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ON FULL COVER PROPERTY OF ORDERED FIELDS

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ABSTRACT. Using the concept of full covering introduced by B. S. Thompson a new property of Archimedean ordered fields is introduced in this paper. This property is equaivalent to the completeness of such fields.

1. Introduction

In [2], [3] and [7], there is introduced and developed the concept of full cover and it is used to unify the proofs of fundamental theorems of elementary analysis. In this note we shall show that this concept enables us to introduce a new property of ordered fields which is in the class of Archimedean ordered fields equivalent to the completeness. Finally we give some further applications of this property that are different from those given in [2], [3] and [7].

Let us remark that fundamental notions and notations from the theory of ordered fields will be used in agreement with the monograph [4].

If $\langle A, +, \cdot, < \rangle$ (briefly A) is an ordered field, then by 0_A we denote the neutral element of A with respect to $+$. Let $a, b \in A$, $a < b$. Then $[a, b]$ and (a, b) denote the closed and open interval with endpoints a and b, respectively, the length of which is the positive element $b - a \in A$. The symbol $|I|$ will stand for the length of the interval I.

2. Full cover property of ordered fields

The following definition is a natural extension of the concept of full cover from [7].

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DEFINITION 1. Let A be an ordered field, let $a, b \in A$, $a < b$. A class S of closed intervals $I \subset [a, b]$ is said to be a full cover of $[a, b]$ if for each $x \in [a, b]$ there is an element $\delta(x) > 0_A$ of A such that S contains each closed interval $I \subset [a, b]$ containing x of length less than $\delta(x)$.

DEFINITION 2. An ordered field A is said to have the full cover property (briefly FC-property) provided that for each two elements $a, b \in A$, $a < b$, the following statement holds: If S is a full cover of $[a, b]$, then there exists a partition $a = x_0 < x_1 < \cdots < x_m = b$ $(x_j \in A,\ j = 0, 1, \dots, m)$ of $[a, b]$ such that $[x_{k-1}, x_k] \in S$ $(k = 1, 2, \dots, m)$.

It is well known that in an Archimedean ordered field concepts of fundamental (Cauchy) sequence and convergent sequence of elements of A can be introduced (cf. [4; p. 73–74]). The completeness of A can be defined in a usual manner (see [4; p. 85]). There are several properties of Archimedean ordered fields that are equivalent to the completeness (cf. [4; p. 95–101], [5]). Such a property is e.g. the existence of the least upper bound for every non-empty from above bounded subset of A. The following theorem shows that among these properties also the FC-property can be arranged.

THEOREM 1. *An Archimedean ordered field A is complete if and only if it has the FC-property.*

Proof. Suppose that A has the FC-property. Let M be a non-empty from above bounded subset of A. Suppose that M has not the least upper bound in A and choose elements $a, b \in A$ such that a is not an upper bound of M while b is an upper bound of M. Denote by S the class of all closed intervals $I \subset [a, b]$ satisfying one of the following conditions:

(C_1) Each $x \in I$ is an upper bound of M.

(C_2) No $x \in I$ is an upper bund of M.

Let $x \in [a, b]$. If x is not an upper bound of M, then $x < b$ and there exists a $t \in M$ such that $x < t$. We put $\delta(x) = t - x > 0_A$. It is easy to verify that if $I \subset [a, b]$ is a closed interval, $x \in I$ and $|I| < \delta(x)$, then I satisfies (C_2) and so $I \in S$. If x is an upper bound of M, then $a < x$, and according to our assumption x is not the least upper bound of M. Therefore there exists an $y < x$ such that y is an upper bound of M. Put $\delta(x) = x - y > 0_A$. It is easy to verify that if $I \subset [a, b]$ is a closed interval, $x \in I$ and $|I| < \delta(x)$, then I satisfies (C_1), and so $I \in S$.

