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MEASURE DENSITY OF SOME SETS

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ABSTRACT. In this paper, measurability and Buck's measure density of some sets of positive integers are studied.

Introduction

The aim of this paper is to determine the value of density of some sets of positive integers. Denote by $\mathbb{N}$ the set of all positive integers. For $A \subset \mathbb{N}$ and $x > 0$ put $A(x) = |\{k \in A;\ k \le x\}|$. The value

$$\overline{d}(A) = \limsup_{x\to\infty} \frac{A(x)}{x},$$

will be called the *upper asymptotic density* of A. If there exists the limit

$$d(A) = \lim_{x\to\infty} \frac{A(x)}{x},$$

we say that A has the asymptotic density, and the value $d(A)$ is called *asymptotic density.* This notion is well known [HAR; p. 269]. Another type of the density of a subset of $\mathbb{N}$ was introduced in 1946 by R. C. Buck [BUC] in the following way:

Put $a + \langle m\rangle = \{a + k\cdot m,\ k = 0,1,\dots\}$, for $m \in \mathbb{N}$ and a nonnegative integer. Then the value

$$\nu^*(A) = \inf\left\{ \sum_{i=1}^{k} \frac{1}{m_i};\ A \subset \bigcup_{1\le i\le k} a_i + \langle m_i\rangle \right\}$$

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will be called the *measure density* of A. Clearly

$$\overline{d}(A) \leq \nu^*(A)\,, \tag{1}$$

for $A \subset \mathbb{N}$. Moreover, the class of sets

$$\mathcal{D}_\nu = \big\{A \subset \mathbb{N};\ \ \nu^*(A) + \nu^*(\mathbb{N} \setminus A) = 1\big\}$$

is an algebra of sets, the restriction $\nu = \nu^*\Big|_{\mathcal{D}_\nu}$ is a finitely additive measure on $\mathcal{D}_\nu$ and $\nu(A) = d(A)$ for each $A \in \mathcal{D}_\nu$ (cf. [PAS1]). Sets from $\mathcal{D}_\nu$ will be called *measurable*. Clearly, each set $a + \langle m\rangle$ belongs to $\mathcal{D}_\nu$. We shall write $\langle m\rangle$ instead of $0 + \langle m\rangle$. Remark that the standard notation for measure density in [BUC] is μ. But we shall use this symbol for Möbius function.

Density of sets without a certain type of divisors

In what follows, let $\mathbf{P}$ be a subset of the set of prime numbers. We start with calculation of the density of the following set

$$Q(n,\mathbf{P}) = \big\{a \in \mathbb{N};\ \ \forall p{\in}\mathbf{P}\ p^n \nmid a\big\}\,.$$

If $n = 2$ and $\mathbf{P}$ is the set of all prime numbers, then this set is the well-known set of square free integers (cf. [HAR; p. 269] and a lot of books about the elementary number theory.) Later we shall study measurability and the measure density of other type of sets given by divisibility.

In this part we prove following statement:

THEOREM 1. *For $n > 1$ and $\mathbf{P}$ a subset of the prime numbers we have*

(a) $d\big(Q(n,\mathbf{P})\big) = \prod\limits_{p\in\mathbf{P}} \Big(1 - \dfrac{1}{p^n}\Big)$.

(b) $\nu^*\big(Q(n,\mathbf{P})\big) = \prod\limits_{p\in\mathbf{P}} \Big(1 - \dfrac{1}{p^n}\Big)$.

(c) *The set $Q(n,\mathbf{P})$ for $n > 1$ is measurable if and only if the set $\mathbf{P}$ is finite. Moreover, if $\mathbf{P}$ is infinite, then $\nu^*\big(\mathbb{N} \setminus Q(n,\mathbf{P})\big) = 1$.*

For the proof of Theorem 1 we shall use following results. Denote $M(\mathbf{P})$ the set of all positive integers which are divisible only by primes from $\mathbf{P}$. Clearly $M(\mathbf{P})$ is a semigroup which contains 1, and satisfies the following property

$$\forall a \in M(\mathbf{P}),\, b \in \mathbb{N};\ b\,|\,a \implies b \in M(\mathbf{P})\,.$$

Let μ be the well-known Möbius function, then the above implication gives for $1 < a \in M(\mathbf{P})$

$$\sum_{\substack{b|a\\ b\in M(\mathbf{P})}} \mu(b) = 0\,. \tag{2}$$

Lemma 1. *Let F, G be functions, defined on $(0,\infty)$. Then for $x>0$*

$$F(x)=\sum_{\substack{n\le x\\ n\in M(\mathbf{P})}} G\Big(\frac{x}{n}\Big) \iff G(x)=\sum_{\substack{n\le x\\ n\in M(\mathbf{P})}} \mu(n)F\Big(\frac{x}{n}\Big)$$

Proof. The statement can be verified by an easy computation using (2).

