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ON STRONG SUPERLATTICES

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(Communicated by Tibor Katriňák)

ABSTRACT. In this paper we deal with a question proposed by Mittas and Konstantinidou concerning strong superlattices.

The notion of superlattice was introduced by Mittas and Konstantinidou [11]. A superlattice is defined to be a partially ordered set with two binary multioperations $\vee$ and $\wedge$ satisfying certain axioms; the resulting structure is a generalization of the notion of lattice.

An alternative (equivalent) definition of superlattice given in [11] uses only properties of multioperations $\vee$ and $\wedge$ without assuming that the underlying set is partially ordered.

Other generalizations of lattices constructed by means of multioperations are multilattices (Benado [1]) and hyperlattices (Konstantinidou and Mittas [7]). Hyperlattices were studied also in [8] and [9]; for multilattices, see e.g. [2] and [3].

Hyperlattices can be considered as to be "near" to superlattices. The notions of superlattice and multilattice essentially differ with regard to the associativity condition. Namely, one of the axioms for superlattices requires both the operations $\vee$ and $\wedge$ to be associative.

On the other hand, Benado [1] constructed an example of a multilattice M (containing 15 elements) such that neither $\vee$ nor $\wedge$ was associative in M. In this paper, Benado proposed the question whether there exist associative multilattices which fail to be lattices.

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The author [6] proved that (i) the answer is positive, and (ii) if M is a multilattice such that M is not a lattice and $a \vee b \neq \emptyset \neq a \wedge b$ for each $a, b \in M$, then neither $\vee$ nor $\wedge$ is associative in M.

In the present paper, a question proposed by M i t t a s and K o n s t a n t i - n i d o u [11] on strong superlattices is dealt with (a superlattice is said to be strong if the corresponding partially ordered set is a lattice).

Next we investigate congruence relations on a superlattice $\mathcal{S}$. It will be shown that if ρ is a congruence relation on $\mathcal{S}$, then the factor structure $\mathcal{S}/\rho$ need not be a superlattice. Consequently, the class of all superlattices fails to be a variety.

Analogous results are valid for quasigroups and for existence algebras (the notion of existence algebra was introduced in [5]). This can be proved by examples, but in the case of quasigroups, it also follows from general results of M a l t s e v [10] and T r e v i s a n [12] concerning permutable congruence relations (recall that T r e v i s a n 's result in [12] solved B i r k h o f f 's Problem 31 from [4]); cf. also [5].

1. Preliminaries

We recall some notations and definitions from [11].

For a set E we denote by $P(E)$ the system of all subsets of E. Let $\vee$ be a binary multioperation on E; i.e. for each $a, b \in E$, $a \vee b$ is an element of $P(E)$. If no ambiguity can occur, the element $a \in E$ will be identified with the corresponding singleton $\{a\}$. For $A, B \in P(E)$ we put

$$A \vee B = \bigcup_{a \in A,\, b \in B} (a \vee b).$$

If $\wedge$ is another binary multioperation on E, then $A \wedge B$ is defined analogously.

1.1. DEFINITION. *A superlattice is a partially ordered set S (the partial order being denoted by $\leqq$) which is endowed with two binary multioperations $\vee$ and $\wedge$ such that the following conditions are satisfied for each $a, b, c \in S$:*

- (S_1) $a \in (a \vee a) \cap (a \wedge a)$;
- (S_2) $a \vee b = b \vee a$, $a \wedge b = b \wedge a$;
- (S_3) $(a \vee b) \vee c = a \vee (b \vee c)$, $(a \wedge b) \wedge c = a \wedge (b \wedge c)$;
- (S_4) $a \in \big[(a \vee b) \wedge a\big] \cap \big[(a \wedge b) \vee a\big]$;
- (S_5) *If $a \leqq b$, then $b \in a \vee b$ and $a \in a \wedge b$.*
- (S_6) *If $b \in a \vee b$ or $a \in a \wedge b$, then $a \leqq b$.*

If for each $a \in S$ the element a is identified with $\{a\}$, then each lattice turns out to be a superlattice.

Now assume that S is a nonempty set with two binary multioperations $\vee$ and $\wedge$. Consider the following conditions $S_1' - S_4'$, S_6', S_7', S_8' for these multioperations (where a, b and c run through S):

$$S_1' \equiv S_1, \qquad S_2' \equiv S_2, \qquad S_3' \equiv S_3, \qquad S_4' \equiv S_4;$$

(S_6') $b \in a \vee b \Longleftrightarrow a \in a \wedge b$;
(S_7') $a, b \in a \vee b \Longrightarrow a = b$;
(S_8') $b \in a \vee b$ and $c \in b \vee c \Longrightarrow c \in a \vee c$.

