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QUASICONTINUOUS SELECTIONS FOR COMPACT–VALUED MULTIFUNCTIONS

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ABSTRACT. In this paper quasicontinuous selection theorems for multifunctions $F\colon X \to Y$ with compact values in special metric spaces are presented. The method used here enables us to work with an arbitrary topological space X.

1. Introduction

The research in selection theory was started by M i c h a e l in 1956 (see for example [6, 7]) by proving several continuous selection theorems. Then, the problem of the existence of selections of various types (measurable, Carathéodory, Darboux, etc.) was studied in many papers. The first paper dealing with the problem of existence of quasicontinuous selections for multifunctions was the paper of M a t e j d e s published in 1987 ([5]). The paper gives some conditions for the existence of quasicontinuous selections for multifunctions $F\colon X \to Y$ with compact values, where X is a Baire space and Y is a compact metric space. Quasicontinuous selection theorems for one-to-finite multifunctions are proved in [2].

In this paper we prove some quasicontinuous selection theorems for multifunctions with compact values. Our method is different from that of Matejdes. We show, how to find a quasicontinuous selection when the space X is an arbitrary topological one. Assuming nothing about our space X we must suppose more about F than Matejdes did. Our results offer some new contribution for the theory of selections in hyperspaces, too.

2. Preliminaries

Let X and Y be two sets; let $P(Y)$ denote the set of all nonempty subsets of Y. A *multifunction* from X to Y is a function F from X to $P(Y)$. We

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write $F\colon X \to Y$. A *selection* for a multifunction $F\colon X \to Y$ is a function $f\colon X \to Y$ such that for each x in X $f(x)$ is an element of $F(x)$.

In what follows X and Y will be topological spaces. A multifunction F from X to Y is called *lower* (*upper*) *semicontinuous* – briefly l.s.c. (u.s.c.) – if for each open subset G of Y the set $F^-(G) = \{x;\ F(x) \cap G \neq \emptyset\}$ ($F^+(G) = \{x;\ F(x) \subset G\}$) is an open subset of X.

A subset B of X is said to be *semiopen* if there exists an open set G such that $G \subset B \subset \operatorname{cl} G$ or, which is equivalent, $B \subset \operatorname{cl}(\operatorname{int} B)$ ([4]). $\operatorname{int} B$ and $\operatorname{cl} B$ denote the interior and the closure of the set B respectively. We will use the following properties of semiopen sets ([4]):

1. The union of any family of semiopen sets is semiopen.
2. The intersection of an open set and a semiopen one is semiopen.

We say that a set S is a *semineighbourhood of a point* x if there exists a semiopen set A such that $x \in A \subset S$ holds.

A function $f\colon X \to Y$ is said to be *quasicontinuous at* $x \in X$ if for each open $G \subset Y$ such that $f(x) \in G$ the set $f^-(G)$ is a semineighbourhood of x. A function $f\colon X \to Y$ is quasicontinuous at $x \in X$ if and only if for any open set V such that $f(x) \in V$ and any open set U such that $x \in U$, there exists a nonempty open set $W \subset U$ such that $f(W) \subset V$. ([1, 4, 10]).

By [9] a multifunction $F\colon X \to Y$ is said to be *upper quasicontinuous* – briefly u-quasicontinuous or u.q.c. (*lower quasicontinuous* – briefly l-quasicontinuous or l.q.c.) *at a point* $x \in X$ if for any open set V containing $F(x)$ (for any point z from $F(x)$ and for any open set V containing z) and any neighbourhood U of x, there exists a nonempty open set $W \subset U$ such that $F(t) \subset V$ ($F(t) \cap V \neq \emptyset$) for every $t \in W$.