By the FC-property of A there is a partition $a = x_0 < x_1 < \cdots < x_m = b$ of $[a, b]$ with $[x_{k-1}, x_k] \in S$ $(k = 1, 2, \dots, m)$. Consequently by the definition of S we get that $x_m = b$ is not an upper bound of M, which contradicts the definition of the element b.

Conversely, suppose that A is a complete Archimedean ordered field. Then A has the property described in [4; p. 96, Statement VII] (i.e. if $I_1 \supset I_2 \supset \dots$ are closed intervals in A, then $\bigcap_{k=1}^{\infty} I_k \neq \emptyset$). The FC-property of A can be now proved by the reasoning used in the proof of [7; Lemma 1.1] or of [2; Lemma]. □

3. Further applications of the FC-property

We now introduce two applications of a full covering different from that contained in [2], [3], [7].

In the proofs we shall restrict ourselves only to a construction of suitable full covers, since then the rest of proofs is already easy.

According to Theorem 1, the FC-property of an Archimedean ordered field is equivalent to the completeness. But in fact there exists only one Archimedean ordered complete field (excepting isomorphic fields – see [4; p. 103]). Therefore the following well-known result (cf. [1; p. 270]) will be formulated already for the field $\mathbb{R}$ of real numbers.

PROPOSITION 1. *Let $a, b \in \mathbb{R}$, $a < b$ and $f_n\colon [a,b] \to \mathbb{R}$ $(n = 1, 2, \dots)$ be continuous functions on $[a,b]$. Suppose that for each $x \in [a,b]$ we have*

$$f_1(x) \geq f_2(x) \geq \dots, \qquad \lim_{n\to\infty} f_n(x) = 0. \tag{1}$$

Then the sequence $(f_n)_{n=1}^{\infty}$ converges uniformly on $[a,b]$ to the function identically equal to zero on $[a,b]$.

Proof. Evidently $f_n(x) \geq 0$ for each n and $x \in [a,b]$. Let $\varepsilon > 0$. Denote by $S(\varepsilon)$ the class of all closed intervals $I \subset [a,b]$ for which there exists a $p = p(I) \in \mathbb{N}$ such that for every $n \geq p$ and every $x \in I$ we have $f_n(x) < \varepsilon$. Then $S(\varepsilon)$ is a full cover of $[a,b]$. □

A sequence $(f_n)_{n=1}^{\infty}$ of functions $f_n\colon [a,b] \to \mathbb{R}$ $(n = 1, 2, \dots)$ is said to converge quasi-uniformly to a function $f\colon [a,b] \to \mathbb{R}$ if $f_n \to f$ (pointwise) and for each $\varepsilon > 0$ and a non-negative integer m there exists a $p \in \mathbb{N}$ such that for each $x \in [a,b]$ we have

$$\min\{|f_{m+1}(x) - f(x)|, \dots, |f_{m+p}(x) - f(x)|\} < \varepsilon. \tag{2}$$

It is well known (cf. [6; p. 143, Exercise 3]) that if the functions f_n $(n = 1, 2, \dots)$ are continuous on $[a,b]$ and $(f_n)_{n=1}^{\infty}$ converges to f pointwise on $[a,b]$, then f is continuous on $[a,b]$ if and only if (2) holds. Using the FC-property we prove the necessity part of this statement.

PROPOSITION 2. *Let* $f_n\colon [a,b] \to \mathbb{R}$ $(n = 1,2,\dots)$ *be continuous functions on* $[a,b]$ *and* $f_n \to f$ *on* $[a,b]$. *If* f *is continuous on* $[a,b]$, *then* (2) *holds.*

Proof. Let $\varepsilon > 0$, and m be a non-negative integer. Denote by $S(\varepsilon, m)$ the class of all closed intervals $I \subset [a,b]$ for which there exists a $q = q(I) \in \mathbb{N}$ such that for each $y \in I$ we have

$$\min\{|f_{m+1}(y) - f(y)|, \dots, |f_{m+q}(y) - f(y)|\} < \varepsilon .$$

Then $S(\varepsilon, m)$ is a full cover of $[a,b]$. □

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