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ON GOLOMB BIRULERS AND THEIR APPLICATIONS

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ABSTRACT. Golomb birulers are generalizations of Golomb rulers which have been investigated because of applications in radio astronomy. A survey of Golomb rulers and difference triangle sets is given. All small Golomb birulers are determined. Moreover applications in configuration theory are discussed.

1. Introduction

1.1 Definitions and Notations.

In order to measure all the distances from 1 to 6 it is not necessary to have a ruler with 7 marks $0,1,2,3,4,5,6$. A ruler with 4 marks in the positions $0,1,4,6$ suffices, since all the distances from 1 to 6 can be expressed as differences in the set $\{0,1,4,6\}$ as follows:

$$1 = 1-0, \quad 2 = 6-4, \quad 3 = 4-1, \quad 4 = 4-0, \quad 5 = 6-1, \quad 6 = 6-0.$$

However, this is the biggest example of a ruler with $s+1$ marks which measures the first $s(s+1)/2$ natural numbers exactly once. The more general question is: What is the shortest length of a ruler with a given number of marks such that all the occurring differences are distinct?

This question is motivated by applications in several fields, e.g. radar pulses, radio antennae signals, crystallography, and coding theory. For more details and further references see [2].

1.2 Contents of the sections.

In section 2 the known results on Golomb rulers and difference triangle sets are surveyed. All Golomb rulers with size at most 15 are known. The natural generalization of Golomb rulers are DTS (difference triangle sets). These structures have a lot of applications in mathematics (cyclic configurations) and in

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other sciences (radar pulses, radio astronomy, x-ray diffraction crystallography). Another possible generalization, the so-called Costas arrays, which can be regarded as two-dimensional Golomb rulers are not discussed here. For further details see [1] and [2].

Birulers are discussed in this paper for the first time. Section 3 contains the existence results for small Golomb birulers with size at most 8. Also some 2-DTS are presented there. In section 4 these results are applied to construct 2-configurations.

This paper should be understood as a first step into the investigation of birulers rather than a large collection of deep results. Its main aim is to get the reader interested in the problems which are exhibited here.

2. Golomb rulers and difference triangle sets

2.1 Perfect rulers.

DEFINITION 2.1. *A ruler of size S and length L is a series of S numbers $a_1, a_2, \dots, a_S$ such that $a_1 + a_2 + \cdots + a_S = L$ and all sums of i consecutive numbers ($1 \le i \le S$) are different.*

If $L = S(S+1)/2$, the ruler is called perfect.

A ruler of size S and length L is usually described by a difference triangle, where the first row (or base row) contains the S numbers $a_1, a_2, \dots, a_S$, the second row contains the sums of 2 consecutive numbers $a_1+a_2, a_2+a_3, \dots, a_{S-1}+a_S$ and so on, and the last row contains only L.

A ruler of size 3 and length 6 can be described as follows:

$$\begin{array}{ccccc} 1 & & 3 & & 2 \\ & 4 & & 5 & \\ & & 6 & & \end{array}$$

Of course, this ruler is perfect. Moreover, it is the unique perfect ruler of size 3. As usual a ruler $a_1, a_2, \dots, a_S$ and its mirror ruler $a_S, a_{S-1}, \dots, a_1$ are called equivalent.

The proof of the following theorem contains the construction of all perfect rulers and is due to S. Golomb.

THEOREM 2.2. *There is no perfect ruler of size $S > 3$.*

P r o o f. For a perfect ruler the following holds:

$$L = S(S+1)/2 = \sum_{i=1}^{S} i\,.$$

Hence the base row consists of the numbers from 1 to S in a certain order. For $S=1$ the ruler consists of the number 1.

If $S>1$, the only possible neighbour of 1 in the base row is S. Otherwise the sum of 1 and its neighbour would be a number which also occurs in the base row. If $S=2$, the ruler is $1,2$ (or equivalently $2,1$).

If $S>2$, the only possible neighbour of 2 is again S, since the number $S+1$ occurs already as sum of 1 and S. Thus the numbers 1 and 2 occur at the 2 ends of the ruler. Since they both are neighbours of S it follows that $S=3$ and the ruler is $1,3,2$ (or equivalently $2,3,1$).

2.2 Golomb rulers.

The fact that there are no bigger perfect rulers leads to the following definition.

