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CONVEXITY OF THE ORIENTOR FIELD AND THE SOLUTION SET OF A CLASS OF EVOLUTION INCLUSIONS

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ABSTRACT. In this paper we examine semilinear evolution inclusions and show that if the set of solutions is weakly closed in the space of absolutely continuous functions, then the orientor filed for almost all times is convex valued. In establishing this result we obtain a new existence theorem for the nonconvex problem, a new relaxation theorem and a new Filippov-Gronwall type theorem, all of which are of independent interest and can be useful in the study of infinite dimensional control systems.

1. Introduction

In a recent interesting paper, Cellina–Ornelas [4], considered differential inclusions in $\mathbb{R}^n$, driven by a Lipschitz orientor field and established an equivalence between the closedness in AC_w (the space of absolutely continuous $\mathbb{R}^n$-valued functions defined on $T = [0, b]$ and endowed with the weak topology) and the convexity of the values of the orientor filed. The purpose of this note is to extend this result to evolution inclusions, that appear often in the study of distributed parameter systems (see [14]).

Let $T = [0, b]$ and let X be a separable Banach space having the Radon-Nikodym Property (RNP; see Diestel–Uhl [6, Definition 3, p. 61]). Let $AC(X)$ be the space of all absolutely continuous functions from T into X. It is well known (see for example Diestel–Uhl [6, p. 217]), that such functions are almost everywhere differentiable and furthermore that $x(t) = x(0) + \int_0^t \dot{x}(s)\, ds$,

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$t \in T$. So $AC(X)$ can be identified with $X \times L^1(X)$, where $L^1(X)$ is the Lebesgue-Bochner space of all functions $h\colon T \to X$ s.t. $\|h(\cdot)\| \in L^1$. Now let $F\colon T \to 2^X \setminus \{\emptyset\}$ be a multifunction with closed values and assume that $F(\cdot)$ is measurable (see Section 2) and that $t \mapsto |F(t)| = \sup\{\|x\| : x \in F(t)\} \in L^1$. Let $S(x_0) \subseteq AC(X)$ be the set of solutions of $\dot{x}(t) \in F(t)$ a.e., $x(0) = x_0$. Given the identification of $AC(X)$ with $X \times L^1(X)$, we see that $S(x_0)$ is identified with $\{x_0\} \times S_F^1$, where $S_F^1 = \{h \in L^1(X) : h(t) \in F(t) \text{ a.e.}\}$. From Theorem 4.3 of [19], we know that S_F^1 is weakly closed in $L^1(X)$ (and so $S(x_0)$ is weakly closed in $AC(X)$) if and only if $F(\cdot)$ is convex-valued. In this paper we extend this fact to the case where F depends also on x and is the orientor field of an evolution inclusion defined in a separable Banach space. In doing this, we also prove a new Filippov-type approximation theorem and a new relaxation theorem, which are interesting by themselves. Our result on the automatic convexity of the orientor field, is another instance of the "principle" that the weak topology and convexity go together. Recall Mazur's theorem that says that a convex set is closed if and only if is w-closed and from nonlinear analysis the result that says that the integral functional $I_f(x) = \int_0^b f(t, x(t))\, dt$ is w-l.s.c. on $L^1(X)$ if and only if $f(t, \cdot)$ is convex for almost all $t \in T$. This result was first proved by Rockafellar [21, Theorem 1] for $X = \mathbb{R}^n$, it was extended to separable reflexive Banach spaces by Bismut [3, Theorem 1] and finally it was proved for X a general separable Banach space by this author in [16, Theorem 5.1].

2. Preliminaries

Let (Ω, Σ) be a measurable space and X a separable Banach space. Throughout this paper we will be using the following notations:

$$P_{f(c)}(X) = \{A \subseteq X : \text{ nonempty, closed (and convex)}\}$$

and

$$P_{(w)k(c)}(X) = \{A \subseteq X : \text{ nonempty, (weakly-) compact (and convex)}\}.$$

A multifunction $F\colon \Omega \to P_f(X)$ is said to be measurable, if for all $z \in X$ $\omega \mapsto d(z, F(\omega)) = \inf\{\|z - x\| : x \in F(\omega)\}$ is measurable. A multifunction $G\colon \Omega \to 2^X \setminus \{\emptyset\}$ is said to be graph measurable if $\operatorname{Gr} G = \{(\omega, x) \in \Omega \times X : x \in G(\omega)\} \in \Sigma \times B(X)$, with $B(X)$ being the Borel σ-field of X. In general, for $P_f(X)$-valued multifunctions measurability implies graph measurability, and the converse is true if there exists a σ-finite measure $\mu(\cdot)$ on Σ, with respect to which Σ is complete. Other equivalent definitions of measurability of closed-valued multifunctions and additional results on the subject, can be found in the survey paper of Wagner [23].

Now let (Ω, Σ, μ) be a σ-finite measure space. Given a multifunction $F\colon \Omega \to 2^X \setminus \{\emptyset\}$, by S_F^p $(1 \le p \le \infty)$ we will denote the set of selectors of $F(\cdot)$ that belong in the Lebesgue-Bochner space $S_F^p = \{f \in L^p(X) : f(\omega) \in F(\omega)\ \mu\text{-a.e.}\}$. This set may be empty. It is nonempty if $F(\cdot)$ is graph measurable and $\omega \mapsto \inf\{\|x\| : x \in F(\omega)\}$ belongs in L^1. Note that S_F^p is a decomposable set; i.e., if $A \in \Sigma$ and $f_1, f_2 \in S_F^p$, then $f = \chi_A f_1 + \chi_{A^c} f_2 \in S_F^p$. Decomposable sets in Lebesgue-Bochner spaces were studied in [19]. Using S_F^1 we can define a set-valued integral for $F(\cdot)$ by setting $\int\limits_\Omega F(\omega)\, \mathrm{d}\mu(\omega) = \Big\{ \int\limits_\Omega f(\omega)\, \mathrm{d}\mu(\omega) : f \in S_F^1 \Big\}$. The properties of this integral were studied in [11].

Let Y, Z be Hausdorff topological spaces and let $G\colon Y \to 2^Z \setminus \{\emptyset\}$ be a multifunction. We will say that $G(\cdot)$ is upper-semicontinuous (u.s.c.) (resp. lower-semicontinuous (l.s.c.)), if for all $U \subseteq Z$ open, $G^+(U) = \{y \in Y : G(y) \subseteq U\}$ (resp. $G^-(U) = \{y \in Y : G(y) \cap U \neq \emptyset\}$), is open in Y. Additional properties of upper and lower semicontinuous multifunctions, can be found in DeBlasi–Myjak [5] and Klein–Thompson [12]. Suppose V is a metric space. On $P_f(V)$ we can define a generalized metric, known in the literature as the Hausdorff metric, by setting

$$h(A,B) = \max\Big[\sup_{a\in A} d(a,B),\ \sup_{b\in B} d(b,A)\Big] \qquad \text{for all} \quad A, B \in P_f(V).$$

If V is complete, then so is $\big(P_f(V), h\big)$.

Next let H be a separable Hilbert space and X a dense subspace of H carrying the structure of a separable, reflexive Banach space. Assume that X embeds continuously into H; i.e., $X \to H$ continuously. Then identifying H with its dual (pivot space), we have $X \to H \to X^*$, with all embeddings being continuous and dense (see Zeidler [24, p. 416]). Such a triple of spaces is known in the literature as evolution triple or Gelfand triple. To have a concrete example in mind let Z be a bounded domain in $\mathbb{R}^n$, let $H = L^2(Z)$ and $X = W_0^{m,p}(Z)$ with $m \in \mathbb{N}$, $m \ge 1$ and $p \ge 2$. Then $X^* = W^{-m,q}(Z)$, $\frac{1}{q} + \frac{1}{p} = 1$ and from the Sobolev embedding theorem, we know that (X, H, X^*) is an evolution triple and in fact all embeddings are compact. By $\langle \cdot, \cdot \rangle$ we will be denoting the duality brackets for the pair (X, X^*) and by $(\cdot, \cdot)$ the inner product in H. The two are compatible in the sense that $\langle \cdot, \cdot \rangle|_{X \times H} = (\cdot, \cdot)$. Also by $\|\cdot\|$ (resp. $|\cdot|$, $\|\cdot\|_*$), we will denote the norm of X (resp. of H, X^*).

