called an *assignment* on G, and for an edge e of G, $w(e)$ is called the *weight* of e. We say that w is of *strength* $s(w)$ if $s(w) = \max\{w(e) : \ e \in E(G)\}$. The *weight of a vertex* $x \in V(G)$ is the sum of the weights of its incident edges, and is denoted by $wt(x)$. We call an assignment w *irregular* if distinct vertices have distinct weights. The *irregularity strength* $s(G)$ of G is defined as $s(G) = \min\{s(w) : \ w$ is an irregular assignment on $G\}$.

The problem of studying $s(G)$ was proposed by Chartrand et al. in [1]. It proved to be rather hard, even for very simple graphs ([2], [3], [4], [5], [6], [7], and [8]). An excellent survey on subject was written by Lehel [9].

In this note we continue the study of irregular assignments by determining the irregularity strength of generalized Petersen graphs.

Let n and k be positive integers, $n \geq 3$ and $1 \leq k < \frac{n}{2}$. The *generalized Petersen graph* $P(n,k)$ is a graph with vertex set $\{a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n\}$ and edge set consisting of all edges of the form $a_i a_{i+1}$, $a_i b_i$ and $b_i b_{i+k}$, where $1 \leq i \leq n$; the subscripts are reduced modulo n.

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Generalized Petersen graphs were first defined by W a t k i n s [13]. Various properties of $P(n,k)$ have been found out ever since (see e.g. M c Q u i l l a n - R i c h t e r [10], N e d e l a - Š k o v i e r a [11], S c h w e n k [12], where other references can be found).

We prove in the next sections our main result, the following theorem:

THEOREM. *Let $P(n,k)$, $n \geq 3$, $1 \leq k < \frac{n}{2}$, be a generalized Petersen graph; then*

$$s(P(n,k)) = \begin{cases} \left\lceil \frac{2n+2}{3} \right\rceil & \text{if } n \not\equiv 5 \pmod 6, \\ \left\lceil \frac{2n+2}{3} \right\rceil + 1 & \text{if } n \equiv 5 \pmod 6. \end{cases}$$

2. Lower bounds on $s(P(n,k))$

Since the graph $P(n,k)$ is a cubic graph on $2n$ vertices, it can be easily seen that (compare with [1], [2], and [9]):

LEMMA 1. $s(P(n,k)) \geq \left\lceil \frac{2n+2}{3} \right\rceil$.

LEMMA 2. *Let w be an irregular assignment of $P(n,k)$; then*

$$2\sum_{i=1}^{n}\left[w(a_i a_{i+1}) + w(a_i b_i) + w(b_i b_{i+k})\right] = \sum_{i=1}^{n}\left[wt(a_i) + wt(b_i)\right].$$

LEMMA 3. *If $n \equiv 5 \pmod 6$, then* $s(P(n,k)) \geq \left\lceil \frac{2n+2}{3} \right\rceil + 1$.

P r o o f . If it is not true, then, by Lemma 1, the vertices of $P(n,k)$ must have weights $3, 4, 5, \dots, 12t+11$ and $12t+12$, where $n = 6t+5$, $t \geq 1$. However, note that the sum $[3 + 4 + 5 + \dots + (12t+11) + (12t+12)]$ is odd, which is a contradiction with Lemma 2. □

3. An assignment w of $P(n,k)$ and its strength

To abbreviate the explanation, let us put

$$r = \begin{cases} \left\lceil \frac{2n+2}{3} \right\rceil & \text{for } n \not\equiv 5 \pmod 6, \\ \left\lceil \frac{2n+2}{3} \right\rceil + 1 & \text{for } n \equiv 5 \pmod 6, \end{cases}$$

$$d = \begin{cases} n - r & \text{for } n \equiv 2, 3, 4 \text{ or } 5 \pmod 6, \\ n - r + 1 & \text{for } n \equiv 0 \text{ or } 1 \pmod 6, \end{cases}$$

and

$$c = \left\lfloor \frac{d}{2k} \right\rfloor \qquad \text{(note that } d \text{ is even and } c \geq 0\text{)}.$$

Define an assignment $w\colon E\big(P(n,k)\big) \to \mathbb{Z}^+$ in the following way:

(1) $w(a_i a_{i+1}) = r$ for $1 \le i \le \min\{r, n-1\}$;

(2) $w(a_{r+i} a_{r+i+1}) = r - i$ for $1 \le i \le n - r - 1$;

(3) $w(a_n a_1) = 2r - n$, $w(a_1 b_1) = 1$;

(4) $w(a_{r+i} b_{r+i}) = i + 1$ for $1 \le i \le n - r$;

(5) $$\left\{\begin{array}{r} w(a_{i+2jk} b_{i+2jk}) = 2i + 2jk - 2 \\ w(a_{i+2jk+k} b_{i+2jk+k}) = 2i + 2jk - 1 \\ w(b_{i+2jk} b_{i+2jk+k} = r \end{array}\right\} \quad \begin{array}{c} \text{for } 2 \le i \le k+1 \\ \text{and} \\ 0 \le j \le c-1; \end{array}$$

