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Mathematica Slovaca, Vol. 44 (1994), No. 5, 505--514

Persistent URL: http://dml.cz/dmlcz/136624

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Mathematica Slovaca

REMARKS ON MAXIMUM AND MINIMUM EXPONENTS IN FACTORING

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(Communicated by Stanislav Jakubec)

ABSTRACT. Let $n > 1$ be an integer, $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_k^{\alpha_k}$ (standard for n). Put $H(n) = \max\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$, $h(n) = \min\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$, $h(1) = 1 = H(1)$. In the paper, asymptotic densities of the sets $M_f = \{n : f(n) \mid n\}$ for $f = H$ and $f = h$ are established. Further some properties of functions h, H are investigated in connection with the concepts of statistical convergence and normal order.

Introduction

In their papers [1], [2], [6], the authors deal with determining the natural (asymptotic) densities of sets of the form $M_f = \{n : f(n) \mid n\}$, where $f : \mathbb{N} \to \mathbb{N}$ is a given function. This aim is achieved in [1], for the function $s(n)$ (the sum of digits of n), $\omega(n)$ (the number of distinct prime factors of n), $\ell(n) = [\log_b n]$ ($b > 1$) and $r(n) = [n^{1/2}]$, and the proof is based on a result derived with the help of the classical Chebyshev inequality from probability theory. A related result covering the functions $s(n)$, $\omega(n)$, $\Omega(n)$ (the number of prime factors of n counted with multiplicities), $\pi(n)$ (the number of primes not exceeding n), $S(n) = \sum_{p \mid n} p$ (the sum of prime factors of n) is proved in [2]. In [6], the density of the set M_τ is evaluated, where $\tau(n)$ denotes the number of divisors of n.

In this note, we shall investigate similar questions for the functions h and H introduced in [7] (see also [13]). Further we shall study the properties of functions h and H from the standpoint of statistical convergence and investigate normal order of these functions.

AMS Subject Classification (1991): Primary 11A25, 11B05.

Key words: Asymptotic density, Statistical convergence, Normal order.

[1] Research on this work was partially supported by the Grant No. 363 of the Slovak Academy of Sciences.

In what follows, we shall use the following usual notations: If $A \subseteq \mathbb{N}$ and $A(x) = \sum_{a \in A,\, a \leqq x} 1$, then we put $\overline{d}(A) = \limsup_{x \to \infty} \frac{A(x)}{x}$ (the upper asymptotic density of A), $\underline{d}(A) = \liminf_{x \to \infty} \frac{A(x)}{x}$ (the lower asymptotic density of A) and $d(A) = \lim_{x \to \infty} \frac{A(x)}{x}$ (the asymptotic density of A), if the limit on the right-hand side exists (cf. [9; p. 71]).

If $T(n)$ is a prediction formula (a property of n) defined for $n \in \mathbb{N}$ and the set of all $n \in \mathbb{N}$ satisfying $T(n)$ (having the property $T(n)$) has the asymptotic density 1, then we briefly say that almost all $n \in \mathbb{N}$ satisfy $T(n)$ (have the property $T(n)$).

1. On sets M_h and M_H

If $n > 1$ is a positive integer, $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \dots \cdot p_k^{\alpha_k}$ is the standard form of n, then we put $h(n) = \min_{1 \leqq j \leqq k} \alpha_j$, $H(n) = \max_{1 \leqq j \leqq k} \alpha_j$ and $h(1) = 1 = H(1)$ (cf. [7]). It is proved in [7] that

$$\lim_{n \to \infty} \frac{h(1) + h(2) + \dots + h(n)}{n} = 1. \tag{1}$$

This result is strengthened in [13].

The equality (1) eliminates the possibility of applying the method of [1] for determining the density of M_h. We shall determine this density in another way.

THEOREM 1.1. *We have $d(M_h) = 1$.*

Proof. In [8], the following result is proved (see [8; p. 254, Theorem 11.7]):

Let $(p_j)_{j=1}^{\infty}$ be a sequence of prime numbers with $\sum_{j=1}^{\infty} p_j^{-1} = +\infty$. Let $A \subseteq \mathbb{N}$, and denote by A_{p_j} $(j = 1, 2, \dots)$ the set of all $a \in A$ such that $p_j \mid a$, but $p_j^2 \nmid a$. If $d(A_{p_j}) = 0$ $(j = 1, 2, \dots)$, then $d(A) = 0$.