We set $W(T) = \{x \in L^2(X) : \dot{x} \in L^2(X^*)\}$, where the derivative is understood in the sense of vectorial distributions on T. Furnished with the norm $\|x\|_{W(T)} = \left(\|x\|^2_{L^2(X)} + \|\dot{x}\|^2_{L^2(X^*)}\right)^{1/2}$, $W(T)$ becomes a separable reflexive Banach space. It is well known (see Zeidler [24, Proposition 23.23, p. 422]) that $W(T) \to C(T, H)$ continuously; i.e, every function in $W(T)$ after possible modification on a Lebesgue-null subset of T, equals an H-valued continuous function on T. Also if $X \to H$ compactly, then $W(T) \to L^2(H)$ compactly (see Zeidler [24, p. 450]). Furthermore, since by definition $W(T) \subseteq H^1(X^*) = W^{1,2}(X^*)$ (= the Sobolev space of X^*-valued distributions) and since $W^{1,2}(X^*) = AC^{1,2}(X^*) = \{x : T \to X^* : x(\cdot)$ absolutely continuous, $\dot{x}(\cdot) \in L^2(X^*)\}$ (recall that since X^* is separable, reflexive, it has the RNP (Phillips' theorem see Diestel–Uhl [6, Corollary 4, p. 82]) and so $\dot{x}(t)$ exists), we have that $W(T) \to AC^{1,2}(X^*)$. For further details on vector distributions and vectorial Sobolev spaces, we refer to Barbu [2].

3. Existence and relaxation theorems

Let $T = [0, b]$ and X be a separable Banach space. Consider the following evolution inclusion:

$$\begin{aligned} \dot{x}(t) + A(t)x(t) &\in F(t, x(t)), \\ x(0) &= x_0 . \end{aligned} \tag{$*$}$$

We will need the following hypotheses on the data of $(*)$.

$H(A)$: $\{-A(t) : t \in T\}$ is a family of densely defined, closed, linear operators, which generates an evolution operator $S : \Delta = \{(t, s) \in T \times T : 0 \le s \le t \le b\} \to \mathcal{L}(X)$, which is assumed to be compact for $t - s > 0$.

Remark. Recall (see Tanabe [22, p. 87]), that $S(t, s)$ is an evolution operator (or fundamental solutions), if $S : \Delta \to \mathcal{L}(X)$ is strongly continuous, $S(t, \tau)S(\tau, s) = S(t, s)$ for $0 \le s \le \tau \le t \le b$ (semigroup property), $S(t, t) = I$ for all $t \in T$, $\frac{\partial}{\partial t} S(t, s) = -A(t)S(t, s)$ and $\frac{\partial}{\partial s} S(t, s) = S(t, s)A(s)$. Conditions for such an operator to exist can be found in Tanabe [22].

$H(F)$: $F : T \times X \to P_f(X)$ is a multifunction s.t.
(1) $(t, x) \mapsto F(t, x)$ is graph measurable,
(2) $x \mapsto F(t, x)$ is l.s.c.,
(3) $|F(t, x)| = \sup\{\|y\| : y \in F(t, x)\} \le a(t) + b(t)\|x\|$ a.e. with $a(\cdot), b(\cdot) \in L^1_+$.

By a solution of $(*)$, we understand a mild solution $x(\cdot) \in C(T, X)$ of the form $x(t) = S(t, 0)x_0 + \int_0^t S(t, s)f(s)\, ds$, $t \in T$, $f(\cdot) \in S^1_{F(\cdot, x(\cdot))}$.

We start with an existence result that improves Theorem 3.1' of [18], since now our growth hypothesis on the orientor field $F(t,x)$ is more general.

THEOREM 3.1. *If hypotheses $H(A)$ and $H(F)$ hold, then $(*)$ admits a solution.*

P r o o f. Since the proof is similar to that of Theorem 3.1 in [18], we will only present an outline of it, and the details can be found in [18].

For any solution $x(\cdot) \in C(T,X)$ of $(*)$, we get via Gronwall's inequality that $\|x(t)\| \le K = \big(M\|x_0\| + M\|a\|_1\big)\exp\big(M\|b\|_1\big)$, where $\|S(t,s)\| \le M$, $(t,s) \in \Delta$. Define $\hat{F}\colon T \times X \to P_f(X)$ by

$$\hat{F}(t,x) = \begin{cases} F(t,x) & \text{if } \|x\| \le K\,, \\ F\Big(t, \dfrac{Kx}{\|x\|}\Big) & \text{if } \|x\| > K\,. \end{cases}$$

Then $\hat{F}(t,x)$ is graph measurable, l.s.c. in x and $|\hat{F}(t,x)| = \sup\big\{\|y\| : y \in \hat{F}(t,x)\big\} \le a(t) + b(t)K = \varphi(t)$ a.e. with $\varphi(\cdot) \in L^1_+$. Then define

$$W = \Big\{x(\cdot) \in C(T,X) :\ x(t) = S(t,0)x_0 + \int_0^1 S(t,s)g(s)\,\mathrm{d}s\,,\quad t \in T\,,\ \|g(t)\| \le \varphi(t) \text{ a.e.}\Big\}.$$

As in the proof of Theorem 3.1 in [18], via the Arzela-Ascoli theorem, we can get that $\bar{W}$ is compact in $C(T,X)$. Let $R\colon \bar{W} \to P_f\big(L^1(X)\big)$ be defined by $R(x) = S^1_{\hat{F}(\cdot,x(\cdot))}$. From Theorem 4.1 of [15], we have that $R(\cdot)$ is l.s.c. Since the values of $R(\cdot)$ are decomposable subsets of $L^1(X)$ (see Section 2), we can apply F r y s z k o w s k i 's selection theorem [8], to get $r\colon \bar{W} \to L^1(X)$ continuous s.t. $r(x) \in R(x)$ for all $x \in \bar{W}$. Let $q\colon \bar{W} \to \bar{W}$ be defined by $q(x)(t) = S(t,0)x_0 + \int_0^t S(t,s)r(x)(s)\,\mathrm{d}s$, $t \in T$. Clearly $q(\cdot)$ is continuous and since $\bar{W} \in P_{kc}\big(C(T,X)\big)$, we can apply Schauder's fixed point theorem, to get $x \in \bar{W}$ s.t. $q(x) = x$. Then $x(\cdot)$ solves $(*)$ with $\hat{F}(t,x)$. Using Gronwall's inequality and the definition of $\hat{F}(t,x)$, we can show as in [18], that $\|x(t)\| \le K$ for all $t \in T \implies \hat{F}\big(t,x(t)\big) = F\big(t,x(t)\big)$ $t \in T \implies x(\cdot)$ solves $(*)$ with $F(t,x)$. □

To evolution inclusion $(*)$, we associate its relaxed (convexified) version:

$$\begin{gathered} \dot{x}(t) + A(t)x(t) \in \overline{\text{conv}} F\big(t, x(t)\big) , \\ x(0) = x_0 . \end{gathered} \tag{$(*)_r$}$$

By $P(x_0) \subseteq C(T,H)$ we will denote the solution set of $(*)$ and by $P_r(x_0) \subseteq C(T,X)$ the solution set of $(*)_r$.

We will need the following hypothesis on $F(t,x)$.

$H(F)_1$: $F\colon T \to P_{wkc}(X)$ is a multifunction s.t.
(1) $(t,x) \mapsto F(t,x)$ is graph measurable,
(2) $x \mapsto F(t,x)$ is u.s.c. from X into X_w,
$X_w = \{$the space X equipped with the w-topology$\}$,
(3) $|F(t,x)| \le a(t) + b(t)\|x\|$ a.e. with $a(\cdot), b(\cdot) \in L^1_+$.