(6) $$\left\{\begin{array}{r} w(a_{i+2ck} b_{i+2ck}) = 2i + 2ck - 2 \\ w(a_{i+2ck+k} b_{i+2ck+k}) = 2i + 2ck - 1 \\ w(b_{i+2ck} b_{i+2ck+k}) = r \end{array}\right\} \quad \text{for } 2 \le i \le 1 + \frac{d}{2} - ck;$$

(7) $w(a_{i+2ck} b_{i+2ck}) = \frac{d}{2} + ck + i$ for $2 + \frac{d}{2} - ck \le i \le k + 1$;

(8) $w(a_{i+2ck+k} b_{i+2ck+k}) = k + 2ck + i$
for $2 + \frac{d}{2} - ck \le i \le r - k - 2ck$;

(9) for all other edges e of $P(n,k)$ put $w(e) = 1$.

Note that (5) is used only if $c \ge 1$. It is easy to check that no edge gets two different assignments, and hence w is well defined. For an illustration of w, see the graph $P(8,3)$ in Fig. 1.

LEMMA 4.

(i) $d + 1 \le r$,
(ii) $n - r + 3 < d + 4$,
(iii) $1 + ck + k + \frac{d}{2} \le r$,
(iv) $r + d + 2 < 3r - n + 1$.

Proof. Consider six cases according to the residue of n modulo 6. Since the same procedure can be used in every case, we shall only investigate the case $n \equiv 1 \pmod 6$. Details for the remaining cases are left to the reader.

If $n \equiv 1 \pmod 6$, then $r = \frac{2n+4}{3}$ and $d = \frac{n-1}{3}$. The inequalities (i), (ii) and (iv) are now obvious. Rewriting (iii) in terms of n and k, we get $\left\lfloor \frac{n-1}{6k} \right\rfloor k + k \le \frac{n+1}{2}$. This is clearly true for $k \ge \frac{n-1}{6}$. For $1 \le k < \frac{n-1}{6}$ we have $\left\lfloor \frac{n-1}{6k} \right\rfloor k + k < \frac{n-1}{6} + \frac{n-1}{6} = \frac{n-1}{3} < \frac{n+1}{2}$. □

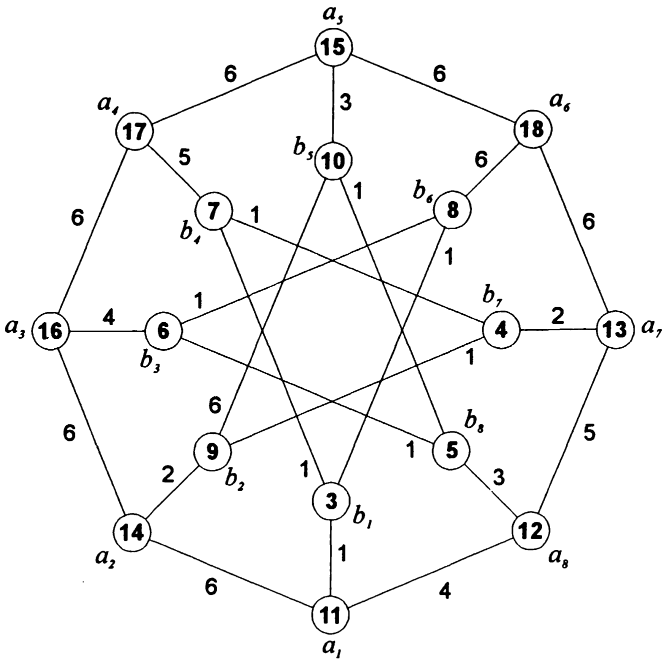

Figure 1.

Lemma 5. *The strength of the assignment w is $s(w) = r$.*

Proof. We need to prove that the weight of every edge e of the graph $P(n,k)$ is at most r, i.e. $1 \le w(e) \le r$. This is obvious for the cases (1), (2), (3), (4), (8) and (9) of the above list.

For the assignments of the case (5) we have $2i + 2jk - 2 \le 2i + 2jk - 1 \le 2(k+1) + 2(c-1)k - 1 = 2ck + 1 = 2k\left\lfloor \frac{d}{2k} \right\rfloor + 1 \le d + 1 \le r$. The last inequality is by Lemma 4 (i).

For the assignments of the case (6), one has $2i + 2ck - 2 < 2i + 2ck - 1 < 2\left(1 + \frac{d}{2} - ck\right) + 2ck - 1 = d + 1 \le r$.

