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ON RECIPROCAL SUMS OF TERMS OF LINEAR RECURRENCES

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ABSTRACT. The paper deals with the irrationality of infinite series, where terms are reciprocal of terms of a linear recurrent sequence with constant coefficients.

Let $G = \{G_n\}_{n=0}^{\infty}$ be a linear recursive sequence of order k (> 1) defined by

$$G_n = A_1 G_{n-1} + A_2 G_{n-2} + \cdots + A_k G_{n-k} \qquad (n \geq k),$$

where A_1, $A_2, \dots, A_k$ and the initial terms G_0, $G_1, \dots, G_{k-1}$ are given rational integers – not all zero. Denote by $\alpha = \alpha_1$, $\alpha_2, \dots, \alpha_s$ the distinct roots of the characteristic polynomial

$$G(x) = x^k - A_1 x^{k-1} - A_2 x^{k-2} - \cdots - A_k.$$

Suppose that $s \geq 2$ and the roots have multiplicity $m = m_1$, $m_2, \dots, m_s$. Then, as it is well known, the terms of the sequence G can be expressed by

$$G_n = f(n)\alpha^n + f_2(n)\alpha_2^n + \cdots + f_s(n)\alpha_s^n \tag{1}$$

for any $n \geq 0$, where $f(x)$, $f_2(x), \dots, f_s(x)$ are polynomials over the number field $Q(\alpha, \alpha_2, \dots, \alpha_s)$ of degree $m-1$, $m_2-1, \dots, m_s-1$, respectively.

In the sequel we suppose that α is a dominant root of $G(x)$ (i.e. $|\alpha| > |\alpha_i|$ for $i = 2, \dots, s$) and $G_n \neq 0$ for $n > 0$.

If $k = 2$ and $G_0 = 0$, $G_1 = 1$, we denote the second order linear recursive sequence by R, furthermore if $G_0 = 2$ and $G_1 = A_1$, then the sequence will be denoted by V. For these sequences the characteristic polynomial is $x^2 - A_1 x - A_2$

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and we denote its roots by α and β ($\beta = \alpha_2$), where $|\alpha| > |\beta|$ by the above restriction. The terms of the sequences R and V can be written in the form

$$R_n = (\alpha^n - \beta^n)/(\alpha - \beta) \tag{2}$$

and

$$V_n = \alpha^n + \beta^n \tag{3}$$

for any $n \geq 0$.

In the special case $A_1 = A_2 = 1$ the sequences R, V are the known Fibonacci and Lucas sequences which will be denoted by F and L.

It is known that for the Fibonacci numbers

$$\sum_{n=0}^{\infty} 1/F_{2^n} = (7 - \sqrt{5}\,)/2 \tag{4}$$

(see [5] and [6]) and so the sum of this series is an irrational number. Solving two problems of Erdős and Graham [4], C. Badea [2] has shown that

$$\sum_{n=1}^{\infty} 1/F_{2^n+1} \tag{5}$$

and

$$\sum_{n=1}^{\infty} 1/L_{2^n} \tag{6}$$

are also irrational.

For the sequence V, with restriction $A_1 \geq 1$, R. André-Jeanin [1] obtained the following result. If $\varepsilon = 1$ or $\varepsilon = -1$, then

$$\Phi = \sum_{n=0}^{\infty} \varepsilon^n / V_{2^n} \tag{7}$$

is an irrational number. Furthermore, if $A_1^2 + 4A_2 = D$ is not a perfect square and $|\beta| < 1$, then $1, \alpha, \Phi$ are linearly independent over Q.

As a consequence of a more general theorem P. Bundschuh and A. Pethő [3] obtained a transcendence result for the sequences R with $A_1 > 0$ and $A_2 = 1$: Let $\{B_n\}_{n=0}^{\infty}$ be a sequence of integers such that $|B_n|$ is not a constant for large indices and

$$|B_n| \leq R_{2^n-1}^{1-\varepsilon}$$

for any $\varepsilon > 0$ and $n > n(\varepsilon)$. Then

$$\sum_{n=0}^{\infty} B_n / R_{2^n} \tag{8}$$

is a transcendental number.