THEOREM 3.2. *If hypotheses $H(A)$ and $H(F)_1$ hold, then*

$$P_r(x_0) \in P_k\big(C(T,X)\big) .$$

P r o o f . The nonemptiness of $P_r(x_0)$, can be established exactly as in Theorem 3.3 of [18]. Since $P_r(x_0) \subseteq \bar{W} \subseteq C(T,X)$ (see the proof of Theorem 3.1) and because $\bar{W}$ is compact in $C(T,X)$, it suffices to show that $P_r(x_0)$ is closed in $C(T,X)$. To this end let $\{x_n\}_{n\ge1} \subseteq P_r(x_0)$ and assume that $x_n \to x$ in $C(T,X)$. Then by definition we have for $n \ge 1$,

$$x_n(t) = S(t,0)x_0 + \int_0^t S(t,x) f_n(s) \, \mathrm{d}s$$

for all $t \in T$ and some $f_n \in S^1_{F(\cdot,x_n(\cdot))}$. Because $F(t,\cdot)$ is u.s.c. from X into X_w and it has values in $P_{wkc}(X)$, from Theorem 7.4.2 of K l e i n - T h o m p s o n [12, p. 90], we have that $G(t) = \Big[\bigcup\limits_{n\ge1} F\big(t, x_n(t)\big)\Big] \cup F\big(t, x(t)\big) \in P_{wk}(X)$, while from hypothesis $H(F)_1(1)$, we have that $t \mapsto G(t)$ is graph measurable, hence measurable for the Lebesgue σ-field on T (see Section 2). Then $t \mapsto \overline{\text{conv}} G(t)$ is measurable (see H i m m e l b e r g [10, Theorem 9.1]), $P_{wkc}(X)$-valued (Krein-Smulian theorem; see D i e s t e l - U h l [6, Theorem 11, p. 51]) and $|\overline{\text{conv}} G(t)| \le a(t) + b(t)M_1 = \varphi_1(t)$ a.e., with $\varphi_1(\cdot) \in L^1_+$, and $M_1 = \sup\limits_{n\ge1} \|x_n\|_{C(T,X)}$. Hence invoking Proposition 3.1 of [20], we deduce that S^1_G is w-compact in $L^1(X)$, hence by the Eberlein-Smulian theorem, sequentially w-compact. Since

$\{f_n\}_{n\geq 1} \subseteq S^1_G$, by passing to a subsequence if necessary, we may assume that $f_n \xrightarrow{w} f$ in $L^1(X)$. From Theorem 3.1 of [15], we get

$$f(t) \in \overline{\mathrm{conv}}\, w\text{-}\overline{\lim}\{f_n(t)\}_{n\geq 1} \subseteq \overline{\mathrm{conv}} F\big(t, x_n(t)\big) \subseteq F\big(t, x(t)\big) \quad \text{a.e.}$$

since $x_n(t) \xrightarrow{s} x(t)$ in X, $F(t,\cdot)$ is u.s.c. from X into X_w and the values of $F(\cdot\,,\cdot)$ are in $P_{wkc}(X)$. Hence $f \in S^1_{F(\cdot\,,x(\cdot))}$. Then

$$x(t) = S(t,0)x_0 + \int_0^t S(t,x)f(s)\,\mathrm{d}s \qquad \text{for} \quad t \in T$$

and with $f(\cdot) \in S^1_{F(\cdot,x(\cdot))} \implies x \in P_r(x_0) \implies P_r(x_0)$ is closed in $C(T,X)$, hence compact. □

Remark. Our result extends Theorem 2.7 of Frankowska [25], where X is assumed to be reflexive and the operator A is time independent. Note that in the time invariant case, technically the situation is easier since the semigroup $\{S(t):\ t \in T\}$ generated by A is a function of one variable and in case we assume that $S(t)$ is compact for $t \in (0,b]$, then $t \mapsto S(t)$ is continuous from $(0,b]$ into $\mathcal{L}(X)$ furnished with the uniform operator topology. In the time varying case the evolution operator $\{S(t,s):\ (t,s) \in \Delta\}$ is a function of two variables and if we assume that $S(t,s)$ is compact for $t-s>0$, then $t \mapsto S(t,s)$ is continuous from $(s,b]$ into $\mathcal{L}(X)$ with the uniform operator topology. Note that this continuity property in general depends on s and is only uniform with respect to s in sets bounded away from t; i.e. $t-s \geq \beta$ for $\beta > 0$ (see Proposition 2.1 of [18]). This makes our arguments more involved and different from those of Frankowska [25]. Also in view of what we said above, in Theorem 2.7 of Frankowska [25], condition (ii) is actually covered by condition (i) since equicontinuity at $t=0$ is an immediate consequence of the absolute continuity of the Lebesgue integral.

The next result relates solution sets $P(x_0)$ and $P_r(x_0)$ (relaxation theorem). For this we will need the following stronger hypothesis on the orientor field $F(t,x)$.

$H(F)_2$: $F\colon T \times X \to P_{wk}(X)$ is a multifunction s.t.
(1) $t \mapsto F(t,x)$ is measurable,
(2) $h\big(F(t,x), F(t,y)\big) \leq k(t)\|x-y\|$ a.e. with $k(\cdot) \in L^1_+$,
(3) $|F(t,x)| \leq a(t) + b(t)\|x\|$ a.e. with $a(\cdot), b(\cdot) \in L^1_+$.

R e m a r k . Note that $H(F)_2(2)$ above implies that $x \mapsto F(t,x)$ is l.s.c. (see D e B l a s i – M y j a k [5]). Also because of $H(F)_2(2)$, the support function $x \mapsto \sigma\big(x^*, F(t,x)\big) = \sigma(x^*, \overline{\mathrm{conv}}F(t,x)\big) = \sup\big\{(x^*, z) : \ z \in \overline{\mathrm{conv}}F(t,x)\big\}$ is continuous and so Theorem 10 of A u b i n – E k e l a n d [1, p. 128], tells us that $x \mapsto \overline{\mathrm{conv}}F(t,x)$ is u.s.c. from X into X_w. Also by the Krein-Smulian theorem, $\overline{\mathrm{conv}}F(t,x) \in P_{wkc}(X)$. Furthermore hypotheses $H(F)_2(1)$ and (2) and Theorem 3.3 of [17], tell us that $(t,x) \mapsto F(t,x)$ is measurable. Note that Theorem 3.3 of [17] gives us a much simpler proof for Lemma 1.4 of F r a n k o w s k a [25]. Furthermore, it shows that the Lipschitz property of $F(t,\cdot)$ is not necessary for the lemma to be valid. It suffices to have Hausdorff continuity.

THEOREM 3.3. *If hypotheses $H(A)$ and $H(F)_2$ hold, then $P_r(x_0) = \overline{P(x_0)}$, the closure taken in $C(T,X)$.*

P r o o f . Let $x(\cdot) \in P_r(x_0)$. Let $\eta\colon L^1(X) \to C(T,X)$ be the map, which to each $h \in S^1_{\overline{\mathrm{conv}}F(\cdot,x(\cdot))}$ assigns the unique mild solution $\eta(h)(\cdot) \in C(T,X)$ of the Cauchy problem $\dot{x}(t) + A(t)x(t) = h(t)$, $x(0) = x_0$. We claim that $\eta(\cdot)$ is continuous from $S^1_{\overline{\mathrm{conv}}F(\cdot,x(\cdot))}$ equipped with the relative weak $L^1(X)$-topology into $C(T,X)$. Note that $S^1_{\overline{\mathrm{conv}}F(\cdot,x(\cdot))}$ with the relative weak $L^1(X)$-topology, is compact metrizable (see Proposition 3.1 of [20] and note $L^1(X)$ is separable, since X is). So we may work with sequences. Hence let $\{h_n\}_{n\ge1} \subseteq S^1_{\overline{\mathrm{conv}}F(\cdot,x(\cdot))}$ s.t. $h_n \xrightarrow{w} h$ in $L^1(X)$. Set $z_n = \eta(f_n) \subseteq P_r(x_0) \subseteq C(T,X)$. Because of Theorem 3.2 and by passing to a subsequence if necessary, we may assume that $z_n \to z$ in $C(T,X)$. By definition we have

$$z_n(t) = S(t,0)x_0 + \int_0^t S(t,s)h_n(s)\,\mathrm{d}s\,, \qquad t \in T$$

and $z_n(t) \xrightarrow{s} z(t)$ in X, while $\int_0^t S(t,s)h_n(s)\,\mathrm{d}s \xrightarrow{w} \int_0^t S(t,s)h(s)\,\mathrm{d}s$. So in the limit as $n \to \infty$, we get

$$z(t) = S(t,0)x_0 + \int_0^t S(t,s)h(s)\,\mathrm{d}s\,, \quad t \in T$$
$$\Longrightarrow\ z = \eta(h)\,.$$

Hence $\eta(\cdot)$ is indeed continuous as claimed. Since $x(\cdot) \in P_r(x_0)$, there exists $f \in S^1_{\overline{\mathrm{conv}}F(\cdot,x(\cdot))}$ s.t. $x = \eta(f)$. Having the continuity of $\eta(\cdot)$, we know that given $\varepsilon > 0$, we can find U a symmetric, weak neighbourhood of the origin s.t. $f - f_1 \in U \implies \|\eta(f) - \eta(f_1)\|_{C(T,X)} = \|x - \eta(f_1)\|_{C(T,X)} < \varepsilon$. Furthermore Theorem 4.1 of [16], tells us that we can pick $f_1 \in S^1_{\overline{\text{conv}}\, F(\cdot,x(\cdot))}$. Set $z_1 = \eta(f_1)$.

