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CONSTRUCTIVE APPROXIMATION OF A BALL BY POLYTOPES

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ABSTRACT. In this paper, we give an explicit construction of m unit vectors in the n-dimensional Euclidean space such that the convex hull of them contains a ball of radius const $\sqrt{n^{-1}\log(m/n)}$, where $2n \le m \le c^n$. This construction is asymptotic optimal. Finally we discuss some algorithmical consequences of our result.

1. Introduction

Approximation of a ball by polytopes is a well-studied subject in convexity theory. Let

$$V(n,m) = \frac{\max\{\operatorname{vol}(\operatorname{conv}\{x_1,\dots,x_m\}); x_1,\dots,x_m \in S_n(1)\}}{\operatorname{vol}(S_n(1))},$$

where $S_n(\delta)$ denotes the n-dimensional ball of radius δ with centre at the origin, and $\operatorname{conv}(K)$ denotes the convex hull of K. The behavior of $V(n,m)^{1/n}$ has been investigated in [2] (see also [1], [3], [4]). It was proved that, if m is a function of n (linear, polynomial, exponential) and $n \to \infty$, then

$$c_1\sqrt{\frac{\log(m/n)}{n}} \le V(n,m)^{1/n} \le c_2\sqrt{\frac{\log(m/n)}{n}}, \tag{1}$$

where c_1, c_2 are constants. For further details and information on approximation, see [2].

A similar question is to determine $\rho(n,m)$, the maximal radius of a ball (with centre at the origin) which is contained in the convex hull of m points chosen

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from $S_n(1)$. Clearly, $\rho(n,m) \le V(n,m)^{1/n}$, and one would expect asymptotic equality here. For this it is enough to show that

$$c_3\sqrt{\frac{\log(m/n)}{n}} \le \rho(n,m) . \tag{2}$$

In the second paragraph, we shall give an explicit construction that proves (2) for any $n \ge 1$ and $2n \le m \le c^n$ ($c > 1$ is a constant). This result is asymptotic optimal because the upper bound of (1) holds in fact for any $m > n$ and n sufficiently large (it remains to set $k = \log(n/d)$ in [2; Theorem 3]). In the third paragraph, we shall apply this result and sketch how it can be used for an improvement of some algorithms from computational geometry.

2. The construction

Let $\|\boldsymbol{u}\|$ denote the Euclidean norm for $\boldsymbol{u} \in \mathbb{R}^r$ and $S^{r-1} = \{\boldsymbol{x} \in \mathbb{R}^r : \|\boldsymbol{x}\| = 1\}$. By a 1-*net* N_r in S^{r-1}, we mean a subset of S^{r-1} such that for any $\boldsymbol{x} \in S^{r-1}$ there exists a $\boldsymbol{v} \in N_r$ satisfying $\|\boldsymbol{x} - \boldsymbol{v}\| \le 1$. We shall frequently use the following well-known fact:

$$\text{If} \quad \boldsymbol{x} = (x_1, \dots, x_r) \in S^{r-1}, \quad \text{then} \quad \sum_{i=1}^{r} |x_i| \le \sqrt{r} . \tag{3}$$

LEMMA 1. *If N_r is a 1-net, then $S_r(1/2) \subseteq \operatorname{conv}(N_r)$.*

Proof. If $\boldsymbol{z} \in S_r(1/2)$ does not belong to $\operatorname{conv}(N_r)$, then separating $\boldsymbol{z}$ from $\operatorname{conv}(N_r)$ by a hyperplane $p_{\boldsymbol{z}}$ we get a cap of $S_r(1)$ which is disjoint from N_r and its "top" $\boldsymbol{t}$ ($\boldsymbol{t}$ is one of the unit vectors perpendicular to $p_{\boldsymbol{z}}$) satisfies $\|\boldsymbol{t} - \boldsymbol{v}\| > 1$ for any $\boldsymbol{v} \in N_r$ – a contradiction. Thus $S_r(1/2) \subseteq \operatorname{conv}(N_r)$. □

In the sequel, we shall need a 1-net in S^{r-1} of cardinality at most d^r for any integer r. The existence of such 1-nets can be proved in several ways: By random construction, or by greedy algorithm, or just choosing a subset X which is maximal with respect to the property that two distinct elements of X are at least 1 apart. Explicit constructions are, as usual, of greatest interest. This will be done in the following lemma.