Remark that Lemma 1 is an easy generalization of summation formula cf. [HAR; p. 237]. The following statement gives a possibility to use Lemma 1 for a direct calculation of $d\big(Q(n,\mathbf{P})\big)$.

Lemma 2. *Let $y\in\mathbb{N}$. Put $Q=Q(n,\mathbf{P})$. Then for $n\in\mathbb{N}$*

$$y^n=\sum_{\substack{b\le y\\ b\in M(\mathbf{P})}} Q\Big(\frac{y^n}{b^n}\Big).$$

Proof. Consider $b\in M(\mathbf{P})$. Let S_b be the set of all numbers from the set $\{1,\dots,y^n\}$ for which b^n is their greatest divisor from the set $\{a^n;\ a\in M(\mathbf{P})\}$. Clearly these sets are disjoint and

$$\{1,\dots,y^n\}=\bigcup_{\substack{b\le y\\ b\in M(\mathbf{P})}} S_b.$$

Hence it is enough to prove $|S_b|=Q\Big(\frac{y^n}{b^n}\Big)$. This equation can be easily verified if we take into account that

$$a\in S_b \iff a=k\cdot b^n\wedge k\in Q,\quad k\le\frac{y^n}{b^n}.$$

□

Proof of Theorem 1. Lemma 1 gives for $Q=Q(n,\mathbf{P})$

$$Q(y^n)=\sum_{\substack{b\le y^n\\ b\in M(\mathbf{P})}} \frac{y^n}{b^n}\,\mu(b)$$

and so

$$\frac{Q(y^n)}{y^n}=\sum_{\substack{b\le y^n\\ b\in M(\mathbf{P})}} \frac{\mu(b)}{b^n}=\prod_{\substack{p\le y^n\\ p\in P}}\Big(1-\frac{1}{p^n}\Big)+O\Bigg(\sum_{\substack{b>y^n\\ b\in M(\mathbf{P})}}\frac{1}{b^n}\Bigg)$$

For $y \to \infty$ we obtain part (a) of Theorem.

For the proof of part (b), we shall recall some facts from the theory of polyadic numbers. The ring of polyadic numbers was constructed in 1960 by E. V. Novoselov [NOV], as a completion of the set $\mathbb{N}$ with respect to a special metric given by divisibility. Denote this ring by Ω. There exists a uniquely determined Haar probability measure π on this ring. If (m) is a principal ideal in Ω generated by $m \in \mathbb{N}$, then for each class $a + (m)$

$$\pi\big(a + (m)\big) = \frac{1}{m} \tag{3}$$

holds (cf. [NOV]). Further it will be important that for a set $S \subset \mathbb{N}$ we have [OKU], [PAS]

$$\nu^*(S) = \pi\big(\overline{S}\big), \tag{4}$$

where $\overline{S}$ is a topologic closure of S. A class $a + (m)$ is an open set, [NOV]. Put $Q = Q(n, \mathbf{P})$ for $n > 1$. Clearly

$$Q \subset \Omega \setminus \bigcup_{p \in \mathbf{P}} (p^n),$$

and by (4)

$$\nu^*(Q) = \pi\big(\overline{Q}\big) \le \pi\Big(\Omega \setminus \bigcup_{p \in \mathbf{P}} (p^n)\Big). \tag{5}$$

Considering (3) and Exclusion-inclusion principle we obtain

$$\pi\Big(\Omega \setminus \bigcup_{p \in \mathbf{P}} (p^n)\Big) = \prod_{p \in \mathbf{P}} \Big(1 - \frac{1}{p^n}\Big),$$

which implies the statement (b) according to part (a) and (5).