Let $L(t) = \{v \in F(t,z_1(t)) : d(f(t), F(t,z_1(t))) = \|f_1(t) - v\|\}$. Since $t \mapsto F(t,z_1(t))$ is measurable (recall that $(t,x) \mapsto F(t,x)$ is measurable, see remark above), we can easily check that $L(\cdot)$ is graph measurable, with nonempty values since $F(\cdot,\cdot)$ is $P_{wk}(X)$-valued. So using Theorem 5.10 of Wagner [23], we get $f_2 \colon T \to X$ measurable s.t. $f_2(t) \in L(t)$ a.e. Then we have

$$f_2(\cdot) \in S^1_{F(\cdot,z_1(\cdot))}$$

and

$$\|f_1(t) - f_2(t)\| \le h(F(t,x(t)), F(t,z_1(t))) \le k(t)\|x(t) - z_1(t)\| \le k(t)\varepsilon \quad \text{a.e.}$$

Set $z_2 = \eta(f_2)$ and as before let $M > 0$ be s.t. $\|S(t,s)\|_{\mathcal{L}} \le M$ for all $(t,s) \in \Delta$. We have

$$\begin{aligned}
\|z_2(t) - x(t)\| &\le \|z_2(t) - z_1(t)\| + \|z_1(t) - x(t)\| \\
&\le \left\| \int_0^t S(t,s) f_2(s)\, \mathrm{d}s - \int_0^t S(t,s) f_1(s)\, \mathrm{d}s \right\| + \varepsilon \\
&\le \int_0^t M \|f_2(s) - f_1(s)\|\, \mathrm{d}s + \varepsilon \\
&\le M\varepsilon \int_0^t k(s)\, \mathrm{d}s + \varepsilon = \varepsilon \left[M \int_0^t k(s)\, \mathrm{d}s + 1 \right].
\end{aligned}$$

Now suppose that we have obtained $f_1, \dots, f_n \in L^1(X)$ s.t.

$$\|f_{r+1}(t) - f_r(t)\| \le \varepsilon k(t) \frac{M^{r-1}}{(r-1)!} \left[\int_0^t k(s)\, \mathrm{d}s \right]^{r-1} \tag{1}$$

and

$$f_{r+1}(t) \in F(t, z_r(t)) \quad \text{a.e.,} \qquad \text{with} \quad z_r = \eta(f_r), \quad r = 1, 2, \dots, n-1. \tag{2}$$

Then we can write that

$$\begin{aligned}\|z_{r+1}(t)-z_r(t)\| &\le \int_0^t M\|f_{r+1}(s)-f_r(s)\|\,\mathrm{d}s\\ &\le \int_0^t M\varepsilon k(s)\frac{M^{r-1}}{(r-1)!}\left[\int_0^s k(\tau)\,d\tau\right]^{r-1}\mathrm{d}s\\ &\le \frac{M^r\varepsilon}{r!}\int_0^t \mathrm{d}\left(\int_0^s k(\tau)\,\mathrm{d}\tau\right)^r = \frac{M^r\varepsilon}{r!}\left[\int_0^t k(s)\,\mathrm{d}s\right]^r.\end{aligned}$$

So from the triangle inequality we get

$$\|z_{r+1}(t)-x(t)\| \le \varepsilon\sum_{\ell=0}^{r}\frac{M^\ell}{\ell!}\left[\int_0^t k(s)\,\mathrm{d}s\right]^\ell \le \varepsilon\exp\left[M\|k\|_1\right]. \tag{3}$$

As above, via Theorem 5.10 of Wagner [23], we can choose $f_{n+1} \in S^1_{F(\cdot,z_n(\cdot))}$ s.t.

$$\begin{aligned}&\|f_{n+1}(t)-f_n(t)\| = d\big(f_n(t),F\big(t,z_n(t)\big)\big) \quad \text{a.e.,} \qquad z_n=\eta(f_n)\\ \Longrightarrow\ &\|f_{n+1}(t)-f_n(t)\| \le h\big(F\big(t,z_n(t)\big),F\big(t,z_{n-1}(t)\big)\big)\\ &\qquad\le k(t)\|z_n(t)-z_{n-1}(t)\|\\ &\qquad\le \frac{M^{n-1}\varepsilon k(t)}{(n-1)!}\left[\int_0^t k(s)\,\mathrm{d}s\right]^{n-1}.\end{aligned}$$

Thus we have completed the construction of $\{f_n\}_{n\ge1} \subseteq L^1(X)$ and $\{z_n\}_{n\ge1} \subseteq C(T,X)$, satisfying (1), (2), and (3) above. From (1), we deduce that $\{f_n\}_{n\ge1} \subseteq L^1(X)$ is Cauchy. Thus $f_n \xrightarrow{s} f$ in $L^1(X)$. Then $z_n=\eta(f_n)\to z=\eta(f)$ in $C(T,X)$ and $f(t)\in h-\lim F\big(t,z_n(t)\big) = F\big(t,z(t)\big)$ a.e. So $z=\eta(f) \in P(x_0)$. Then since (3) holds for all $r\ge 0$, we get in the limit as $r\to\infty$ that

$$\begin{aligned}&\|z(t)-x(t)\| \le \varepsilon\exp\big(M\|k\|_1\big)\\ \Longrightarrow\ &\|x-z\|_{C(T,X)} \le \varepsilon\exp\big(M\|k\|_1\big).\end{aligned}$$

Since $\varepsilon>0$ was arbitrary, we conclude that $P_r(x_0)=\overline{P(x_0)}$ the closure taken in $C(T,X)$. □

R e m a r k . Relaxation results for evolution inclusions can also be found in F r a n k o w s k a [25, Theorem 2.5 and Corollary 2.6] and in T o l s t o n o g o v (Theorem 3.1). In Theorem 2.5, F r a n k o w s k a [25] using an idea of Clarke, proves that $P_r(x_0) \subseteq \overline{P(x_0)}$ the closure taken in $C(T,X)$. Appropriately modifying her arguments (which are based on properties of the set-valued integral) we can extend her result to the broader class of systems considered here, without assuming that $S(t,s)$ is compact for $t-s>0$. Note that Lemma 2.4 of F r a n k o w s k a [25], which is the main tool in the proof of Theorem 2.5 is an immediate consequence of the more general Theorem 3.1 of K a n d i l a k i s – P a p a g e o r g i o u [11]. That general result on set-valued integration permits us to extend Frankowska's Theorem 2.5 to our more general setting. In Corollary 2.6, F r a n k o w s k a [24] proves that $P_r(x_0) = \overline{P(x_0)}$, the closure taken in $C(T,X)$ under the assumption that X is reflexive and the unbounded operator A is time invariant. Again remark that condition (ii) is covered by condition (i) in that corollary. T o l s t o n o g o v [26], considers evolution inclusions where A is a time invariant, generally nonlinear m-accretive operator. However, he assumes that X^* is strictly convex and that $F(t,x)$ has compact values. We believe that this last hypothesis (see also condition (iii) in Corollary 2.6 of F r a n k o w s k a [25]) is very restrictive and is not satisfied in most applied problems (like control systems). Finally, our proof differs from those of F r a n k o w s k a [25] and T o l s t o n o g o v [26].

4. A Filippov-Gronwall type theorem

In this section we prove a Filippov-Gronwall type theorem, which we will use in the proof of our main result in Section 5 (Theorem 5.2) and which we believe can be useful in the study of infinite dimensional control systems. So we show that given an approximate mild solution of the evolution inclusion, we can find an exact mild solution, so that the difference of the two satisfies a Filippov-Gronwall type inequality. This result was first proved for differential inclusions in $\mathbb{R}^n$ by F i l i p p o v [7], and here we extend it to evolution inclusions, following his proof. Analogous results were recently proved by F r a n k o w s k a [25, Theorem 1.2] and T o l s t o n o g o v [26, Theorem 2.1] for evolution inclusions with time-independent unbounded operators. Note that T o l s t o n o g o v [26] assumes that $\delta = \|x_0 - z_0\| = 0$. The proofs of these results as well as our proof, follow closely the original one due to F i l i p p o v . Our result is true with the following weaker hypothesis on $A(t)$.