LEMMA 2. *Let r be a positive integer and*

$$A_r := \mathbb{Z}^r \cap S_r(3\sqrt{r}), \qquad B_r := \{\boldsymbol{b} = \boldsymbol{a}/\|\boldsymbol{a}\| ;\ \boldsymbol{a} \in A_r,\ \|\boldsymbol{a}\| \ne 0\} .$$

Then B_r is a 1-net in S^{r-1} of cardinality at most d^r, where d is a constant independent on r.

Proof. Let $\boldsymbol{x} = (x_1, \dots, x_r) \in S^{r-1}$, then, by (3), $\sum_{i=1}^{r} |x_i| \le \sqrt{r}$. Denote $u_i := \lfloor 3\sqrt{r}x_i \rfloor$ for any $i \in \{1, \dots, r\}$, and $\boldsymbol{u} = (u_1, \dots, u_r) \in \mathbb{R}^r$. Let $\boldsymbol{v} = (v_1, \dots, v_r) := \boldsymbol{u}/\|\boldsymbol{u}\|$. Then

$$\begin{aligned} 3\sqrt{r}x_i - 1 < u_i &\le 3\sqrt{r}x_i\,, \\ 9rx_i^2 - 6\sqrt{r}x_i + 1 < u_i^2 &\le 9rx_i^2\,, \\ 9r\sum_{i=1}^{r} x_i^2 - 6\sqrt{r}\sum_{i=1}^{r}|x_i| + r < \sum_{i=1}^{r} u_i^2 &\le 9r\sum_{i=1}^{r} x_i^2\,, \\ 9r - 6r + r < \|\boldsymbol{u}\|^2 &\le 9r\,, \\ 2\sqrt{r} < \|\boldsymbol{u}\| &\le 3\sqrt{r}\,. \end{aligned}$$

Thus $\boldsymbol{u} \in A_r$ and $\boldsymbol{v} \in B_r$. Furthermore, if $x_i \ge 0$, then $-\sqrt{r}x_i \le 2\sqrt{r}x_i - \lfloor 3\sqrt{r}x_i \rfloor < \|\boldsymbol{u}\|x_i - \lfloor 3\sqrt{r}x_i \rfloor \le 3\sqrt{r}x_i - \lfloor 3\sqrt{r}x_i \rfloor < 1$. Thus

$$\big|\|\boldsymbol{u}\|x_i - u_i\big| = \big|\|\boldsymbol{u}\|x_i - \lfloor 3\sqrt{r}x_i \rfloor\big| \le \sqrt{r}|x_i| + 1\,.$$

This inequality holds also if $x_i < 0$ because then $1 - \sqrt{r}x_i > 2\sqrt{r}x_i - \lfloor 3\sqrt{r}x_i \rfloor > \|\boldsymbol{u}\|x_i - \lfloor 3\sqrt{r}x_i \rfloor \ge 3\sqrt{r}x_i - \lfloor 3\sqrt{r}x_i \rfloor \ge 0$.

Let $\boldsymbol{y} = (y_1, \dots, y_r) := \boldsymbol{x} - \boldsymbol{v}$. Then

$$|y_i| = |x_i - v_i| = \left|\frac{\|\boldsymbol{u}\|x_i - u_i}{\|\boldsymbol{u}\|}\right| \le \frac{\sqrt{r}|x_i| + 1}{2\sqrt{r}}\,,$$

$$\|\boldsymbol{y}\|^2 = \sum_{i=1}^{r} y_i^2 \le \frac{r\Big(\sum_{i=1}^{r} x_i^2\Big) + 2\sqrt{r}\Big(\sum_{i=1}^{r}|x_i|\Big) + r}{4r} \le \frac{4r}{4r} = 1\,.$$

Thus $\|\boldsymbol{y}\| \le 1$, and therefore B_r is a 1-net.