Put $W_1 = \{n : h(n) = 1\}$. Then we have evidently

$$W_1 \subseteq M_h. \tag{2}$$

Set $A = \mathbb{N} \setminus W_1$. If $n \in A$ and p is an arbitrary prime number such that $p \mid n$, then $p^2 \mid n$, as well. Hence $A_p = \emptyset$ for each prime and applying the quoted result of [8] we get $d(A) = 0$. This yields $d(W_1) = 1$, and the assertion follows from (2). □

As it is remarked in [7], the equality $d(W_1) = 1$ can be deduced also from (1), and in this way one can obtain another proof of Theorem 1.1.

It is proved in [7] that

$$\lim_{n\to\infty} \frac{H(1) + H(2) + \cdots + H(n)}{n} = 1 + \sum_{k=2}^{\infty} \left(1 - \xi^{-1}(k)\right) \in (1,2). \tag{3}$$

This result is improved in [13]. The equality (3) eliminates a possibility of applying the method from [1] for determining the density of the set M_H. Therefore we shall proceed in another way.

THEOREM 1.2. *We have*

$$d(M_H) = \xi^{-1}(2) + \sum_{\alpha=2}^{\infty} \left(\xi^{-1}(\alpha+1) \prod_{p\mid\alpha} \frac{p^{\alpha - \mathrm{ord}_p \alpha + 1} - 1}{p^{\alpha+1} - 1} - \xi^{-1}(\alpha) \prod_{p\mid\alpha} \frac{p^{\alpha - \mathrm{ord}_p \alpha} - 1}{p^{\alpha} - 1} \right),$$

where p runs over all primes.

The proof will be based on the following lemmas.

LEMMA 1.1. *The density $d_1(\alpha)$ of numbers n such that $\alpha \mid n$, $H(n) \leqq \alpha$ is*

$$\xi^{-1}(\alpha+1) \prod_{p\mid\alpha} \frac{p^{\alpha - \mathrm{ord}_p \alpha + 1} - 1}{p^{\alpha+1} - 1}.$$

P r o o f. We have

$$\sum_{m^{\alpha+1} \mid n} \mu(m) = \begin{cases} 1 & \text{if } H(n) \leqq \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Hence the number of numbers in question, not exceeding x, equals

$$\begin{aligned}
&\sum_{1\leqq k\leqq \frac{x}{\alpha}} \ \sum_{m^{\alpha+1}\,|\,k\alpha} \mu(m) \\
&= \sum_{1\leqq m\leqq \sqrt[\alpha+1]{x}} \mu(m) \sum_{1\leqq k\leqq \frac{x}{\alpha},\ m^{\alpha+1}\,|\,k\alpha} 1 = \sum_{1\leqq m\leqq \sqrt[\alpha+1]{x}} \mu(m)\left[\frac{x(\alpha,m^{\alpha+1})}{\alpha m^{\alpha+1}}\right] \\
&= \frac{x}{\alpha}\sum_{1\leqq m\leqq \sqrt[\alpha+1]{x}} \frac{\mu(m)(\alpha,m^{\alpha+1})}{m^{\alpha+1}} + O(\sqrt[\alpha+1]{x}) \\
&= \frac{x}{\alpha}\left(\sum_{m=1}^{\infty} \frac{\mu(m)(\alpha,m^{\alpha+1})}{m^{\alpha+1}} + O(\sqrt[\alpha+1]{x^{-\alpha}})\right) + O(\sqrt[\alpha+1]{x}) \\
&= \frac{x}{\alpha}\sum_{m=1}^{\infty} \frac{\mu(m)(\alpha,m^{\alpha+1})}{m^{\alpha+1}} + O(\sqrt[\alpha+1]{x})\,.
\end{aligned}$$

It follows that the density $d_1(\alpha)$ exists, and

$$d_1(\alpha) = \frac{1}{\alpha}\sum_{m=1}^{\infty} \frac{\mu(m)(\alpha,m^{\alpha+1})}{m^{\alpha+1}}\,.$$