3. Results

Let (X, ρ) be a metric space. In what follows we denote $d(x, A) = \inf\{\rho(x, a);\ a \in A\}$ for $x \in X$ and $A \subset X$. A metric space will be called "*b-space*" if every bounded closed subset of this space is compact. It will be called "*convex*" (analogically to concepts mentioned in [12 p. 5 and 114]) if

(∗) for every $a, b \in X$ $\forall \lambda \in (0,1)$ $\exists c \in X$ such that $\rho(a,c) = \lambda \cdot \rho(a,b)$ and $\rho(b,c) = (1-\lambda) \cdot \rho(a,b)$ holds.

LEMMA 1. *Let X be a topological space, (Y, ρ) a metric space. Let $F\colon X \to Y$ be a l.q.c. multifunction. Then for every $r > 0$ there exist an open dense subset U of X and a function $f\colon U \to Y$ continuous on U such that $\forall u \in U$ $d(f(u), F(u)) < r$.*

Proof. Let

$$Z = \big\{(g, D)\,;\ D \text{ is a nonempty open subset of } X \text{ and } g\colon D \to Y \text{ is continuous and such that } \forall\, t \in D \quad d\big(g(t),\, F(t)\big) < r\big\}\,.$$

The set Z is nonempty. It suffices to take an arbitrary $x \in X$ and an $y \in F(x)$ and to define a function g on the set $G = \operatorname{int} F^{-}\big(B(y, \frac{r}{2})\big)$ by $g(t) = y \quad \forall t \in G$. Then $(g, G) \in Z$ holds.

Define a partial order on Z as follows: $(h, A) \le (g, B)$ if and only if $A \subset B$ and $\forall\, a \in A \quad h(a) = g(a)$. Let L be a linearly ordered subset of Z. We will show that Z has an upper bound in Z. Let us define an ordered pair (p, P) as follows:

$$P = \bigcup_{(g,D)\in L} D \text{ and } p\colon P \to Y \text{ is a function such that } \forall\, x \in P \quad p(x) = g(x)$$

holds for every g such that there exists $(g, D) \in L$ and $x \in D$. From definition of p it follows: p is continuous and $\forall\, x \in P \quad d\big(p(x),\, F(x)\big) < r$. Therefore $(p, P) \in Z$ and we see that (p, P) is an upper bound of L in Z.

By Zorn's lemma the set Z has at least one maximal element. Let us denote one of these maximal elements by (m, M). We will show that M is dense in X. Suppose, to the contrary, that $\operatorname{cl} M \neq X$. Then $W = X - \operatorname{cl} M$ is a nonempty open set. Let us choose an $x \in W$ and an $y \in F(x)$. We denote $O = \operatorname{int}\big(F^{-}\big(B(y, \frac{r}{2})\big) \cap W\big)$. O is an interior of a nonempty semiopen set. Let us define $h\colon O \to Y$; $h(t) = y \quad \forall t \in O$. Then $(h, O) \in Z$. Next define a pair $(v, V) \in Z$ as follows: let $V = O \cup M$ and $v(x) = m(x)$ if $x \in M$, $v(x) = h(x)$ if $x \in O$. Then $(v, V) > (m, M)$ holds and this is a contradiction to the maximality of (m, M). Therefore M is dense in X and $m\colon X \to Y$ satisfies the assertion of our lemma.

LEMMA 2. *Let X be a topological space. Let (Y, ρ) be a convex metric space and a b-space. Let $F\colon X \to Y$ be an u.q.c. (u.s.c.) multifunction with compact values. Let $r > 0$. Then a multifunction $P\colon X \to Y$ defined as follows:*

$$\forall\, x \in X \quad P(x) = \overline{B}\big(F(x), r\big) = \{t \in Y\,;\ \exists\, y \in F(x) \quad \rho(t, y) \le r\}$$

is u.q.c. (u.s.c.), too.

Proof. We will consider the case when F is u.q.c.. Let $x \in X$, let $W \subset Y$ be an open set such that $P(x) \subset W$. Since $P(x)$ is a compact set, there exists $t > 0$ such that $V = B\big(P(x), t\big) \subset W$. F is u.q.c., so there exists a semiopen set, S such that $x \in S$ and $\forall\, s \in S \quad F(s) \subset B\big(F(x), \frac{t}{4}\big)$ holds. From the convexity of Y it follows that $\forall s \in S \quad P(s) = \overline{B}(F(s), r) \subset B(F(x), \frac{t}{2} + r) \subset B(\overline{B}(F(x), r), t) = B(P(x), t) \subset W$. P is proved to be u.q.c..