Then we have (cf. [11; p. 64]):

1.2. Proposition. *Let $(S; \leqq, \vee, \wedge)$ be a superlattice. Then the conditions S_1' – S_4', S_6', S_7' and S_8' are satisfied.*

1.3. Proposition. *Let S be a nonempty set, and let $\vee$, $\wedge$ be binary multioperations on S satisfying the conditions $S_1' - S_4'$, S_6', S_7' and S_8'. For $a, b \in S$ we put $a \leqq b$ if $b \in a \vee b$. Then $(S; \leqq)$ is a partially ordered set, and $(S; \leqq, \vee, \wedge)$ is a superlattice.*

In view of 1.2 and 1.3, we can consider a superlattice $\mathcal{S}$ to be a nonempty set S with two binary multioperations $\vee$ and $\wedge$ satisfying the conditions $S_1' - S_4'$, S_6', S_7' and S_8'.

2. Strong superlattices

In this section we apply Definition 1.1 of the notion of superlattice.

2.1. Definition. *Let $\mathcal{S} = (S; \leqq, \vee, \wedge)$ be a superlattice. $\mathcal{S}$ is said to be strong if for each $a, b \in S$ there exist $\sup\{a, b\}$ and $\inf\{a, b\}$ in S, i.e., if $(S; \leqq)$ is a lattice. If, moreover,*

$$\sup\{a, b\} \in a \vee b \quad \textit{and} \quad \inf\{a, b\} \in a \wedge b$$

for each $a, b \in S$, then $\mathcal{S}$ is called strictly strong.

In [11; p. 70] it was remarked that examples of strong superlattices which fail to be strictly strong are not known. In the present section we shall construct a proper class of nonisomorphic types of such superlattices.

Let P be a lattice as in Fig. 1, and let Q be a chain which has no least element, $P \cap Q = \emptyset$. Put $S = Q \oplus P$, where $\oplus$ denotes the ordinal sum (i.e. $S = P \cup Q$; for $p_1, p_2 \in P$, $q_1, q_2 \in Q$ the relations $p_1 \leqq p_2$ and $q_1 \leqq q_2$ in S have the original meaning inherited from P and Q, respectively; next, $q < p$

for each $p \in P$, $q \in Q$). Then S is a lattice. We define binary multioperations $\vee$ and $\wedge$ on S as follows.

1) $a \wedge b = \inf\{a, b\}$ for each $a, b \in S$;
2) $a \vee a = S$ for each $a \in S$;
3) $a \vee b = b \vee a = S - \{a\}$ if $a, b \in S$ and $a < b$;
4) $x \vee y = y \vee x = S - \{x, y, v\}$.

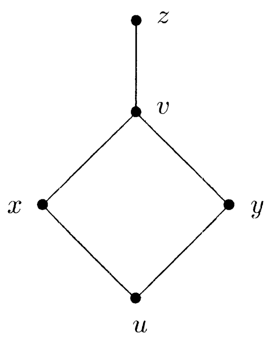

Figure 1.

2.2. LEMMA. $(S; \leqq, \vee, \wedge)$ *is a superlattice.*

Proof. The verification of the conditions S_1, S_2, S_4, S_5 and S_6 is easy. Also, the relation $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ is obviously valid. Thus we have only to verify that the relation

$$(a \vee b) \vee c = a \vee (b \vee c) \tag{$*$}$$

holds.

In view of the definition of the multioperation $\vee$, $a \vee b$ equals to some of the following sets:

$$S, \qquad S - \{a\}, \qquad S - \{b\}, \qquad S - \{a, b, v\}.$$

Let F be a finite subset of S, and let $t \in S$. Then there exist $b_1, b_2 \in Q$ such that $b_1, b_2 \notin F$, $b_i < t$ $(i = 1, 2)$, and $b_1 \neq b_2$. Hence

$$(S - F) \vee t \supseteq b_1 \vee t = S - \{b_1\},$$

and similarly, $(S - F) \vee t \supseteq S - \{b_2\}$. Thus $(S - F) \vee t = S$. Therefore $(a \vee b) \vee c = S$ for each $a, b, c \in S$. Analogously, $a \vee (b \vee c) = S$. Hence $(*)$ is valid. □

2.3. LEMMA. *The superlattice $\mathcal{S} = (S; \leqq, \vee, \wedge)$ is strong, but it fails to be strictly strong.*

Proof. We already observed above that $(S; \leqq)$ is a lattice, hence $\mathcal{S}$ is strong. Since $v = \sup\{x, y\}$ and $v \notin x \vee y$, the superlattice $\mathcal{S}$ fails to be strictly strong. □

Since any linearly ordered set which is dual to some infinite ordinal can be taken in the place of Q, we obtain

2.4. THEOREM. *There exists a proper class C of superlattices such that:*

(i) *if $\mathcal{S} \in C$, then $\mathcal{S}$ is strong and fails to be strictly strong;*
(ii) *if $\mathcal{S}_1$ and $\mathcal{S}_2$ are distinct elements of C, then they are not isomorphic.*

3. Congruences on superlattices

In this section we consider a superlattice $\mathcal{S}$ to be a nonempty set S with two binary multioperations $\vee$ and $\wedge$ satisfying the conditions $S_1' - S_4'$, S_6', S_7' and S_8'.