DEFINITION 2.3. *A ruler of size S with smallest possible length is called a Golomb ruler.*

Consider the following ruler of size 4 and length 11:

$$\begin{array}{ccccccc}
1 & & 3 & & 5 & & 2 \\
& 4 & & 8 & & 7 & \\
& & 9 & & 10 & & \\
& & & 11 & & &
\end{array}$$

Since there is no perfect ruler with length 10 the above example shows a ruler of smallest possible length, a Golomb ruler. In fact, there are 2 non-equivalent Golomb rulers of size 4. The second one is the following:

$$\begin{array}{ccccccc}
2 & & 5 & & 1 & & 3 \\
& 7 & & 6 & & 4 & \\
& & 8 & & 9 & & \\
& & & 11 & & &
\end{array}$$

There are even 4 non-equivalent Golomb rulers of size 5. Two of them which are exhibited below even produce the same set of differences: $\{1,2,\ldots,17\}\backslash\{14,15\}$.

$$\begin{array}{ccccccccc}
1 & & 3 & & 6 & & 2 & & 5 \\
& 4 & & 9 & & 8 & & 7 & \\
& & 10 & & 11 & & 13 & & \\
& & & 12 & & 16 & & & \\
& & & & 17 & & & &
\end{array}
\qquad
\begin{array}{ccccccccc}
1 & & 7 & & 3 & & 2 & & 4 \\
& 8 & & 10 & & 5 & & 6 & \\
& & 11 & & 12 & & 9 & & \\
& & & 13 & & 16 & & & \\
& & & & 17 & & & &
\end{array}$$

All lengths of Golomb rulers of size at most 15 are known. The following table shows the size S, the length of a Golomb ruler $L(S)$, the difference of two consecutive lengths, the number of non-equivalent Golomb rulers of this length, and one example for each length.

S	$L(S)$	$L(S)-L(S-1)$	Number	Example
1	1		1	1
2	3	2	1	1, 2
3	6	3	1	1, 3, 2
4	11	5	2	1, 3, 5, 2
5	17	6	4	1, 3, 6, 2, 5
6	25	8	5	1, 3, 6, 8, 5, 2
7	34	9	1	1, 3, 5, 6, 7, 10, 2
8	44	10	1	1, 4, 7, 13, 2, 8, 6, 3
9	55	11	1	1, 5, 4, 13, 3, 8, 7, 12, 2
10	72	17	2	1, 3, 9, 15, 5, 14, 7, 10, 6, 2
11	85	13	1	2, 4, 18, 5, 11, 3, 12, 13, 7, 1, 9
12	106	21	1	2, 3, 20, 12, 6, 16, 11, 15, 4, 9, 1, 7
13	127	21	1	5, 23, 10, 3, 8, 1, 18, 7, 17, 15, 14, 2, 4
14	151	24	1	6, 1, 8, 13, 12, 11, 24, 14, 3, 2, 27, 10, 16, 4
15	177	26	1	1, 3, 7, 15, 6, 24, 12, 8, 39, 2, 17, 16, 13, 5, 9

The most astonishing length seems to be $L(10) = 72$. The numbers $L(S)$ grow quite regularly for $S \leq 9$ while $L(10) - L(9) > L(11) - L(10)$. For further details see [2].

I would like to mention the fact that there seems to be no general way of describing the solutions. In some sense there has not been any kind of a mathematical theory of Golomb rulers. The only way to obtain the above results which is known currently is more or less an extensive use of computing time which makes further progress in this field rather difficult.

2.3 Difference triangle sets.

The concept of a ruler can be generalized in the following way.

DEFINITION 2.4. *A difference triangle set or an* (I, J)-DTS *is a set of* I *rulers of size* J *such that all the* $IJ(J+1)/2$ *elements in their triangles are distinct. If the largest element (called the length of the* DTS*) is smallest possible the* DTS *is called optimal. If the length is* $IJ(J+1)/2$ *the* DTS *is called perfect (or a perfect system of difference sets* (PSDS)*).*

Of course, perfect rulers are perfect $(1, J)$-DTS whereas Golomb rulers are optimal $(1, J)$-DTS.

The lengths of difference triangle sets have been investigated by engineers who needed the results for applications. Most of the results have been published in non-mathematical journals. However, there is a good survey in [9] which contains a lot of details and references. Many explicit solutions are exhibited in [8]. Some results about optimal and perfect DTS can also be found in [4]. The known lengths of optimal DTS are shown in the following table [9]. The bold entries indicate perfect DTS.