The series on the right-hand side is absolutely convergent and the function $\dfrac{\mu(m)(\alpha,m^{\alpha+1})}{m^{\alpha+1}}$ is multiplicative. Hence

$$d_1(\alpha) = \frac{1}{\alpha}\prod_p\left(1-\frac{(\alpha,p^{\alpha+1})}{p^{\alpha+1}}\right) = \prod_p\left(1-\frac{1}{p^{\alpha+1}}\right)\,.$$

$$\prod_{p|\alpha}\left(\frac{1}{(\alpha,p^{\alpha+1})}-\frac{1}{p^{\alpha+1}}\right)\cdot\frac{p^{\alpha+1}}{p^{\alpha+1}-1} = \xi^{-1}(\alpha+1)\prod_p\frac{p^{\alpha-\operatorname{ord}_p\alpha+1}-1}{p^{\alpha+1}-1}\,.$$

□

Lemma 1.2. *For $\alpha \geqq 2$, the density $d_2(\alpha)$ of numbers n such that $\alpha\,|\,n$, $H(n)<\alpha$ is*

$$\xi^{-1}(\alpha)\prod_{p|\alpha}\frac{p^{\alpha-\operatorname{ord}_p\alpha}-1}{p^{\alpha}-1}\,.$$

Proof. Arguing as in the proof of Lemma 1.1, we obtain

$$d_2(\alpha) = \frac{1}{\alpha}\sum_{m=1}^{\infty}\frac{\mu(m)(\alpha,m^{\alpha})}{m^{\alpha}} = \frac{1}{\alpha}\prod_p\left(1-\frac{(\alpha,p^{\alpha})}{p^{\alpha}}\right)$$

which gives the lemma. □

Proof of Theorem 1.2. By Lemmas 1.1 and 1.2, the density of numbers n such that $\alpha \mid n$ and $H(n) = \alpha$ is $d_1(1) = \xi^{-1}(2)$ for $\alpha = 1$,

$$d_1(\alpha) - d_2(\alpha) = \xi(\alpha+1)^{-1} \prod_{p \mid \alpha} \frac{p^{\alpha - \operatorname{ord}_p \alpha + 1} - 1}{p^{\alpha+1} - 1} - \xi(\alpha)^{-1} \prod_{p \mid \alpha} \frac{p^{\alpha - \operatorname{ord}_p \alpha} - 1}{p^{\alpha} - 1}$$

for $\alpha \geqq 2$. Hence all terms of the series occurring in the theorem are nonnegative and its partial sums are bounded by 1. It follows that the series is convergent.

Take $\varepsilon > 0$ and an integer $a > 2\varepsilon^{-1}$ such that

$$\sum_{\alpha > a} (d_1(\alpha) - d_2(\alpha)) < \frac{\varepsilon}{2}.$$

Then the number $M_H(x)$ is

$$\left(d_1(1) + \sum_{\alpha=2}^{a} (d_1(\alpha) - d_2(\alpha)) \right) \cdot x + o(x) + O\left(\sum_{n \leqq x,\ H(n) > a} 1 \right).$$

However

$$\sum_{n \leqq x,\ H(n) > a} 1 \leqq \sum_{p \text{ prime}} \left[\frac{x}{p^{a+1}} \right] \leqq x\left(\xi(a+1)^{-1} - 1\right) < \frac{x}{a} < \frac{\varepsilon}{2}\, x.$$

It follows that

$$\left| M_H(x) - x\left(\xi(2)^{-1} + \sum_{\alpha=2}^{\infty} \left(\xi^{-1}(\alpha+1) \prod_{p \mid \alpha} \frac{p^{\alpha - \operatorname{ord}_p \alpha + 1} - 1}{p^{\alpha} - 1} - \xi^{-1}(\alpha) \prod_{p \mid \alpha} \frac{p^{\alpha - \operatorname{ord}_p \alpha} - 1}{p^{\alpha} - 1} \right) \right) \right|$$

$$\leqq \varepsilon x + o(x),$$

which gives the desired formula for $d(M_H)$. □

2. Functions h and H, statistical convergence and normal order

At first we shall deal with sequences

$$\left(\frac{h(n)}{\log n} \right)_{n=2}^{\infty}, \quad \left(\frac{H(n)}{\log n} \right)_{n=2}^{\infty}. \tag{4}$$

THEOREM 2.1. *Each of the sequences* (4) *is dense in the interval* $\left(0, \frac{1}{\log 2}\right)$.

P r o o f. We prove the density of $\left(\frac{H(n)}{\log n}\right)_{n=2}^{\infty}$. Let $t \in \left(0, \frac{1}{\log 2}\right)$. Then $t = \frac{1}{\log 2 + u}$, where $0 < u < +\infty$.