Let ρ be an equivalence on the set S. For $x \in S$ we denote

$$\overline{x}^{\rho} = \{y \in S : \ x\rho y\}.$$

If $A \subseteq S$, then we put

$$\overline{A}^{\rho} = \{\overline{a}^{\rho} : \ a \in A\}.$$

3.1. DEFINITION. *Let $\mathcal{S} = (S; \leqq, \vee, \wedge)$ be a superlattice. An equivalence ρ on S will be called a congruence on $\mathcal{S}$ if, whenever $x_i, y_i \in S$ $(i = 1, 2)$ and $\overline{x}_1^{\rho} = \overline{x}_2^{\rho}$, $\overline{y}_1^{\rho} = \overline{y}_2^{\rho}$, then*

$$\overline{x_1 \vee y_1}^{\rho} = \overline{x_2 \vee y_2}^{\rho} \quad \text{and} \quad \overline{x_1 \wedge y_1}^{\rho} = \overline{x_2 \wedge y_2}^{\rho}.$$

In such a case, we define a binary multioperations $\vee$ and $\wedge$ on $\overline{S}^{\rho}$ by putting

$$\overline{x}^{\rho} \vee \overline{y}^{\rho} = \overline{x \vee y}^{\rho}, \qquad \overline{x}^{\rho} \wedge \overline{y}^{\rho} = \overline{x \wedge y}^{\rho}$$

for each $\overline{x}^{\rho}, \overline{y}^{\rho} \in \overline{S}^{\rho}$. We denote $\left(\overline{S}^{\rho}; \vee, \wedge\right) = \mathcal{S}/\rho$.

It is clear that the multioperations $\vee$ and $\wedge$ on $\overline{S}^{\rho}$ are correctly defined.

If no ambiguity can occur, then we write $\overline{x}$ and $\overline{A}$ instead of $\overline{x}^{\rho}$ and $\overline{A}^{\rho}$.

The natural question arises whether $\mathcal{S}/\rho$ is a superlattice (i.e., whether it satisfies the conditions $S_1' - S_4'$ and $S_6' - S_8'$) for each congruence ρ on $\mathcal{S}$. The following example shows that the answer is "No".

3.2. Example. Let $\mathbb{R}$ be the set of all reals with the natural linear order. Further, let S be the set of all pairs (x,y) with $x,y \in \mathbb{R}$. For (x_1,y_1) and (x_2,y_2) in S we put $(x_1,y_1) \leqq (x_2,y_2)$ if either $(x_1,y_1) = (x_2,y_2)$ or $y_1 < y_2$. Thus $(S;\leqq)$ is a partially ordered set. We define binary multioperations $\vee$ and $\wedge$ on S as follows.

Let $a,b \in S$. We denote by $a \wedge b$ the set of all lower bounds of the set $\{a,b\}$. Next we put

$$a \vee b = b \vee a = \begin{cases} S & \text{if } a = b\,, \\ S - \{a\} & \text{if } a < b\,, \\ S - \{a,b\} & \text{if } a \text{ and } b \text{ are incomparable.} \end{cases}$$

Then $(S;\vee,\wedge) = \mathcal{S}$ satisfies the conditions S'_1 – S'_4 and S'_6 – S'_8 (for the verification of S'_3, we can apply the fact that $(a \vee b) \vee c = S$ for each $a,b,c \in S$). Thus $\mathcal{S}$ is a superlattice.

For (x,y) and (x',y') in S we put $(x,y)\,\rho\,(x',y')$ if $x = x'$. Let $(x_i,y_i) \in S$, $i = 1,2,3$. There are $y,y' \in \mathbb{R}$ such that $y' < y_i < y''$ for $i = 1,2,3$. Then

$$(x_3,y_3) \in \big[(x_1,y') \vee (x_2,y')\big] \cap \big[(x_1,y'') \wedge (x_2,y'')\big]\,,$$

whence

$$\overline{(x_3,y_3)} \in \big[\,\overline{(x_1,y_1)} \vee \overline{(x_2,y_2)}\,\big] \cap \big[\,\overline{(x_1,y_1)} \wedge \overline{(x_2,y_2)}\,\big]\,.$$

Therefore $\overline{(x_1,y_1)} \vee \overline{(x_2,y_2)} = \overline{(x_1,y_1)} \wedge \overline{(x_2,y_2)} = \overline{S}$. Hence ρ is a congruence on $\mathcal{S}$ and we can construct the structure $\mathcal{S}/\rho$. The condition S'_7 fails to be valid for $\mathcal{S}/\rho$.