The aim of this paper is to give similar results for general sequences. For the most general linear recurrences G (with restriction $|\alpha| > |\alpha_i|$ for $i = 2, \ldots, s$) we prove:

THEOREM 1. *Le G be a linear recurrence of order k defined by (1) and let $\{b_n\}_{n=1}^{\infty}$ be a sequence of non-zero integers. If $\{k_n\}_{n=1}^{\infty}$ is a sequence of positive integers such that*

$$\lim_{n\to\infty} \left(b_n \cdot \prod_{i=1}^{n-1} f(k_i)\alpha^{k_i} \right) \Big/ \left(f(k_n)\alpha^{k_n} \right) = 0, \tag{9}$$

then the sum of the series

$$\sum_{n=1}^{\infty} b_n / G_{k_n} \tag{10}$$

is an irrational number.

This theorem implies some consequences for second order linear recurrences R and V. Let $k = 2$ and denote by D the discriminant of the characteristic polynomial $x^2 - A_1 x - A_2$ of sequences R and V. Thus

$$\sqrt{D} = \sqrt{A_1^2 + 4A_2} = |\alpha - \beta|$$

and by (2) and (3) for the sequences R and V the function $f(n)$, defined in (1), is $f(n) = 1/\sqrt{D}$ or $-1/\sqrt{D}$ (according to $\alpha > 0$ or $\alpha < 0$) and $f(n) = 1$ respectively. Substituting these values into (9) from Theorem 1 we immediately obtain:

COROLLARY 1. *Let t (> 0) and k be fixed integers with $t > k$ and let $\{b_n\}_{n=1}^{\infty}$ be a sequence of non-zero integers. Define the sequence $\{k_n\}_{n=1}^{\infty}$ by $k_n = t\,2^n - k$. If*

$$\lim_{n\to\infty} b_n / \left(\alpha^k \sqrt{D}\,\right)^n = 0,$$

then the sum of the series

$$\sum_{n=1}^{\infty} b_n / R_{t2^n - k}$$

is an irrational number.

COROLLARY 2. *If $k > 0$ and*

$$\lim_{n\to\infty} b_n/\alpha^{kn} = 0,$$

then

$$\sum_{n=1}^{\infty} b_n/V_{t2^n-k}$$

is irrational for any fixed integer t with $t > k$.

In the case $k = 0$ we can give a weaker condition for the sequence $\{b_n\}_{n=1}^{\infty}$.

THEOREM 2. *If $\{b_n\}_{n=1}^{\infty}$ is a sequence of non-zero integers such that*

$$\lim_{n\to\infty} b_n/\alpha^{t2^{n-1}} = 0, \tag{11}$$

then the sum of the series

$$\sum_{n=1}^{\infty} b_n/R_{t2^n}$$

is an irrational number.

THEOREM 3. *Let V be a generalized Lucas sequence defined by (3), such that $|\beta| < 1$. Then the sum of the series*

$$\sum_{n=1}^{\infty} 1/V_{t2^n}$$

is an irrational number for any fixed positive integer t.

Notes. The irrationality of the sums (4) and (5) follows from Corollary 1 with $t = 1$, and with $k = 0$ and $k = -1$, respectively, since $D = 5$, $\alpha = \left(1+\sqrt{5}\,\right)/2$ and $0 < |\alpha/\sqrt{D}| < 1$ in this case. Theorem 3 with $t = 1$ implies the irrationality of sums (6) and (7) with $\varepsilon = 1$. The irrationality of (8) follows from Theorem 2 with $t = 1$ since

$$\left|R_{2^{n-1}}^{1-\varepsilon}\right| / \left|\alpha^{2^{n-1}}\right| < \varepsilon_1$$

for any ε, $\varepsilon_1 > 0$ if n is sufficiently large.

For the proof of the theorems we need the following result which can be found in [7] in a more general form.

LEMMA. *Let* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *be sequences of integers such that* $0 < |b_n| < |a_n| < |a_{n+1}|$ *for every* $n > n_0$. *If* $\lim_{n\to\infty} b_n/a_n = 0$, *then the sum of the series*

$$\sum_{n=1}^{\infty} \frac{b_n}{a_1 \cdot a_2 \cdot \ldots \cdot a_n}$$

is an irrational number.