Now we show that $|B_r|$ is bounded by an exponential function of r. Let H_r denote the cube in $\mathbb{R}^r$ whose vertices have all coordinates equal to $\pm 1/2$. H_r has centre at the origin, and its volume is 1. Let $H_r + \boldsymbol{a}$ denote the image of H_r under the translation by $\boldsymbol{a} \in \mathbb{R}^r$. If $\boldsymbol{a} = (a_1, \dots, a_r) \in A_r$, then we show that $H_r + \boldsymbol{a} \subseteq S_r(7\sqrt{r}/2)$. Really, since $\boldsymbol{a} \in A_r$, then $\sum_{i=1}^{r} a_i^2 \le 9r$, and, by (3), $\sum\limits_{i=1}^{r} |a_i| \le \sqrt{r}\|\boldsymbol{a}\| \le 3r$ and

$$\sum_{i=1}^{r}(a_i \pm 1/2)^2 \le \sum_{i=1}^{r} a_i^2 + \sum_{i=1}^{r} |a_i| + r/4 \le 49r/4\,,$$

i.e. all vertices of $H_r + \boldsymbol{a}$ are from $S_r(7\sqrt{r}/2)$, therefore, if $\boldsymbol{a} \in A_r$, then $H_r + \boldsymbol{a} \subseteq S_r(7\sqrt{r}/2)$.

Clearly, if $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{Z}^r$ and $\boldsymbol{a} \neq \boldsymbol{b}$, then the set $(H_r + \boldsymbol{a}) \cap (H_r + \boldsymbol{b})$ is not full-dimensional, i.e. its volume is 0. Then $|A_r| = \sum\limits_{\boldsymbol{a}\in A_r} \mathrm{vol}(H_r + \boldsymbol{a}) \le \mathrm{vol}\big(S_r(7\sqrt{r}/2)\big) = (7/2)^r r^{r/2} \mathrm{vol}\big(S_r(1)\big)$. It is known that

$$\mathrm{vol}\big(S_r(1)\big) = \pi^{r/2}/\Gamma(r/2+1)\,,$$

where $\Gamma(x) = \int\limits_0^\infty \mathrm{e}^{-t}\, t^{x-1}\,\mathrm{d}t$ $(x > 0)$ is the gamma-function. By the Stirling formula,

$$\Gamma(r/2+1) = \sqrt{\pi r}(r/2\,\mathrm{e})^{r/2}\,\mathrm{e}^{\theta(r/2)}\,,$$

where $0 < \theta(x) < 1/12x$. Therefore $|B_r| \le |A_r| \le d^r$, where d is a constant independent on r. □

Now we can formulate the main theorem.

THEOREM 1. *Let n, m be integers, $2n \le m \le c^n$, where $c > 1$ is a constant. Then there exist m unit vectors in $\mathbb{R}^n$ such that the convex hull of them contains a ball with centre at the origin and of radius $c_3\sqrt{n^{-1}\log(m/n)}$, where the constant c_3 does not depend on n.*

Proof. Let m, n be integers such that $2n \le m \le c^n$. We also suppose that $n \ge 2$. By Lemma 2, for any integer r there exists a 1-net B_r in S^{r-1} of cardinality at most d^r, where $d > 1$ is a constant. If $2n \le m \le d^2 n$, take $C_n := \{\pm\boldsymbol{e}_i\,;\ i = 1,\dots,n\}$. Then $|C_n| = 2n \le m$, and, by (3), $S_n(1/\sqrt{n}) \subseteq \mathrm{conv}(C_n)$, what proves (2) in this special case. If $m > d^2 n$, then choose

$$r := \lfloor \log_d(m/n) \rfloor\,, \qquad s := \lceil n/r \rceil\,.$$

Clearly, $r \ge 2$. If $n = rs$, then take s copies of B_r in pairwise orthogonal r-dimensional subspaces of $\mathbb{R}^n$, the set we obtain is denoted by C_n. Then $|C_n| \le sd^r \le \left(\frac{n}{r} + 1\right)\frac{m}{n} \le \frac{m}{r} + \frac{m}{n} \le m$ because $n, r \ge 2$. We show that