$H(A)_1$: $\{-A(t) : t \in T\}$ is a family of densely defined, closed linear operators that generates an evolution operator $S\colon \Delta \to \mathcal{L}(X)$.

THEOREM 4.1. *If hypotheses* $H(A)_1$ *and* $H(F)_2$ *hold,* $g \in L^1(X)$, $z(\cdot) \in C(T,X)$ *is the mild solution of* $\dot z(t) + A(t)z(t) = g(t)$, $z(0) = z_0$

and $t \mapsto p(t) = d\big(g(t), F(t, z(t))\big) \in L^1_+$, *then there exists a mild solution* $x(\cdot) \in C(T, X)$ *of* $(*)$ *s.t.*

$$\|x(t) - z(t)\| \le \delta \exp\Big(M \int_0^t k(s)\, \mathrm{d}s\Big) + \int_0^t p(\tau) \exp\Big(M \int_\tau^t k(s)\, \mathrm{d}s\Big)\, \mathrm{d}\tau\,,$$

where $\delta = \|z_0 - x_0\|$.

Proof. Let $m(t) = \int_0^t k(s)\, \mathrm{d}s$. Through a straightforward application of Theorem 5.10 of Wagner [23], we can find $v_0 \colon T \to X$ measurable s.t. $v_0(t) \in F\big(t, z(t)\big)$ a.e. and $d\big(g(t), F(t, z(t))\big) = \|g(t) - v_0(t)\| = p(t)$ a.e. Clearly $v_0(\cdot) \in L^1(X)$. Set $x_1(t) = S(t,0)x_0 + \int_0^t S(t,s) v_0(s)\, \mathrm{d}s$. Then we have $\|x_1(t) - z(t)\| \le M\|x_0 - z_0\| + M \int_0^t p(s)\, \mathrm{d}s$ (recall $\|S(t,s)\|_{\mathcal{L}} \le M$ for all $(t,s) \in \Delta$).

Suppose that we have constructed $x_1(\cdot), \dots, x_n(\cdot) \in C(T, X)$, so that for $r \in \{1, \dots, n\}$ we have:

(a) $x_r(t) = S(t,0)x_0 + \int_0^t S(t,s) v_{r-1}(s)\, \mathrm{d}s$, $t \in T$
with $v_{r-1}(s) \in F\big(s, x_{r-1}(s)\big)$ a.e.,

(b) $\|v_{r-1}(t) - v_{r-2}(t)\|$
$$\le k(t)\left[\delta \frac{M^{r-2} m(t)^{r-2}}{(r-2)!} + \int_0^t \frac{M^{r-2}\big(m(t) - m(s)\big)^{r-2}}{(r-2)!} p(s)\, \mathrm{d}s\right].$$

Observe that from (b) above, we have

$$\begin{aligned} &\|x_{n-1}(t) - x_n(t)\| \\ &\le \Big\| \int_0^t S(t,s)\big(v_{n-2}(s) - v_{n-1}(s)\big)\, \mathrm{d}s \Big\| \\ &\le M \int_0^t \|v_{n-2}(s) - v_{n-1}(s)\|\, \mathrm{d}s \end{aligned}$$

$$\leq M\left[\int_0^t k(s)\frac{\delta M^{n-2}m(s)^{n-2}}{(n-2)!}\,\mathrm{d}s\right.$$
$$\left.+\int_0^t\int_0^s \frac{M^{n-2}\big(m(s)-m(\tau)\big)^{n-2}}{(n-2)!}k(s)p(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\right]$$
$$=\int_0^t \frac{M^{n-1}\delta}{(n-1)!}\,\mathrm{d}\big(m(s)\big)^{n-1}+\frac{M^{n-1}}{(n-1)!}\int_0^t p(\tau)\int_\tau^t \frac{\mathrm{d}}{\mathrm{d}s}\big(m(s)-m(\tau)\big)^{n-1}\,\mathrm{d}s\,\mathrm{d}\tau$$
$$=\frac{\delta M^{n-1}}{(n-1)!}m(t)^{n-1}+\frac{M^{n-1}}{(n-1)!}\int_0^t p(\tau)\big(m(t)-m(\tau)\big)^{n-1}\,\mathrm{d}\tau\,. \tag{1}$$

Via a new, easy application of Theorem 5.10 of W a g n e r [23], we can get a measurable function $v_n\colon T\to X$ s.t. $v_n(t)\in F\big(t,x_n(t)\big)$ a.e. and

$$\|v_n(t)-v_{n-1}(t)\|=d\big(v_{n-1}(t),F\big(t,x_n(t)\big)\big)\leq h\big(F\big(t,x_{n-1}(t)\big),F\big(t,x_n(t)\big)\big)$$
$$\leq k(t)\|x_{n-1}(t)-x_n(t)\|\quad\text{a.e.}$$

Using (1) above we get that

$$\|v_n(t)-v_{n-1}(t)\|\leq k(t)\left[\frac{\delta M^{n-1}}{(n-1)!}m(t)^{n-1}+\frac{M^{n-1}}{(n-1)!}\int_0^t p(\tau)\big(m(t)-m(\tau)\big)^{n-1}\,\mathrm{d}\tau\right],$$

which gives us part (b) of the induction process.

Set $x_{n+1}(t)=S(t,0)x_0+\int_0^t S(t,s)v_n(s)\,\mathrm{d}s$, $t\in T$, $v_n(t)\in F\big(t,x_n(t)\big)$ a.e. to get also part (a) of the induction process.

By setting $x_0=z\in C(T,X)$, we have

$$\|x_{n+1}(t)-z(t)\|\leq\sum_{k=0}^n\|x_{k+1}(t)-x_k(t)\|$$
$$\leq\sum_{k=0}^n\left[\frac{\delta M^k}{k!}m(t)^k+\frac{M^k}{k!}\int_0^t p(\tau)\big(m(t)-m(\tau)\big)^k\,\mathrm{d}\tau\right]$$
$$\leq\delta\exp\Big(M\int_0^t k(s)\,\mathrm{d}s\Big)+\int_0^t p(\tau)\exp\Big(M\int_\tau^t k(s)\,\mathrm{d}s\Big)\,\mathrm{d}\tau\,. \tag{2}$$

From (a) and (b) above, we have that $\{x_n(\cdot)\}_{n\geq 1}$ is Cauchy in $C(T,X)$ while $\{v_n(\cdot)\}_{n\geq 1}$ is Cauchy in $L^1(X)$. So let $x_n \xrightarrow{s} x$ in $C(T,X)$ and $v_n \xrightarrow{s} v$ in $L^1(X)$. We have

$$v(t) \in h - \lim F\big(t, x_n(t)\big) = F\big(t, x(t)\big) \quad \text{a.e.}$$

and

$$x(t) = S(t,0)x_0 + \int_0^t S(t,s)v(s)\,\mathrm{d}s\,, \qquad t \in T; \quad \text{i.e. } x(\cdot) \in P(x_0)\,.$$

Furthermore, from (2) above, we get

$$\|x(t) - z(t)\| \leq \delta \exp\Big(M \int_0^t k(s)\,\mathrm{d}s\Big) + \int_0^t p(\tau) \exp\Big(M \int_\tau^t k(s)\,\mathrm{d}s\Big)\,\mathrm{d}\tau\,,$$

which proves the theorem. □

R e m a r k . If $A(t) \equiv 0$ and $X = \mathbb{R}^n$, we recover the result of F i l i p p o v [7]. If $A(t) = A$ (i.e. independent of t), we recover the result of F r a n k o w s k a [25].

Another result that we will need in the proof of our main theorem in Section 5 (Theorem 5.2) is the following. Assume (Ω, Σ, μ) is a σ-finite measure space, while as before X is a separable Banach space.