Take $n = 2^{\alpha} \cdot q^{\beta}$, $\beta \leqq \alpha$, q being an odd prime. Then

$$\frac{H(n)}{\log n} = \left(\log 2 + \frac{\beta}{\alpha} \log q\right)^{-1}. \tag{5}$$

According to (5), it suffices to prove that for each $u \in (0, +\infty)$ there exist positive integers α_k, β_k with $\beta_k \leqq \alpha_k$ $(k = 1, 2, \dots)$ and a sequence $(q_k)_{k=1}^{\infty}$ of odd prime numbers such that

$$\lim_{k \to \infty} \frac{\beta_k}{\alpha_k} \log q_k = u. \tag{6}$$

This can be seen as follows. Choose an odd prime q such that

$$0 < \frac{u}{\log q} \leqq 1.$$

Owing to density of rational numbers in $(0, +\infty)$, there are positive integers α_k, β_k, $\beta_k \leqq \alpha_k$ $(k = 1, 2, \dots)$ such that

$$\frac{\beta_k}{\alpha_k} \to \frac{u}{\log q}.$$

Thus (6) is satisfied choosing $q_k = q$ $(k = 1, 2, \dots)$.

The density of $\left(\frac{h(n)}{\log n}\right)_{n=2}^{\infty}$ can be proved similarly. □

In [4], the concept of statistical convergence is introduced (see also [12]). A sequence $(x_n)_{n=1}^{\infty}$ of real numbers is said to converge statistically to $x \in \mathbb{R}$ (shortly: $\lim\operatorname{stat} x_n = x$) provided that for each $\varepsilon > 0$ we have $d(A_{\varepsilon}) = 0$, where $A_{\varepsilon} = \{n : |x_n - x| \geqq \varepsilon\}$.

In connection with Theorem 2.1, the question arises whether the sequences (4) converge statistically. The answer is positive.

THEOREM 2.2. *We have*

$$\lim\operatorname{stat} \frac{h(n)}{\log n} = \lim\operatorname{stat} \frac{H(n)}{\log n} = 0.$$

We shall not give any proof of this theorem since it is a simple consequences of stronger Theorem 2.3. The latter theorem implies that

$$\lim\operatorname{stat}\frac{H(n)}{g(n)}=0$$

for every positive function $g\colon \mathbb{N}\to\mathbb{R}$ with $\lim_{n\to\infty} g(n)=+\infty$.

THEOREM 2.3. *For any function* $g(n)\to\infty$ *we have*

$$\lim\operatorname{stat}\frac{\Omega(n)-\omega(n)}{g(n)}=0\,.$$

COROLLARY 2.1. *For every* $g(n)\to\infty$

$$\lim\operatorname{stat}\frac{H(n)}{g(n)}=0$$

$$(\textit{and so}\quad \lim\operatorname{stat}\frac{h(n)}{g(n)}=0)\,.$$

(Hint: Observe that $H(n)\leqq\Omega(n)-\omega(n)+1$.)

Proof of Theorem 2.3. By a theorem of Rényi [11] (see also Delange [3]) for every integer $q\geqq 0$ the set of numbers n with $\Omega(n)-\omega(n)=q$ has a density d_q and

$$\sum_{q=0}^{\infty} d_q=1\,. \tag{7}$$

Put

$$A_\varepsilon=\left\{n:\ \frac{\Omega(n)-\omega(n)}{g(n)}\geqq\varepsilon\right\}$$

and take an arbitrary $\eta>0$. By (7) there exists an r such that $\sum_{q=r}^{\infty} d_q<\eta$. Therefore the number of integers $n\leqq x$ such that $\Omega(n)-\omega(n)>r$ is less than $\eta x+o(x)$. Now take n_0 such that for $n>n_0$ we have $g(n)\varepsilon\geqq r$. It follows that $A_\varepsilon(x)<n_0+\eta x+o(x)$. Hence $\overline{d}(A_\varepsilon)\leqq\eta$, and since η is arbitrary, $d(A_\varepsilon)=0$. □

Now, recall the concept of normal order of an arithmetical function. A function F defined on $\mathbb{N}$ is said to be a normal order of an arithmetical function f

provided that for each $\varepsilon > 0$ there is a set B_ε of positive integers with $d(B_\varepsilon) = 1$ such that for each $n \in B_\varepsilon$ we have

$$(1-\varepsilon)F(n) < f(n) < (1+\varepsilon)F(n) \tag{8}$$

(i.e. inequalities (8) hold for almost all $n \in \mathbb{N}$) – cf. [5; p. 356].