Clearly $Q(n, \mathbf{P}) = \mathbb{N} \setminus \bigcup\limits_{p \in \mathbf{P}} \langle p^n \rangle$. This yields $Q(n, \mathbf{P}) \in \mathcal{D}_\nu$ if $\mathbf{P}$ is finite. Let $\mathbf{P}$ be infinite. Consider an arithmetic progression $a + \langle m \rangle$. There exists such $p \in \mathbf{P}$ that $(m, p) = 1$. Hence

$$p^{\varphi(m)} \equiv 1 \pmod{m}.$$

And so $a \cdot p^{\varphi(m) \cdot n} \in a + \langle m \rangle \cap \big(\mathbb{N} \setminus Q(n, \mathbf{P})\big)$. This implies that $\mathbb{N} \setminus Q(n, \mathbf{P})$ has a nonempty intersection with arbitrary arithmetic progression and so $\nu^*\big(\mathbb{N} \setminus Q(n, \mathbf{P})\big) = 1$ (cf. [PAS1]). The proof of (c) is complete. $\square$

Measurability and measure density of two types of sets

Let p be a prime number. The arithmetical function a_p is defined in the following way:

$$a_p(n) = \max\{k\,;\ p^k \mid n\}$$

for $n = 1, 2, \dots$. Put $T_k = \{n\,;\ a_p(n) = k\}$. Clearly $T_k = \langle p^k \rangle \setminus \langle p^{k+1} \rangle$ and so $T_k \in \mathcal{D}_\nu$ and $\nu(T_k) = \dfrac{1}{p^k} - \dfrac{1}{p^{k+1}}$.

In the sequel we shall determine the measure density of the sets

$$M^p = \{n \in \mathbb{N};\ a_p(n) \mid n\}$$

for an arbitrary prime number p. The asymptotic density of these sets is determined in [SAL].

THEOREM 2. *For each prime number p we have that $M^p \in \mathcal{D}_\nu$ and*

$$\nu(M^p) = \frac{p-1}{p^2} \cdot \sum_{j=0}^{\infty} \log \frac{p^{p^{j+1}} - 1}{(p^{p^j} - 1)^p}\,.$$

We shall use the following statements:

THEOREM A. *Let $\{H_i\}$ be a system of disjoint measurable sets. Let*

$$\lim_{n\to\infty} \nu^*\left(\bigcup_{k=n}^{\infty} H_k\right) = 0\,.$$

Then the set $H = \bigcup\limits_{k=1}^{\infty} H_k$ belongs to $\mathcal{D}_\nu$ and

$$\nu\left(\bigcup_{k=1}^{\infty} H_k\right) = \sum_{k=1}^{\infty} \nu(H_k)\,.$$

P r o o f. See [PAS1].

LEMMA 3. *For arbitrary prime number p there holds*

$$\sum_{k=1}^{\infty} \frac{p-1}{k \cdot p^{k-a_p(k)+1}} = \frac{p-1}{p^2} \cdot \sum_{j=0}^{\infty} \log \frac{p^{p^{j+1}} - 1}{(p^{p^j} - 1)^p}$$

Proof. We shall use the following equation for p prime and $x \in [0,1)$:

$$\sum_{(p,q)=1} \frac{x^q}{q} = \frac{1}{p} \log \frac{1-x^p}{(1-x)^p} \,. \tag{6}$$

This equation can be easily verified considering that for $x \in [0,1)$ we have $\sum\limits_{q=1}^{\infty} \frac{x^q}{q} = -\log(1-x)$ and for p prime

$$\sum_{(p,q)=1} \frac{x^q}{q} = \sum_{q=1}^{\infty} \frac{x^q}{q} - \sum_{r=1}^{\infty} \frac{x^{rp}}{rp} \,.$$

Denote the sum on the left side of the statement by S. Then we have

$$S = \frac{p-1}{p} \sum_{k=1}^{\infty} \frac{1}{k \cdot p^{k-a_p(k)}} = \frac{p-1}{p} \sum_{j=0}^{\infty} \sum_{a_p(k)=j} \frac{1}{k \cdot p^{k-j}} \,.$$

But for $j = 0,1,\dots$ it holds $a_p(k) = j \iff k = p^j \cdot q$, $(p,q) = 1$ and so

$$\sum_{a_p(k)=j} \frac{1}{k \cdot p^{k-j}} = \sum_{(q,p)=1} \frac{1}{p^j \cdot q \cdot p^{(p^j)q-j}} = \sum_{(q,p)=1} \frac{1}{q(p^{(p^j)})^q} \,.$$