PROPOSITION 4.2. *If $F\colon \Omega \to P_f(X)$ is a measurable multifunction s.t. $S^1_{\overline{\mathrm{conv}}F} \neq \emptyset$ and $F(\omega) \neq \overline{\mathrm{conv}}F(\omega)$ for all $\omega \in \Omega_0$, $\mu(\Omega_0) > 0$, then there exists $f \in S^1_{\overline{\mathrm{conv}}F}$ s.t. $d\big(f(\omega), F(\omega)\big) \geq \beta > 0$ for all $\omega \in \Omega_1$, $\mu(\Omega_1) > 0$.*

P r o o f . Let $R\colon \Omega_0 \to 2^X \setminus \{\emptyset\}$ be defined by

$$\begin{aligned} R(\omega) &= \big\{x \in \overline{\mathrm{conv}}F(\omega) :\ x \notin F(\omega)\big\} \\ &= \big\{x \in X :\ d\big(x, \overline{\mathrm{conv}}F(\omega)\big) = 0 \text{ and } d\big(x, F(\omega)\big) > 0\big\}\,. \end{aligned}$$

Since $F(\cdot)$ is measurable, Theorem 9.1 of H i m m e l b e r g [10], tells us that $\overline{\mathrm{conv}}F(\cdot)$ is measurable too. Thus $\mathrm{Gr}\,R \in (\Sigma \cap \Omega_0) \times B(X)$ and so Aumann's selection theorem gives us $g\colon \Omega_0 \to X$ measurable s.t. $g(\omega) \in R(\omega)$ $\mu|_{\Omega_0}$-a.e.

Let $\Omega_n = \left\{\omega \in \Omega_0 : d\big(g(\omega), F(\omega)\big) \geq \frac{1}{n}\right\}$. Then clearly $\mu(\Omega_n) \uparrow \mu(\Omega_0) > 0$. So for some $n \geq 1$, we have $\mu(\Omega_n) > 0$. Since Ω is σ-finite we have an increasing sequence of Σ-sets $\hat{\Omega}_m$ s.t. $\mu(\hat{\Omega}_m) < \infty$ $m \geq 1$ and $\Omega = \bigcup\limits_{m\geq 1} \hat{\Omega}_m$. Set $\hat{\Omega}_n = \bigcup\limits_{m\geq 1} \Omega_n \cap \hat{\Omega}_m$. Again as before, we can find $m \geq 1$ s.t. $0 < \mu(\Omega_n \cap \hat{\Omega}_m) = \mu(\Omega_{nm}) < \infty$. Then set $\Omega_{nmk} = \big\{\omega \in \Omega_{nm} : \|g(\omega)\| \leq k\big\}$. Since $\Omega_{nm} = \bigcup\limits_{k\geq 1} \Omega_{nmk}$, for some $k \geq 1$ $\mu(\Omega_{nmk}) > 0$. Let $h \in S^1_{\mathrm{conv}F}$ (such a function exists by hypothesis) and define

$$f = \chi_{\Omega_{nmk}} g + \chi_{\Omega^c_{nmk}} h .$$

Clearly $f \in S^1_{\mathrm{conv}F}$ and $d\big(f(\omega), F(\omega)\big) \geq \frac{1}{n} > 0$ for all $\omega \in \Omega_{nmk}$,. $\mu(\Omega_{nmk}) > 0$. □

5. Convexity of the orientor field

In this section, let (X, H, X^*) be an evolution triple of spaces (see Section 2), and assume in addition that $X \to H$ compactly (hence $H \to X^*$ compactly too; see Zeidler [24]). Recall $\langle \cdot , \cdot \rangle$ denotes the duality brackets for (X, X^*) and $(\cdot , \cdot)$ the inner product in H. Also $\|\cdot\|$ (resp. $|\cdot|$, $\|\cdot\|_*$) denotes the norm of X (resp. of H, X^*). We will need the following hypothesis on $A(t)x$.

$H(A)_2$: $A\colon T \to \mathcal{L}(X, X^*)$ is a map s.t.
(1) $\|A(t) - A(t')\|_{\mathcal{L}} \leq \ell|t - t'|$ for some $\ell > 0$,
(2) $\|A(t)x\|_* \leq \gamma\|x\|$ for all $t \in T$ and some $\gamma > 0$,
(3) $\langle A(t)x, x\rangle \geq c_1\|x\|^2$ for all $t \in T$ and some $c_1 > 0$
(i.e. $A(t)(\cdot)$ is strongly monotone, uniformly in $t \in T$).

Then given $f \in L^2(H)$, from Tanabe [22] and Zeidler [24], we know that there exists $x(\cdot) \in W(T)$ s.t. $\dot{x}(t) + A(t)x(t) = f(t)$ a.e., $x(0) = x_0$. Furthermore from Proposition 5.5.1 of Tanabe [22, p. 153], we know that $\big\{-A(t) : t \in T\big\}$ generates an evolution operator $S\colon \Delta \to \mathcal{L}(H)$ s.t.

$$x(t) = S(t,0)x_0 + \int_0^t S(t,s)f(s)\,\mathrm{d}s , \qquad t \in T .$$

Now we turn our attention to evolution inclusion $(*)$, and we make the following hypothesis concerning the orientor field $F(t,x)$.

$H(F)_3$: $F\colon T \times H \to P_{fc}(H)$ is a multifunction s.t.
(1) $t \mapsto F(t,x)$ is measurable,

(2) $h_H\big(F(t,x),F(t,x')\big) \le k(t)|x-x'|$ a.e., with $k(\cdot) \in L^1_+$
(here $h_H(\cdot\,,\cdot)$ denotes the Hausdorff metric on $P_f(H)$),
(3) $|F(t,x)| \le a(t)+c(t)|x|$ a.e. with $a(\cdot),c(\cdot) \in L^2_+$.

Denote the solution set by $P(x_0)$. Recall (see Section 2), that $P(x_0) \subseteq W(T)$.

PROPOSITION 5.1. *If hypotheses $H(A)_2$ and $H(F)_3$ hold and $x_0 \in H$, then $P(x_0)$ is a weakly compact subset of $W(T)$.*

Proof. Let $x(\cdot) \in P(x_0)$. Then by definition we have

$$\dot{x}(t)+A(t)x(t)=f(t) \quad \text{a.e.}, \qquad x(0)=x_0$$

with $f \in S^2_{F(\cdot,x(\cdot))}$. So using Proposition 23.23 of Zeidler [24, p. 422], we get

$$\langle \dot{x}(t),x(t)\rangle + \langle A(t)x(t),x(t)\rangle = \big(f(t),x(t)\big) \quad \text{a.e.}$$

$$\Longrightarrow \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|x(t)|^2 + c_1\|x(t)\|^2 \le |f(t)||x(t)| \le |f(t)|\beta\|x(t)\| \quad \text{a.e.,} \tag{1}$$

where $\beta>0$ is such that $|\cdot| \le \beta\|\cdot\|$. It exists since by hypothesis $X \to H$ continuously. Then using Cauchy's inequality with $\varepsilon>0$, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|x(t)|^2 + c_1\|x(t)\|^2 \le \frac{\varepsilon}{2}|f(t)|^2 + \frac{\beta}{2\varepsilon}\|x(t)\|^2 \quad \text{a.e.}$$

$$\Longrightarrow |x(t)|^2 + 2c_1\int_0^t \|x(s)\|^2\,\mathrm{d}s \le \varepsilon\int_0^t |f(s)|^2\,\mathrm{d}s + \frac{\beta}{\varepsilon}\int_0^t \|x(s)\|^2\,\mathrm{d}s + |x_0|^2$$

$$\le \varepsilon\int_0^t \big(2a(s)^2+2c(s)^2|x(s)|^2\big)\,\mathrm{d}s + \frac{\beta}{\varepsilon}\int_0^t \|x(s)\|^2\,\mathrm{d}s + |x_0|^2.$$

Let $\varepsilon = \dfrac{\beta}{2c_1}$. We get

$$|x(t)|^2 \le \frac{\beta}{2c_1}\|a\|_2^2 + \frac{\beta}{2c_1}\int_0^t c(s)^2|x(s)|^2\,\mathrm{d}s + |x_0|^2.$$

Invoking Gronwall's inequality we get that there exists $M_1>0$ s.t.

$$|x(t)| \le M_1 \tag{2}$$

for all $t \in T$ and all $x(\cdot) \in P(x_0)$.

Using (2) in (1), we get

$$2c_1 \int_0^b \|x(t)\|^2 \, dt \le \int_0^b \big(a(s) + b(s)M_1\big)M_1 \, ds$$

$$\Longrightarrow \|x\|_{L^2(X)} \le M_2$$

for some $M_2 > 0$ and all $x(\cdot) \in P(x_0)$.