Remark that if $0 < \varepsilon < 1$ in (8), then for $n \in B_\varepsilon$ we obtain $F(n) > 0$, $f(n) > 0$.

The definitions of statistical convergence and of normal order suggest that there is a strong connection between these two concepts. The following simple theorem confirms that.

THEOREM 2.4. *Let f, F, be two functions defined on $\mathbb{N}$. Then F is a normal order of f if and only if*

$$\lim\operatorname{stat} \frac{f(n)}{F(n)} = 1. \tag{9}$$

P r o o f. Suppose that F is a normal order of f. If $0 < \varepsilon < 1$, then there is a set $B_\varepsilon \subseteq \mathbb{N}$ such that $d(B_\varepsilon) = 1$, $F(n) > 0$ for $n \in B_\varepsilon$, and

$$(1-\varepsilon)F(n) < f(n) < (1+\varepsilon)F(n). \tag{10}$$

From this we obtain

$$\left| \frac{f(n)}{F(n)} - 1 \right| < \varepsilon$$

for $n \in B_\varepsilon$. Therefore the inequality

$$\left| \frac{f(n)}{F(n)} - 1 \right| \geqq \varepsilon$$

holds at most for all $n \in \mathbb{N} \setminus B_\varepsilon = A_\varepsilon$, where $d(A_\varepsilon) = 0$. Thus (9) holds.

Conversely, if (9) holds, then it can be easily checked that (10) holds for almost all $n \in \mathbb{N}$. □

It is well known that each of the functions ω, Ω has a normal order F, where $F(1) = F(2) = 1$, $F(n) = \log\log n$ $(n > 2)$ – cf. [5; p. 356–358]. Hence we get

COROLLARY 2.2. *We have*

$$\lim\operatorname{stat} \frac{\omega(n)}{\log\log n} = \lim\operatorname{stat} \frac{\Omega(n)}{\log\log n} = 1. \tag{11}$$

In connection with (11), the question about normal order of functions h, H arises. Evidently, the constant function $F(n) = 1$ for all $n \in \mathbb{N}$ is a normal order of h (see the proof of Theorem 1.1). We shall show that H cannot have any non-decreasing normal order.

THEOREM 2.5. *If F is any non-decreasing function on $\mathbb{N}$, then F is not a normal order of H.*

Proof. Owing to the monotonicity of F, there exists $\lim\limits_{n\to\infty} F(n)$. If $\lim\limits_{n\to\infty} F(n) = +\infty$, then Corollary 2.1 and Theorem 2.4 show that F is not a normal order of H. Suppose that $\lim\limits_{n\to\infty} F(n) = d < +\infty$. Suppose that F is a normal order of H. Then for each $\varepsilon > 0$ the inequalities

$$(1-\varepsilon)\,d < H(n) < (1+\varepsilon)\,d \tag{12}$$

are satisfied for almost all $n \in \mathbb{N}$.

Let $d > 1$. Then for $\varepsilon = 1 - \frac{1}{d}$ we get from (12)

$$\Omega(n) - \omega(n) \geqq H(n) - 1 > 0\,.$$

Therefore the density of all numbers n satisfying (12) in this case is not greater than $\sum\limits_{q=1}^{\infty} d_q < 1$.

Let $d = 1$. If $n > 1$ satisfies (12) and $\varepsilon < 1$, then $H(n) < 1 + \varepsilon < 2$. Hence $H(n) = 1$ and n is a square-free number. Therefore the density of all numbers n satisfying (12) is $\frac{6}{\pi^2} < 1$ in this case (cf. [10; p. 21]).

Let $d < 1$. If ε is a small number such that $(1+\varepsilon)d < 1$, then no positive integer satisfies (12). □

In the end we remark that the method of the paper [2] for determining densities of sets M_f ($f\colon \mathbb{N} \to \mathbb{N}$) concerns the functions $f\colon \mathbb{N} \to \mathbb{N}$ having non-decreasing normal orders. Theorem 2.5 shows that the density of the set M_H cannot be obtained using this method.

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Received May 26, 1994

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