Example 1. We will show that without the convexity of Y Lemma 2 need not be true. Let $X = \langle 0, 1\rangle$, $Y = \langle 0, 1\rangle \cup (2, 3\rangle$ be two spaces with the natural metric derived from the absolute value. Let $F\colon X \to Y$ be defined as follows:

$$F(x) = \{x\} \quad \forall x \in X .$$

Let us define $P(x) = \overline{B}(F(x), 2) \quad \forall x \in X$. Then $P(0) = \langle 0, 1\rangle$ and for every $x \neq 0 \quad P(x) = \langle 0, 1\rangle \cup (2, 2 + x\rangle$. The multifunction F is u.s.c. but P is not u.q.c. in the point 0.

THEOREM 1. *Let X be a topological space. Let (Y, ρ) be a convex metric space and a b-space. Let $F\colon X \to Y$ be a l.q.c. and u.q.c. multifunction with compact values. Then $\forall r > 0$ there exists an $f\colon X \to Y$ quasicontinuous and such that $d(f(x), F(x)) < r \quad \forall x \in X$.*

Proof. By Lemma 1 there exist an open dense set $U \subset X$ and a continuous function $g\colon U \to Y$ such that $\forall x \in U \quad d(g(x), F(x)) < \frac{r}{2}$. Let us define a multifunction $P\colon X \to Y$ as follows: $\forall x \in X \quad P(x) = \overline{B}(F(x), \frac{r}{2})$. By Lemma 2 P is u.q.c.. Since P has compact values and $\forall u \in U \quad g(u) \in P(u)$ holds: By [2] (or by Lemma 2.3 in [3]) there exists a quasicontinuous extension $f\colon X \to Y$ of g such that f is a selection of P. From the definition of P it follows that $\forall x \in X \quad d(f(x), F(x)) \leq \frac{r}{2} < r$ holds.

LEMMA 3. ([3]) *Let X be a topological space, Y a normal topological space. Let $F\colon X \to Y$ be an u.s.c. multifunction with closed values, $H\colon X \to Y$ an u.q.c. multifunction with closed values. Let $\forall x \in X \quad F(x) \cap H(x) \neq \emptyset$. Then the multifunction $P(x) = H(x) \cap F(x)$ is u.q.c..*

Proof. (according to [3]): Let $x \in X$. Let $V \subset Y$ be open and let $P(x) \subset V$. We have to show that there exists a semiopen set O such that $x \in O$ and $P(O) \subset V$ holds. From the normality of Y we obtain that there exist two open disjoint sets U_1 and U_2 in Y such that $F(x) - V \subset U_1$ and $H(x) - V \subset U_2$ holds. So $F(x) \subset U_1 \cup V$ and $H(x) \subset U_2 \cup V$ holds. Since F is u.s.c. and H is u.q.c., there exist an open set O_1 and a semiopen set O_2 such that $x \in O = O_1 \cap O_2$ and for each point z from O_1 (resp. O_2) $F(z) \subset V \cup U_1$ (resp. $H(x) \subset V \cup U_2$) takes place. Therefore for each $z \in O$ $P(z) = F(z) \cap H(z) \subset V$ holds.

LEMMA 4. *Let X be a topological space. Let (Y, ρ) be a convex metric space. Let $F\colon X \to Y$ be a l.s.c. multifunction. Let $r > 0$ be a real number and*

$g\colon X \to Y$ a quasicontinuous function such that $\forall x \in X \;\; d(g(x), F(x)) < r$ holds. Then there exist an open dense subset $O \subset X$ and a continuous function $f\colon O \to Y$ such that $\forall x \in O \;\; d(f(x), F(x)) < \frac{r}{2}$ and $d(g(x), f(x)) < \frac{r}{2}$.