Finally let $h \in L^2(X)$ and denote by $((\cdot\,,\cdot))_0$ the duality brackets for the pair $\big(L^2(X), L^2(X^*)\big)$; i.e., $((v,h))_0 = \int_0^b \langle v(t), h(t)\rangle \, dt$ for all $v \in L^2(X^*)$, $h \in L^2(X)$. Let $\hat{A}\colon L^2(X) \to L^2(X^*)$ be defined by $(\hat{A}x)(t) = A(t)x(t)$. Clearly $\hat{A}(\cdot)$ is linear, monotone hence continuous (see Zeidler [24, p. 596]). Then we have

$$((\dot{x},h))_0 + ((\hat{A}(x),h))_0 = ((f,h))_0$$

$$\begin{aligned} ((\dot{x},h))_0 &\le \big[\|\hat{A}(x)\|_{L^2(X^*)} + \|f\|_{L^2(X^*)}\big]\|h\|_{L^2(X)} \\ &\le \big(\gamma M_2 + \|a\|_2 + M_1\|b\|_2\big)\|h\|_{L^2(X)} \end{aligned}$$

$$\Longrightarrow \|\dot{x}\|_{L^2(X^*)} \le M_3 \tag{3}$$

for some $M_3 > 0$ and all $x(\cdot) \in P(x_0)$.

From (2) and (3), we deduce that $P(x_0)$ is bounded in $W(T)$ and since the latter is a separable, reflexive Banach space, we deduce that $P(x_0)$ is relatively weakly compact in $W(T)$. To finish the proof, we need to show that $P(x_0)$ is w-closed in $W(T)$. So let $\{x_n\}_{n\ge1} \subseteq P(x_0)$ and assume $x_n \xrightarrow{w} x$ in $W(T)$. Then by definition

$$\dot{x}_n + \hat{A}(x_n) = f_n \tag{4}$$

with $f_n \in S^1_{F(\cdot,x_n(\cdot))}$. Note that $|f_n(t)| \le a(t) + c(t)M_1$ a.e. (hypothesis $H(F)_3(3)$). So by passing to a subsequence if necessary, we may assume that $f_n \xrightarrow{w} f$ in $L^2(H)$. Since $x_n \xrightarrow{w} x$ in $W(T)$ and $W(T) \to L^2(H)$ compactly we have $x_n \xrightarrow{s} x$ in $L^2(H)$. So using this and Theorem 3.1 of [15], we get that $f \in S^2_{F(\cdot,x(\cdot))}$. Also $\dot{x}_n \xrightarrow{w} \dot{x}$ in $L^2(X^*)$ (because $x_n \xrightarrow{w} x$ in $W(T)$) and $\hat{A}(x_n) \xrightarrow{w} \hat{A}(x)$ in $L^2(X^*)$ (because $\hat{A}(\cdot)$ is continuous, linear). So passing to the limit as $n \to \infty$ in (4) above, we get

$$\dot{x} + \hat{A}(x) = f$$

with $f \in S^2_{F(\cdot,x(\cdot))}$ and $x(0) = x_0$. Hence $x \in P(x_0) \implies P(x_0)$ is w-compact in $W(T)$. $\square$

Remark. Since $W(T) \to L^2(H)$ compactly (see Zeidler [24, p. 450]), Proposition 5.1 tells us that $P(x_0)$ is compact in $L^2(H)$. Also if X is a Hilbert space too, then from Nagy [13], we know that $W(T) \to C(T, H)$ compactly, and so $P(x_0)$ is compact in $C(T, H)$. Finally since $W(T) \to W^{1,2}(X^*) = H^1(X^*) = AC^{1,2}(X^*)$, we have that $P(x_0)$ is weakly closed in $AC^{1,2}(X^*)$.

In the next theorem, we will obtain a kind of converse of this last observation concerning the set $P(x_0)$. Our result extends to evolution inclusions Theorem 1 of Cellina–Ornelas [4]. So we will show that if $P(x_0)$ is closed in $AC^{1,2}(X^*)$ endowed with the weak topology (denoted henceforth by $AC^{1,2}(X^*)_w$), then for all $t \in T \setminus N$, $\lambda(N) = 0$ (λ is the Lebesgue measure on T) and all $x \in H$, we have $F(t, x)$ is convex. Our proof was inspired by that of Cellina–Ornelas [4]. We will need the following additional hypothesis.

H_c: For all $(t, s) \in \Delta$, $t - s > 0$, $S(t, s)$ is a compact operator.

THEOREM 5.2. *If hypotheses $H(A)_2, H(F)_3$ (with $F(\cdot,\cdot)$ only $P_f(H)$-valued) and H_c hold, and for ever initial time t_0 and initial state $x_0 \in H$, there exists an interval $T_\delta = [t_0, t_0 + \delta]$ on which the solution set $P(x_0)$ of the evolution inclusion $\dot{x}(t) + A(t)x(t) \in F(t, x(t))$ a.e. on T_δ, $x(t_0) = x_0$ is closed in $AC^{1,2}(T_\delta, X^*)_w$, then $F(t, x) \in P_{fc}(H)$ for all $t \in T \setminus N$, $\lambda(N) = 0$ and all $x \in H$.*

Proof. Suppose not. Then there exists $x_0 \in H$ s.t. $F(t, x_0) \subsetneq \overline{\text{conv}} F(t, x_0)$ for all $t \in T_0 \subseteq T$, $\lambda(T_0) > 0$. Then the Proposition 4.2, we know that there exists $f \in S^2_{\overline{\text{conv}} F(\cdot, x_0)}$ s.t.

$$d_H\big(f(t), F(t, x_0)\big) \geq \theta \geq 0$$

for all $t \in T_1$, $\lambda(T_1) > 0$ and $T_1 \subseteq T_0$ (here $d_H(\cdot, B)$ denotes the distance function from a set B in H).

Let $t_0 \in T_1$ be a point of density of T_1, interior of T s.t.

$$\lim_{\delta \to 0} \frac{1}{\delta} \int_{t_0}^{t_0+\delta} k(t) \exp\left(M \int_{t_0}^{t} k(s)\, \mathrm{d}s\right) \mathrm{d}t = k(t_0), \qquad [t_0, t_0 + \delta] \subseteq T.$$

Such a point exists by Theorem 18.2 of Hewitt–Stromberg [9, p. 274].

Set $\eta(\delta) = M \int\limits_{t_0}^{t_0+\delta} |f(s)| \, \mathrm{d}s$ and $x(\cdot) \in W\big([t_0, b]\big) \subseteq C\big([t_0, b], H\big)$ by

$$x(t) = S(t, t_0)x_0 + \int\limits_{t_0}^{t} S(t, s)f(s) \, \mathrm{d}s\,, \qquad t \in [t_0, b]\,.$$

Then we have

$$|x(t) - S(t, t_0)x_0| \le \eta(\delta) \qquad \text{for} \quad t \in [t_0, t_0 + \delta]$$

and

$$\begin{aligned} d_H\big(f(t), \overline{\mathrm{conv}} F\big(t, x(t)\big)\big) &\le h_H\big(\overline{\mathrm{conv}} F(t, x_0), \overline{\mathrm{conv}} F\big(t, x(t)\big)\big) \le k(t)|x_0 - x(t)| \\ &\le k(t)\big[|x_0 - S(t, t_0)x_0| + |S(t, t_0)x_0 - x(t)|\big]\,. \end{aligned}$$

Since the evolution operator $(t, s) \mapsto S(t, s)$ is continuous from Δ into $\mathcal{L}(H)$ with the strong operator topology and $S(t_0, t_0) = I$, given $\varepsilon > 0$, we can find $\delta(\varepsilon) > 0$ s.t. if $t \in [t_0, t_0 + \delta]$, then $|x_0 - S(t, t_0)x_0| < \varepsilon$. So we have

$$d_H\big(f(t), \overline{\mathrm{conv}} F\big(t, x(t)\big)\big) \le k(t)\big[\eta(\delta) + \varepsilon\big]\,.$$

From Theorem 4.1 we know that we can find $\hat{x}(\cdot) \in W(T_\delta)$, $T_\delta = [t_0, t_0 + \delta]$ s.t.