$$S_n(1/\sqrt{4s}) \subseteq \mathrm{conv}(C_n)\,.$$

To prove this, let $\mathbf{x} \in S^{n-1}$. Then $\mathbf{x}$ can be expressed as sum of its s projections $\mathbf{x}_i$ ($i \in \{1, \dots, s\}$) on the pairwise orthogonal r-dimensional subspaces of $\mathbb{R}^n$, i.e. $\mathbf{x} = \sum_{i=1}^{s} \mathbf{x}_i$, $1 = \|\mathbf{x}\|^2 = \sum_{i=1}^{s} \|\mathbf{x}_i\|^2$, and, by (3), $\sum_{i=1}^{s} \|\mathbf{x}_i\| \leq \sqrt{s}$. By Lemma 1, $S_r(1/2) \subseteq \operatorname{conv}(B_r)$, therefore for any $i \in \{1, \dots, s\}$ there exist $\mathbf{v}_{i,1}, \dots, \mathbf{v}_{i,n_i} \in C_n$ and positive reals $\alpha_{i,1}, \dots, \alpha_{i,n_i}$ such that $\sum_{j=1}^{n_i} \alpha_{i,j} \leq 2$ and $\mathbf{x}_i/\|\mathbf{x}_i\| = \sum_{j=1}^{n_i} \alpha_{i,j} \mathbf{v}_{i,j}$. Then $\mathbf{x} = \sum_{i=1}^{s} \sum_{j=1}^{n_i} \|\mathbf{x}_i\| \alpha_{i,j} \mathbf{v}_{i,j}$ and $\sum_{i=1}^{s} \sum_{j=1}^{n_i} \|\mathbf{x}_i\| \alpha_{i,j} \leq 2\sqrt{s}$, thus $\left(1/\sqrt{4s}\right)\mathbf{x} \in \operatorname{conv}(C_n)$. Since this is true for any $\mathbf{x} \in S^{n-1}$, then $S_n\left(1/\sqrt{4s}\right) \subseteq \operatorname{conv}(C_n)$.

Concluding, C_n has cardinality at most m and contains a ball of radius $1/\sqrt{4s} \geq c_3\sqrt{n^{-1}\log(m/n)}$, what proves (2). If $n < rs$, then take s copies of B_r or B_{r-1} in pairwise orthogonal r- or $(r-1)$-dimensional subspaces of $\mathbb{R}^n$ and continue analogously as if $n = rs$. □

Note that our construction gives in fact a constructive proof of the lower bound of (1).

Finally let us discuss the restriction $2n \leq m \leq c^n$. The upper bound is trivial because, if $n \to \infty$, then the lower bounds given in (1), (2) are valid only if $m \leq c^n$. On the other hand, the optimal ratios $V(n, n+1)$ and $\rho(n, n+1)$ occur if we deal with regular simplex. Then $V(n, n+1)^{1/n} \approx \text{const}\sqrt{1/n}$, and (1) remains true also if $m > n$. It is known that any n-dimensional regular simplex with unit vertices contains a ball (with centre at the origin) of radius at most $1/n$ (see e.g. [7]). Thus $\rho(n, n+1) = 1/n$. But $c_3\sqrt{n^{-1}\log(1 + n^{-1})} \approx \text{const}\sqrt{1/n} > 1/n$ if $n \to \infty$. Therefore (2) is not true for any $m > n$ though it holds for any $m \geq 2n$. It could be of some interest to study the behaviour of $\rho(n, m)$ if $n < m < 2n$. We show that, if $r < n$, then

$$\rho(n, n+r+1) \geq 1/\sqrt{r + (n-r)^2}\,. \tag{4}$$

To prove this, take the set $B_1 = \{\pm\mathbf{e}_i\,;\ i = 1, \dots, r\}$ of unit vectors in $\mathbb{R}^r$ and the set B_2 consisting of unit vertices of a regular simplex in $\mathbb{R}^{n-r}$. Take copies of B_1 and B_2 in two mutually orthogonal r- and $(n-r)$-dimensional subspaces of $\mathbb{R}^n$, respectively. The set we obtain is denoted by B. Then $|B| = n+r+1$. Since $\max\left\{ax + b\sqrt{1-x^2}\,;\ x \in \langle 0, 1\rangle\right\} = \sqrt{a^2 + b^2}$, then using the above methods we can check that $S_n\left(1/\sqrt{r + (n-r)^2}\right) \subseteq \operatorname{conv}(B)$, what proves (4).