P r o o f . We will show that for every open nonempty set $U \subset X$ there exist an open set $W \subset U$ and a continuous function $h\colon W \to Y$ such that $\forall x \in W \;\; d(F(x), h(x)) < \frac{r}{2}$ and $d(g(x), F(x)) < \frac{r}{2}$. Let us take an arbitrary point $u \in U$. Then there exists a point $y \in F(u)$ such that $d(y, g(u)) < r$. Denote $v = d(y, g(u))$. From the convexity of Y we obtain that there exists a point t in Y such that $d(y,t) = \frac{v}{2} < \frac{r}{2}$ and $d(t, g(u)) = \frac{v}{2}$. Let $c > 0$ be such that $c + \frac{v}{2} < \frac{r}{2}$. F is l.s.c. so there exists an open neighbourhood $O(u)$ of u such that $\forall x \in O(u) \;\; F(x) \cap B(y,c) \neq \emptyset$ holds. From the quasicontinuity of g we obtain that there exists a semiopen set S such that $u \in S$ and $\forall x \in S$ $g(x) \in B(g(u), c)$. Let us denote $W = \operatorname{int}(U \cap O(u) \cap S)$. W is a nonempty open subset of U and $\forall x \in W \;\; d(g(x), t) \leq d(g(x), g(u)) + d(g(u), t) < \frac{r}{2}$ holds. Analogously $d(F(x), t) < \frac{r}{2}$ holds. Now, let us define a function $h\colon W \to Y$ such that $\forall w \in W \;\; h(w) = t$.

We omit the rest of the proof. It is analogous to the proof of Lemma 1 – it suffices to use Zorn's lemma for an appropriate system Z; the couple (h, W) will be one of its elements.

LEMMA 5. *Let X be a topological space. Let (Y, ρ) be a convex metric space and a b-space. Let $F\colon X \to Y$ be a l.s.c. and u.s.c. multifunction with compact values. Let $r > 0$. Let $f\colon X \to Y$ be a quasicontinuous function such that $\forall x \in X \;\; d(f(x), F(x)) < r$. Then there exists a quasicontinuous function $g\colon X \to Y$ such that $\forall x \in X \;\; d(f(x), g(x)) \leq \frac{r}{2}$ and $d(g(x), F(x)) \leq \frac{r}{2}$ holds.*

P r o o f . Let us define multifunctions $P\colon X \to Y$ and $H\colon X \to Y$ as follows: $\forall x \in X \;\; P(x) = \overline{B}(F(x), \frac{r}{2})$ and $H(x) = \overline{B}(f(x), \frac{r}{2})$. By Lemma 2 P is u.s.c. and H is u.q.c. (f being quasicontinuous, so u.q.c.). By Lemma 3 the multifunction $D\colon X \to Y$ defined by $D(x) = H(x) \cap P(x) \;\; \forall x \in X$; is u.q.c.. F and f fulfill the assumptions of Lemma 4. Hence there exist an open dense set $U \subset X$ and a continuous function $h\colon U \to Y$ such that $\forall u \in U \;\; h(u) \in D(u)$ holds. The function h is a continuous selection of D on the set U. By [2] the multifunction D has an quasicontinuous selection $g\colon X \to Y$. From the definition of D we obtain: $\forall x \in X \;\; d(f(x), g(x)) \leq \frac{r}{2}$ and $d(g(x), F(x)) \leq \frac{r}{2}$ holds.