$$\left.\begin{aligned} \dot{\hat{x}}(t) + A(t)\hat{x}(t) \in \overline{\mathrm{conv}} F\big(t, \hat{x}(t)\big) \ \text{ a.e. on } T_\delta \\ x(t_0) = \dot{x}_0 \end{aligned}\right\} \qquad (*)'_r$$

and also

$$\begin{aligned} |\hat{x}(t) - x(t)| &\le \int\limits_{t_0}^{t} \exp\bigg(M \int\limits_{s}^{t} k(\tau) \, \mathrm{d}\tau\bigg) k(s)\big(\eta(\delta) + \varepsilon\big) \, \mathrm{d}s \\ &\le \frac{\eta(\delta) + \varepsilon}{M} \int\limits_{t_0}^{t} \mathrm{d}\bigg(\exp\bigg(M \int\limits_{s}^{t} k(\tau) \, \mathrm{d}\tau\bigg)\bigg) \\ &= \frac{\eta(\delta) + \varepsilon}{M} \bigg(\exp\bigg(M \int\limits_{t_0}^{t} k(\tau) \, \mathrm{d}\tau\bigg) - 1\bigg). \end{aligned}$$

We will show that $\hat{x}(\cdot) \in W(T_\delta)$ cannot be a solution of the Cauchy problem $\dot{y}(t) + A(t)y(t) \in F\big(t, y(t)\big)$ a.e. on T_δ, $y(0) = x_0$. To this and we have

$$0 < \theta \le d_*\big(f(t), F(t, x_0)\big) \le \|f(t) - \dot{\hat{x}}(t) - A(t)\hat{x}(t)\|_* \\ + d_*\big(\dot{\hat{x}}(t) + A(t)\hat{x}(t), F\big(t, \hat{x}(t)\big)\big) + h_*\big(F\big(t, \hat{x}(t)\big), F(t, x_0)\big) \text{ a.e. on } T_1 \cap T_\delta, \tag{1}$$

where $d_*(\cdot\,, B)$ denotes the distance function from a set B in X^* and $h_*(\cdot\,,\cdot)$ is the Hausdorff metric on $P_f(X^*)$. From the proof of Theorem 4.1, and in particular part (b) of the induction process, we have:

$$\|f(t) - \dot{\hat{x}}(t) - A(t)\hat{x}(t)\|_* \le \beta' k(t)\big(\eta(\delta) + \varepsilon\big) \exp\Big(M \int_{t_0}^{t} k(\tau)\, \mathrm{d}\tau\Big) \text{ a.e. on } T_\delta \tag{2}$$

where $\beta' > 0$ is such that $\|\cdot\|_* \le \beta'|\cdot|$. It exists since H embeds into X^* continuously. Similarly, if $h_H(\cdot\,,\cdot)$ and $h_*(\cdot\,,\cdot)$ are the Hausdorff metrics on $P_f(H)$ and $P_f(X^*)$ respectively, we have $h_*(\cdot\,,\cdot) \le \beta' h_H(\cdot\,,\cdot)$. So we can write

$$\begin{aligned} &h_*\big(F\big(t, \hat{x}(t)\big), F(t, x_0)\big) \\ &\le \beta' h_H\big(F\big(t, \hat{x}(t)\big), F(t, x_0)\big) \le \beta' k(t)|\hat{x}(t) - x_0| \\ &\le \beta' k(t)\big[|\hat{x}(t) - x(t)| + |x(t) - S(t, t_0)x_0| + |S(t, t_0)x_0 - x_0|\big] \\ &\le \beta' k(t)\left[\frac{\eta(\delta) + \varepsilon}{M}\Big(\exp\Big(M \int_{t_0}^{t} k(\tau)\, \mathrm{d}\tau\Big) - 1\Big) + \eta(\delta) + \varepsilon\right] \\ &= \beta' k(t)\big(\eta(\delta) + \varepsilon\big) \exp\Big(M \int_{t_0}^{t} k(\tau)\, \mathrm{d}\tau\Big) \end{aligned} \tag{3}$$

since we may assume without any loss of generality that $M \ge 1$. Recall $M > 0$ is such that $\|S(t, s)\|_{\mathcal{L}} \le M$ for all $(t, s) \in \Delta$.

Using estimates (2) and (3) in (1), we get

$$\theta - k(t)\big(\eta(\delta) + \varepsilon\big) \exp\Big(M \int_{t_0}^{t} k(\tau)\, \mathrm{d}\tau\Big) \le d_*\big(\dot{\hat{x}}(t) + A(t)\hat{x}(t), F\big(t, \hat{x}(t)\big)\big) \\ \text{a.e. on } \quad T_1 \cap T_\delta.$$

Let $T'_\delta = \left\{t \in T_\delta = [t_0, t_0+\delta] : \ k(t)\big(\eta(\delta)+\varepsilon\big)\exp\Big(M\int_{t_0}^{t} k(\tau)\,\mathrm{d}\tau\Big) > \frac{\theta}{2}\right\}$ and choose $\delta > 0$ small enough, so that $\lambda(T_1 \cap T_\delta) = \lambda\big(T_1 \cap [t_0, t_0+\delta]\big) > \frac{2\delta}{3}$. This is possible because t_0 was chosen to be a density point for T_1. Also note that

$$\big(\eta(\delta)+\varepsilon\big)\frac{1}{\delta}\int_{t_0}^{t_0+\delta} k(t)\exp\Big(M\int_{t_0}^{t} k(t)\,\mathrm{d}\tau\Big)\,\mathrm{d}t$$

$$= \frac{1}{\delta}\int_{t_0}^{t_0+\delta} k(t)\big(\eta(\delta)+\varepsilon\big)\exp\Big(M\int_{t_0}^{t} k(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}t > \frac{\theta}{2}\frac{\lambda(T'_\delta)}{\delta} \tag{4}$$

(since the integral is positive).

Now observe that the left-hand side in inequality (4) above, goes to ε as $\delta \to 0^+$ (recall the choice of $t_0 \in T_1$ in the beginning of the proof). So we get

$$\frac{\theta}{2}\lim_{\delta\downarrow 0}\frac{\lambda(T'_\delta)}{\delta} \le \varepsilon\,.$$

But $\varepsilon > 0$ was arbitrary and $\theta > 0$. So $\lim\limits_{\delta\downarrow 0}\frac{\lambda(T'_\delta)}{\delta} = 0$. In particular then for $\delta > 0$ small enough, we have $\lambda(T'_\delta) < \frac{\delta}{3}$. Hence we get

$$\lambda\big((T_1 \cap T_\delta)\setminus T'_\delta\big) > \frac{\delta}{3}$$

and for $t \in (T_1 \cap T_\delta)\setminus T'_\delta$ we have

$$0 < \frac{\theta}{2} = \theta - \frac{\theta}{2} \le d_*\big(\dot{\hat{x}}(t) + A(t)\hat{x}(t), F\big(t, \hat{x}(t)\big)\big)\,,$$

which means that $\hat{x}(\cdot)$ is not a solution of $(*)$ on T_δ. However $\hat{x}(\cdot)$ is by choice a solution to the relaxed problem $(*)'_r$. So from Theorem 3.3, we can find $\{x_n\}_{n\ge1}$ solutions of the nonconvex problem on T_δ, emanating from x_0 at t_0 s.t. $x_n \xrightarrow{s} \hat{x}$ in $C(T_\delta, H)$. Recall from the a priori bounds derived in the proof of the Proposition 5.1, that $\{x_n\}_{n\ge1}$ is bounded in $W(T_\delta)$. So by passing to a subsequence if necessary, we may assume that $x_n \xrightarrow{w} y$ in $W(T_\delta)$. Clearly since $W(T_\delta) \to C(T_\delta, H)$ continuously, $y = \hat{x}$. Then in particular we have $\dot{x}_n \xrightarrow{w} \dot{\hat{x}}$ in $L^2(T_\delta, X^*) \implies x_n \xrightarrow{w} x$ in $AC^{1,2}(T_\delta, X^*)$. But by hypothesis the solution set of the evolution inclusion $\dot{y}(t) + A(t)y(t) \in F\big(t, y(t)\big)$ a.e., $y(0) = x_0$ on $T_\delta = [t_0, t_0+\delta]$ is closed in $AC^{1,2}(T_\delta, X^*)_w$. So $\dot{\hat{x}}(\cdot)$ is a solution of the Cauchy problem $\dot{y}(t) + A(t)y(t) \in F\big(t, y(t)\big)$ a.e. on T_δ, $y(0) = x_0$, a contradiction.

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