3. Algorithmical consequences of the construction

Now we sketch an application of our result in computational geometry. From Theorem 1 it follows:

Corollary 1. *There exist n^2 unit vectors in $\mathbb{R}^n$ $(n \geq 2)$ such that the convex hull of them contains the ball $S_n(c_3\sqrt{n^{-1}\log n})$, where the constant c_3 does not depend on n.*

Primarily, we suppose that any convex body $K \subseteq \mathbb{R}^n$ is given by a membership oracle, i.e. we have an oracle that decides for any $\boldsymbol{x} \in \mathbb{Q}^n$ whether $\boldsymbol{x} \in K$ or not. Furthermore, we suppose that K is contained in a ball with centre at the origin and of radius R, K contains a ball with centre $\boldsymbol{a} \in \mathbb{Q}^n$ and of radius r and the coordinates of $\boldsymbol{a}$, R and $1/r$ are bounded by a polynomial of 2^n. This model coincides with that of Grötschel, Lovász and Schrijver [7].

It is well known (see [7], [5]) that every convex body $K \subseteq \mathbb{R}^n$ is contained in a unique ellipsoid E of minimal volume. This ellipsoid is called the *Löwner-John ellipsoid of* K. Moreover, K contains the ellipsoid $(1/n)E$ (where $(1/\delta)E$ will denote the ellipsoid obtained from E by shrinking it from its centre by a factor of δ). If K is centrally symmetric, then the component $1/n$ can be improved on $1/\sqrt{n}$ (see [7] for more details).

In general, the Löwner-John ellipsoid of a convex body is hard to compute. Grötschel, Lovász and Schrijver [7] (see also [6]) presented an algorithm bounded by a polynomial of n that approximate the Löwner-John ellipsoid. This algorithm finds an ellipsoid E such that $(c_4/n^{3/2})E \subseteq K \subseteq E$ for any convex set $K \subseteq \mathbb{R}^n$, and, if K is centrally symmetric, then $(c_4/n)E \subseteq K \subseteq E$. Using Corollary 1 and the methods of [7; Theorems 4.6.1 and 4.6.3] (see also [7; Remark 4.6.2]) we can asymptotically improve this algorithm such that the components $c_4/n^{3/2}$ and c_4/n are replaced by $c_5\sqrt{\log n}/n^{3/2}$ and $c_5\sqrt{\log n}/n$, respectively.

It is easy to compare volumes of two concentrical ellipsoids. Thus, the algorithm of Grötschel, Lovász and Schrijver for approximation of the Löwner-John ellipsoid gives in fact an upper bound $\overline{\mathrm{vol}}(K)$ and a lower bound $\underline{\mathrm{vol}}(K)$ for the volume of the convex set K such that $\overline{\mathrm{vol}}(K)/\underline{\mathrm{vol}}(K) \leq (n/c_4)^{3n/2}$ in general case, and, if K is centrally symmetric, then $\overline{\mathrm{vol}}(K)/\underline{\mathrm{vol}}(K) \leq (n/c_4)^n$. Thus our improvement of the algorithm improves the ratio

$$\frac{\overline{\mathrm{vol}}(K)}{\underline{\mathrm{vol}}(K)} \leq \left(\frac{n^{3/2}}{c_5\sqrt{\log n}}\right)^n$$

in general case, and, if K is centrally symmetric, then

$$\frac{\overline{\mathrm{vol}}(K)}{\underline{\mathrm{vol}}(K)} \leq \left(\frac{n}{c_5\sqrt{\log n}}\right)^n.$$

Bárány and Füredi [1] proved the following negative result. For any polynomial time algorithm which gives an upper bound $\overline{\mathrm{vol}}(K)$ and a lower bound $\underline{\mathrm{vol}}(K)$ for the volume of a convex set $K \subseteq \mathbb{R}^n$ the ratio $\overline{\mathrm{vol}}(K)/\underline{\mathrm{vol}}(K)$ is at least $(c_6 n/\log n)^n$ for some convex body $K \subseteq \mathbb{R}^n$, where c_6 is a constant independent of n. Thus our algorithm is very close to being asymptotically optimal for centrally symmetric convex bodies.

Other results from [7; Section 4.6] can be improved similarly.

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