E x a m p l e 2. Let $X = \mathbb{N} = \{1, 2, \dots\}$ be a topological space with the topology $T = \{A;\; A \subset \mathbb{N},\; \mathbb{N} - A \text{ is a finite set}\} \cup \{\mathbb{N}, \emptyset\}$. Let us define a multifunction $F\colon X \to \mathbb{R}$ as follows:

$$F(1) = \{1\}\,, \quad F(2) = \{2\}\,, \quad F(n) = \{1,2\} \qquad \text{if} \quad n \geq 3\,.$$

The multifunction F is l.s.c. and it is not u.s.c. only in the points 1 and 2. All other assumptions of Lemma 5 are satisfied. Let us consider the constant function $f\colon X \to \mathbb{R}$ defined by $f(n) = 1.5$ for every $n \in \mathbb{N}$. Then $\forall n \in \mathbb{N}$ $d\big(f(n),\, F(n)\big) < 0.8$ holds but there is no quasicontinuous function $g\colon X \to \mathbb{R}$ satisfying $d\big(g(n),\, F(n)\big) \leq 0.4 \quad \forall n \in \mathbb{N}$. It is easy to see, because all quasicontinuous functions from (X,T) to $\mathbb{R}$ are constant.

We are ready to prove our main result now.

THEOREM 2. *Let X be an arbitrary topological space. Let (Y,d) be a complete metric space, convex and a b-space. Let Z be a closed subset of Y. Let $F\colon X \to Z$ be a l.s.c. and u.s.c. multifunction with compact values. Then F has a quasicontinuous selection.*

Proof. F can be considered as a multifunction from X to Y. By Theorem 1 there exists a quasicontinuous function $g\colon X \to Y$ such that $\forall x \in X$ $d\big(F(x), g(x)\big) < \frac{3}{4}$ holds. Let us denote $f_1 = g$. Using Lemma 5 we can construct by induction a sequence $\{f_n\}_{n=1}^{\infty}$ of quasicontinuous functions $f_n\colon X \to Y$ such that $f_1 = g$ and $\forall n \geq 2$ the following two inequalities hold:

(a) $\forall x \in X \quad d\big(f_n(x),\, f_{n-1}(x)\big) < \left(\frac{3}{4}\right)^n,$

(b) $\forall x \in X \quad d\big(f_n(x),\, F(x)\big) < \left(\frac{3}{4}\right)^n.$

Hence the sequence $\{f_n\}_{n=1}^{\infty}$ is a fundamental sequence of quasicontinuous functions, it converges by Proposition 2.35 of [11] to a quasicontinuous function $f\colon X \to Y$. It follows from (b) and from the definition of f that f is a quasicontinuous selection of our multifunction F.

Example 3. We show that the assumption "F is u.s.c." in the Theorem 2 cannot be omitted.

Let $X = \{a,b,c\}$, let (X,T) be a topological space with the topology $T = \{\emptyset\} \cup \big\{\{a\},\, \{c,a\},\, \{b,a\},\, X\big\}$. Define $F\colon X \to \mathbb{R}$ as follows:

$$F(a) = \{1,2\}\,, \quad F(b) = \{1\}\,, \quad F(c) = \{2\}$$

F is a l.s.c. multifunction with compact values and F has no quasicontinuous selection.

It is well known that there is no continuous selection for the hyperspace of compact subsets of $\mathbb{R}^n$ (see e.g. [8] – also for more about Hausdorff metric). However, Theorem 2 gives us the following result:

COROLLARY 1. *Let $n \in \mathbb{N}$. Let $Z \subset \mathbb{R}^n$ be a closed set. Let $\mathcal{Z}$ be the hyperspace of compact subsets of Z equipped with Hausdorff metric. Then $\mathcal{Z}$ admits a quasicontinuous selection, i.e. the "identical" multifunction $F\colon \mathcal{Z} \to Z$, ($F(A) = A \subset Z$ for every point $A \in \mathcal{Z}$) has a quasicontinuous selection.*

Addendum. The author wishes to thank the referee for his help to make this paper more reader-friendly. The referee has brought to our attention the possibility of generalization of some results. For example, it is possible to reformulate Lemma 1 using open covers to describe the closeness of functions. In such a case a Moore space Y or a uniform one could be appropriate. But Lemma 2 shows that – using the methods of the paper – the assumption of the convexity of Y is necessary to obtain Theorems 1 and 2.

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