Proof of Theorem 1. By condition (9) it is easy to see that the sequence $\{k_n\}_{n=1}^{\infty}$ is strictly increasing from some index and that the sum (10) is convergent. (10) can be written in the form

$$\sum_{n=1}^{\infty} b_n/G_{k_n} = \sum_{n=1}^{\infty} b_n P(n-1)/P(n),$$

where $P(m)$ denote the product

$$\prod_{i=1}^{m} G_{k_i},$$

and so by the lemma it is enough to prove that

$$C_n = b_n P(n-1)/G_{k_n} \to 0 \qquad \text{as} \quad n \to \infty .$$

By (1) we have

$$C_n = \frac{b_n\left(\prod_{i=1}^{n-1} f(k_i)\alpha^{k_i}\right)\cdot\left(\prod_{i=1}^{n-1}(1+F(k_i))\right)}{f(k_n)\alpha^{k_n}(1+F(k_n))}, \tag{12}$$

where

$$F(m) = \frac{f_2(m)}{f(m)}\cdot\left(\frac{\alpha_2}{\alpha}\right)^m + \cdots + \frac{f_s(m)}{f(m)}\cdot\left(\frac{\alpha_s}{\alpha}\right)^m .$$

But, using that $0 < |\alpha_i/\alpha| < 1$ for $i = 2, \ldots, s$, it can be easily seen that $F(k_n) \to 0$ as $n \to \infty$ and

$$\prod_{i=1}^{n-1}(1+F(k_i)) = c(n) < c$$

for any n, where c depends only on the parameters of the sequence G. So by (9) and (12) we get

$$\lim_{n\to\infty} C_n = 0$$

which proves the theorem.

Proof of Theorem 2. By (2) and (3) we have

$$R_{t2^n} = \frac{\alpha^{t2^n} - \beta^{t2^n}}{\alpha - \beta} = V_{t2^{n-1}} \frac{\alpha^{t2^{n-1}} - \beta^{t2^{n-1}}}{\alpha - \beta} = \cdots = R_t \prod_{i=1}^{n} V_{t2^{i-1}}$$

and so

$$\sum_{n=1}^{\infty} b_n / R_{t2^n} = (1/R_t) \sum_{n=1}^{\infty} b_n \Big/ \left(\prod_{i=1}^{n} V_{t2^{i-1}} \right).$$

By (3) we have

$$b_n / V_{t2^{n-1}} = \left(b_n / \alpha^{t2^{n-1}}\right) \cdot \left(1/(1 + (\beta/\alpha)^{t2^{n-1}})\right)$$

from which, by the Lemma and (11), the theorem follows.

Proof of Theorem 3. By (3) the numbers V_{t2^n} are positive since $t2^n$ is even. C. Badea [2] proved that if $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive integers and

$$a_{n+1} > a_n^2 - a_n + 1$$

for all large n, then the sum of the series $\sum_{n=1}^{\infty} 1/a_n$ is irrational. Thus we have only to prove that

$$V_{t2^{n+1}} > \left(V_{t2^n}\right)^2 - V_{t2^n} + 1 \tag{13}$$

for any sufficiently large n. Using (3), (13) can be written in the form

$$\alpha^{t2^{n+1}} \left(1 + (\beta/\alpha)^{t2^{n+1}}\right) > \alpha^{t2^{n+1}} \left(1 + (\beta/\alpha)^{t2^n}\right)^2 - \alpha^{t2^n} \left(1 + (\beta/\alpha)^{t2^n}\right) + 1.$$

Dividing the inequality by α^{t2^n} we obtain

$$0 > 2 \cdot \alpha^{t2^n} (\beta/\alpha)^{t2^n} - 1 - (\beta/\alpha)^{t2^n} + 1/\alpha^{t2^n} = 2\beta^{t2^n} - 1 - (\beta/\alpha)^{t2^n} + 1/\alpha^{t2^n}$$

which holds for any large n since $|\beta| < 1$ and $|\alpha| > 1$.

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