In the case (7), we apply Lemma 4 (iii) and obtain $\frac{d}{2} + 2ck + i \le \frac{d}{2} + ck + k + 1 \le r$. □

4. The irregularity of the assignment w

LEMMA 6. *The assignment w is irregular.*

Proof. The assignment w yields the below listed weight wt for the vertices of the graph $P(n,k)$. Divide them into ten lists $A(1),\dots,A(10)$ in the following way:

$A(1)$: $wt(b_1)=3$, $wt(b_{r+i})=i+3$ for $1\le i\le n-r$.

These weights create a sequence
$S(1)=\{3,4,\dots,n-r+3\}$.

Similarly,

$A(2)$: $wt(b_{i+2ck})=\frac{d}{2}+ck+i+2$ for $2+\frac{d}{2}-ck\le i\le k+1$,

$S(2)=\left\{d+4,\ d+5,\dots,\frac{d}{2}+ck+k+3\right\}$.

$A(3)$: $wt(b_{i+2ck+k})=2ck+k+i+2$ for $2+\frac{d}{2}-ck\le i\le r-k-2ck$,

$S(3)=\left\{\frac{d}{2}+ck+k+4,\dots,r+1,\ r+2\right\}$.

$$A(4):\quad \left\{\begin{array}{l} wt(b_{i+2jk})=r+2jk+2i-1\\ wt(b_{i+2jk+k})=r+2jk+2i\end{array}\right\}\quad \begin{array}{l}\text{for } 2\le i\le k+1\\ \text{and } 0\le j\le c-1,\end{array}$$

$S(4)=\{r+3,\ r+4,\dots,r+2ck+2\}$.
(Note that $S(4)$ is empty if $c=0$).

$$A(5):\quad \left\{\begin{array}{l} wt(b_{i+2ck})=r+2ck+2i-1\\ wt(b_{i+2ck+k})=r+2ck+2i\end{array}\right\}\quad \text{for } 2\le i\le 1+\frac{d}{2}-ck,$$

$S(5)=\{r+2ck+3,\ r+2ck+4,\dots,r+d+2\}$.

$$A(6):\quad \left\{\begin{array}{ll} wt(a_1)=3r-n+1=2r-i+2 & \text{for } i=n-r+1,\\ wt(a_{r+i})=2r-i+2 & \text{for } 1\le i\le n-r,\end{array}\right.$$

$S(6)=\{3r-n+1,\ 3r-n+2,\dots,2r+1\}$.

$$A(7):\quad \left\{\begin{array}{l} wt(a_{i+2jk})=2r+2jk+2i-2\\ wt(a_{i+2jk+k})=2r+2jk+2i-1\end{array}\right\}\quad \begin{array}{l}\text{for } 2\le i\le k+1\\ \text{and } 0\le j\le c-1,\end{array}$$

$S(7)=\{2r+2,\ 2r+3,\dots,2r+2ck+1\}$.
(Note that $S(7)$ is empty if $c=0$).

$$A(8): \quad \left\{ \begin{array}{c} wt(a_{i+2ck}) = 2r + 2ck + 2i - 2 \\ wt(a_{i+2ck+k}) = 2r + 2ck + 2i - 1 \end{array} \right\} \quad \text{for } 2 \le i \le 1 + \frac{d}{2} - ck\,,$$

$S(8) = \{2r + 2ck + 2,\ 2r + 2ck + 3, \dots, 2r + d + 1\}$.

$$A(9): \quad wt(a_{i+2ck}) = 2r + ck + \tfrac{d}{2} + i \quad \text{for } 2 + \frac{d}{2} - ck \le i \le k + 1\,,$$

$S(9) = \left\{2r + d + 2,\ 2r + d + 3, \dots, 2r + ck + k + \frac{d}{2} + 1\right\}$.

$$A(10): \quad wt(a_{i+2ck+k}) = 2r + 2ck + k + i \quad \text{for } 2 + \frac{d}{2} - ck \le i \le r - k - 2ck\,,$$

$S(10) = \left\{2r + ck + k + \frac{d}{2} + 2, \dots, 3r - 1,\ 3r\right\}$.

Now it is a routine matter to verify that :

(i) every vertex of $P(n,k)$ is in the list $A(m)$ for a suitable m;

(ii) for every $m = 1, 2, \dots, 10$, $S(m)$ is a finite arithmetical sequence with difference 1 (if it is not empty);

(iii) $\bigcup\limits_{m=1}^{10} S(m)$ is the set of $2n$ mutually different values because $\max S(m) < \min S(m+1)$ for every $m = 1, 2, \dots, 9$, and $c \ge 1$ (for $m = 1$ by Lemma 4 (ii), and for $m = 5$ by Lemma 4 (iv)).
In the case $c = 0$, it is also easy to see that $\max S(s) < \min S(t)$ for every s and t, $1 \le s < t \le 10$, for which the sets $S(s)$ and $S(t)$ are not empty.

This completes the proof. □

Now the main theorem immediately follows from Lemmas $1, 3, 5, 6$.

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