By defining the notion of variety for systems with multioperations we apply the analogy to systems with operations. Namely, a class $\mathcal{C}$ of systems with multioperations of the same type will be called a variety if $\mathcal{C}$ is closed with respect to homomorphic images, subalgebras and direct products.

Therefore in view of 3.2 we have

3.3. Proposition. *The class of all superlattices fails to be a variety.*

3.4. Proposition. *Let ρ be a congruence on a superlattice $\mathcal{S}$. Then the factor structure $\mathcal{S}/\rho$ satisfies the conditions S'_1 – S'_4.*

Proof. Since ρ is fixed, for each $x \in S$ we write $\overline{x}$ instead of $\overline{x}^{\rho}$.

a) Let $a \in S$. Then $a \in a \vee a$, whence $\overline{a} \in \overline{a \vee a} = \overline{a} \vee \overline{a}$.

b) Let $a,b \in S$. Then $\overline{a} \vee \overline{b} = \overline{a \vee b} = \overline{b \vee a} = \overline{b} \vee \overline{a}$.

c) Let $a, b, c \in S$. Put $X = a \vee b$, $Y = b \vee c$. Hence

$$(a \vee b) \vee c = X \vee c = \bigcup_{x \in X} (x \vee c),$$

$$a \vee (b \vee c) = a \vee Y = \bigcup_{y \in Y} (a \vee y).$$

Analogously, we have

$$\left(\overline{a} \vee \overline{b}\right) \vee \overline{c} = \bigcup (\overline{x} \vee \overline{c}),$$

$$\overline{a} \vee \left(\overline{b} \vee \overline{c}\right) = \bigcup (\overline{a} \vee \overline{y}),$$

where $\overline{x}$ runs over $\overline{a} \vee \overline{b}$ and $\overline{y}$ runs over $\overline{b} \vee \overline{c}$. Therefore

$$\left(\overline{a} \vee \overline{b}\right) \vee \overline{c} = \bigcup_{\overline{x} \in \overline{a \vee b}} \overline{x \vee c}$$

for $x \in \overline{x}$. There exists $x' \in S$ with $\overline{x'} = \overline{x}$ and $x' \in a \vee b$. Let $z \in x' \vee c$. Since $(a \vee b) \vee c = a \vee (b \vee c)$, there exists $y \in Y$ with $z \in a \vee y$. Hence

$$\bigcup_{\overline{x} \in \overline{a \vee b}} \left(\overline{x \vee c}\right) = \bigcup_{x' \in a \vee b} \left(\overline{x' \vee c}\right) \subseteq \bigcup_{y \in b \vee c} \left(\overline{a \vee y}\right)$$

$$= \bigcup_{\overline{y} \in \overline{b} \vee \overline{c}} (\overline{a} \vee \overline{y}) = \overline{a} \vee \left(\overline{b} \vee \overline{c}\right).$$

Thus $\left(\overline{a} \vee \overline{b}\right) \vee \overline{c} \subseteq \overline{a} \vee \left(\overline{b} \vee \overline{c}\right)$. Analogously, we can verify that $\overline{a} \vee \left(\overline{b} \vee \overline{c}\right) \subseteq \left(\overline{a} \vee \overline{b}\right) \vee \overline{c}$.

d) For each $a, b \in S$ we have $a \in (a \vee b) \wedge a = \bigcup (x \wedge a)$, where x runs over $a \vee b$.

Next,

$$\left(\overline{a} \vee \overline{b}\right) \wedge \overline{a} = \bigcup (\overline{x} \wedge \overline{a}),$$

where $\overline{x}$ runs over $\overline{a} \vee \overline{b} = \overline{a \vee b}$. For each $\overline{x} \in \overline{a \vee b}$ there is $x' \in S$ with $\overline{x'} = \overline{x}$ and $x' \in a \vee b$. This yields that

$$\left(\overline{a} \vee \overline{b}\right) \wedge \overline{e} = \bigcup_{x' \in a \vee b} \left(\overline{x' \wedge a}\right).$$

There exists $x_0 \in a \vee b$ such that $a \in x_0 \wedge a$, whence $\overline{a} \in \overline{x_0 \wedge a}$. We conclude that $\overline{a} \in \left(\overline{a} \vee \overline{b}\right) \wedge \overline{a}$.

The corresponding dual conditions can be verified analogously. □

OPEN QUESTION. Let ρ be a congruence relation on a superlattice $\mathcal{S}$. Does $\mathcal{S}/\rho$ satisfy the conditions S_6' and S_8'?

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