The statement follows now from (6). □

Proof of Theorem 2. Clearly

$$M^p = \bigcup_{k=1}^{\infty} B_k \,,$$

where

$$B_k = \{n \in \mathbb{N};\ a_p(n) = k \wedge k \mid n\}$$

for $k = 1,2,\dots$. We show that $n \in B_k$ if and only if n can be expressed in the form

$$n = k \cdot p^{k-a_p(k)} \cdot m \,, \quad (m,p) = 1 \,. \tag{7}$$

This follows from the fact that B_k is the set of all positive integers which are divisible by k and p^k and not divisible by p^{k+1}. Condition (7) gives

$$B_k = \langle kp^{k-a_p(k)} \rangle \setminus \langle kp^{k-a_p(k)+1} \rangle \,. \tag{8}$$

Now we can use Theorem A. Clearly, $B_{k_1} \cap B_{k_2} = \emptyset$ for $k_1 \neq k_2$. From (8) it follows that $B_k \in \mathcal{D}_\nu$ and

$$\nu(B_k) = \frac{p-1}{k \cdot p^{k-a_p(k)+1}} \tag{9}$$

for $k = 1, 2, \dots$. Moreover, for $k \geq K$ we get $B_k \subset \langle p^{K-a_p(K)} \rangle$, thus $\bigcup\limits_{k=K}^{\infty} B_k \subset \langle p^{K-a_p(K)} \rangle$. Considering the fact $a_p(K) = O(\log K)$ we obtain

$$\lim_{K \to \infty} \nu^* \left(\bigcup_{k=K}^{\infty} B_k \right) = 0 .$$

The statement now follows from (9), Theorem A and Lemma 3. □

If $n > 1$ is a positive integer, $n = p_1^{\alpha_1} \cdot \ldots \cdot p_k^{\alpha_k}$ is the standard form of n, then we put $h(n) = \min\{\alpha_1, \dots, \alpha_n\}$. Define

$$M^h = \{ n \in \mathbb{N}; \quad h(n) \,|\, n \} .$$

In [S-S] it is proved that asymptotic density of M^h is 1. The following theorem is a stronger result.

THEOREM 3. *The set M^h is measurable and $\nu(M^h) = 1$.*

Denote $A_p = A \cap (\langle p \rangle \setminus \langle p^2 \rangle)$ for $A \subset \mathbb{N}$ and p prime. We shall use following statement:

THEOREM B. *Let $A \subset \mathbb{N}$. If $\nu^*(A_p) = 0$ for each prime p, then $A \in \mathcal{D}_\nu$ and $\nu(A) = 0$.*

Proof. See [PAS2].

Proof of Theorem 3. Put $C = \mathbb{N} \setminus M^h$. Clearly, for each p prime and $a \in \langle p \rangle \setminus \langle p^2 \rangle$ we have $h(a) = 1$. Therefore $C_p = \emptyset$. The statement follows now from Theorem B. □

Note on Cantor expansions of positive integers

Let $\{C_n\}$ be an increasing sequence of positive integers, such that $C_0 = 1$ and $C_n \,|\, C_{n+1}$ for $n = 1, 2, \dots$. It is well known that each positive integer k can be uniquely expressed in the form of finite *Cantor series*

$$k = \sum_{n=0}^{N} a_n(k) C_n \tag{10}$$

where $0 \le a_n(k) < \dfrac{C_{n+1}}{C_n}$, $n = 1, 2, \dots, N$, and N is such index that $k < C_{N+1}$. Clearly, the coefficients $a_n(k)$ are functions of k.

Let $a_0^0, \dots, a_n^0$ be such integers that $0 \le a_i^0 < \dfrac{C_{i+1}}{C_i}$, $i = 1, 2, \dots, n$. Put

$$b = a_0^0 C_0 + \dots + a_n^0 C_n .$$

Considering that $k \in b + \langle C_{n+1} \rangle$ if and only if $k \equiv b \pmod{C_j}$, $0 \le j \le n+1$, we obtain

$$\forall k \in \mathbb{N} \;\; k \in b + \langle C_{n+1} \rangle \iff a_i(k) = a_i^0 , \quad i = 0, \dots, n . \tag{11}$$

This property gives possibility to establish some properties analogical to the properties of Cantor series of real numbers.

THEOREM 4. *Let $0 \le r < \dfrac{C_{n+1}}{C_n}$, for n fixed. Then the set $F_r^n = \{k \in \mathbb{N};\ a_n(k) = r\}$ is measurable and*

$$\nu(F_r^n) = \frac{C_n}{C_{n+1}} .$$

Proof. Let $b_1, \dots, b_K$ be all numbers in the form

$$a_0^0 C_0 + \dots + a_{n-1}^0 C_{n-1} + r C_n ,$$

where $0 \le a_i^0 < \dfrac{C_{i+1}}{C_i}$, $i = 1, 2, \dots, n-1$. Then (11) gives $F_r^n = \bigcup\limits_{j=1}^{K} b_j + \langle C_{n+1} \rangle$. Thus F_r^n is measurable. Moreover, arithmetic sequences on the right-hand side are disjoint and

$$K = \frac{C_1}{C_0} \cdot \ldots \cdot \frac{C_n}{C_{n-1}} = C_n .$$

Hence $\nu(F_r^n) = \dfrac{C_n}{C_{n+1}}$. □

Finally we shall study the following set:

$$\mathbb{N}^{(r)} = \bigl\{ k \in \mathbb{N};\ \exists n \; a_n(k) = r \bigr\}$$

for r nonnegative integer.

THEOREM 5. *Let n_0 be the smallest index such that $r < \frac{C_{n_0+1}}{C_{n_0}}$, and $r < \frac{C_{n+1}}{C_n}$, $n \geq n_0$. Then*

$$\nu^*(\mathbb{N} \setminus \mathbb{N}^{(r)}) = \prod_{n=n_0}^{\infty} \left(1 - \frac{C_n}{C_{n+1}}\right).$$

P r o o f. Put $M = \mathbb{N} \setminus \mathbb{N}^{(r)}$. For $n \leq n_0$ we have

$$M \subset \bigcup_{\substack{b < C_{n+1}, \\ b \in M}} b + \langle C_{n+1} \rangle .$$

The number of such nonnegative integers $b < C_{n+1}$, $b \in M$, is due to (11) exactly

$$M(C_{n+1}) = \frac{C_1}{C_0} \cdot \ldots \cdot \frac{C_{n_0}}{C_{n_0-1}} \cdot \left(\frac{C_{n_0+1}}{C_{n_0}} - 1\right) \cdot \ldots \cdot \left(\frac{C_{n+1}}{C_n} - 1\right).$$

Moreover,

$$C_{n+1} = \prod_{j=0}^{n} \frac{C_{j+1}}{C_j} . \tag{12}$$

Hence

$$\nu^*(M) \leq \frac{1}{C_{n+1}} M(C_{n+1}) = \prod_{j=n_0}^{n} \left(1 - \frac{C_j}{C_{j+1}}\right). \tag{13}$$

Thus for $n \to \infty$ we obtain

$$\nu^*(M) \leq \prod_{n=n_0}^{\infty} \left(1 - \frac{C_n}{C_{n+1}}\right).$$

From (12) we get $\lim_{n\to\infty} \frac{M(C_{n+1})}{C_{n+1}} = \prod_{n=n_0}^{\infty} \left(1 - \frac{C_n}{C_{n+1}}\right)$, and so $\overline{d}(M) \geq \prod_{n=n_0}^{\infty} \left(1 - \frac{C_n}{C_{n+1}}\right)$. Now from (13) and (1) we obtain the statement. □

PROPOSITION. *Let $\{C_n\}$ satisfy the following condition*

$$\forall d \in \mathbb{N}\ \exists n_0\ \forall\ n > n_0 :\ d \,|\, C_n .$$

Let $r < \frac{C_{n+1}}{C_n}$, for $n > m$. Then $\nu^*\big(\mathbb{N}^{(r)}\big) = 1$.

P r o o f. Consider an arithmetic progression $a + \langle d \rangle$. Let n be such an index that $a < C_n$, $d \,|\, C_n$ and $r < \frac{C_{n+1}}{C_n}$. Then $a + rC_n \in a + \langle C_n \rangle \cap \mathbb{N}^{(r)}$. Thus $\mathbb{N}^{(r)}$ has a nonempty intersection with an arbitrary arithmetic progression, and so $\nu^*\big(\mathbb{N}^{(r)}\big) = 1$ (cf. [PAS1]). $\square$

Remark that a sequence $\{C_n\}$ satisfying condition from Proposition is called *complete* (cf. [PAS1]). Theorem 5 and Proposition give:

E x a m p l e. If $C_n = n!$, then $\mathbb{N}^{(r)}$ is a measurable set for each $r = 0, 1, \dots$. If $C_n = (n!)^a$, $a > 1$ – an integer, then $\mathbb{N}^{(r)}$ is not a measurable set.

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