$J=$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$I=1$	**1**	**3**	**6**	11	17	25	34	44	55	72	85	106	127	151	177
2	**2**	7	13	22	34	51									
3	**3**	10	19	32	49	77									
4	**4**	**12**	**24**	41	65										
5	**5**	**15**	**30**	51	83										
6	**6**	19	**36**	**60**											
7	**7**	22	**42**	71											
8	**8**	**24**	**48**	**80**											
9	**9**	**27**	**54**												
10	**10**	31	**60**	**100**											
11	**11**	34	**66**												
12	**12**	**36**	**72**												
13	**13**	**39**	**78**												
14	**14**	43	**84**												
15	**15**	46	**90**												
16	**16**	**48**													
17	**17**	**51**													
18	**18**	55	**108**												

Of course, the first row of this table corresponds to the table of Golomb rulers in 2.2.

In the following some especially interesting explicit solutions of [8] are exhibited. The biggest known length of an optimal DTS with $I > 1$ and maximal J is 77 for a $(3,6)$-DTS. The 3 triangles contain 63 distinct numbers. Those which are not contained are $36, 41, 44, 47, 50, 51, 57, 60, 65, 66, 67, 69, 71, 73$.

1 2 13 18 22 21
3 15 31 40 43
16 33 53 61
34 55 74
56 76
77

4 6 20 19 9 14
10 26 39 28 23
30 45 48 42
49 54 62
58 68
72

5		7		17		8		27		11
	12		24		25		35		38	
		29		32		52		46		
			37		59		63			
				64		70				
					75					

Especially remarkable is the existence of a perfect $(10,4)$-DTS. This means that in the following 10 triangles the 100 smallest natural numbers occur exactly once. This is the biggest known perfect DTS with $J = 4$.

1		44		53		2
	45		97		55	
		98		99		
			100			

7		51		23		14
	58		74		37	
		81		88		
			95			

15		31		30		17
	46		61		47	
		76		78		
			93			

6		43		20		22
	49		63		42	
		69		85		
			91			

3		29		33		24
	32		62		57	
		65		86		
			89			

9		27		41		19
	36		68		60	
		77		87		
			96			

10		25		48		11
	35		73		59	
		83		84		
			94			

12		28		39		13
	40		67		52	
		79		80		
			92			

8		26		38		18
	34		64		56	
		72		82		
			90			

4		50		16		5
	54		66		21	
		70		71		
			75			

3. Birulers

3.1 Perfect birulers.

While the previous section surveys already known results on difference triangle sets, this section introduces the new concept of a biruler and contains the first results on small birulers. While it is clear that the application of birulers is also possible in configuration theory (compare [4] and section 4) there is also hope that birulers are of good use in the fields mentioned in the beginning as radio astronomy etc.

The name biruler is used as a generalization of the word ruler in the same sense as a biplane is a generalization of a finite projective plane.

DEFINITION 3.1. *A biruler of size S and length L is a series of S numbers $a_1, a_2, \ldots, a_S$ such that $a_1 + a_2 + \cdots + a_S = L$ and no number occurs more than twice as a sum of i consecutive numbers.*

If L is smallest possible the biruler is called perfect.

Again the same description as above is used. The difference triangle of a biruler must not contain a number more than twice. There are the following perfect birulers of size at most 5 (up to equivalence). They are due to S. Golomb [3].

$$
\begin{array}{c} 1 \end{array}
\qquad
\begin{array}{ccc} 1 & & 1 \\ & 2 & \end{array}
\qquad
\begin{array}{ccccc} 1 & & 2 & & 1 \\ & 3 & & 3 & \\ & & 4 & & \end{array}
\quad \text{and} \quad
\begin{array}{ccccc} 1 & & 1 & & 2 \\ & 2 & & 3 & \\ & & 4 & & \end{array}
$$

$$
\begin{array}{ccccccc} 1 & & 2 & & 2 & & 1 \\ & 3 & & 4 & & 3 & \\ & & 5 & & 5 & & \\ & & & 6 & & & \end{array}
\qquad
\begin{array}{ccccccccc} 1 & & 2 & & 2 & & 3 & & 1 \\ & 3 & & 4 & & 5 & & 4 & \\ & & 5 & & 7 & & 6 & & \\ & & & 8 & & 8 & & & \\ & & & & 9 & & & & \end{array}
$$

The following theorem is similar to Theorem 2.2 and states that there are no big perfect birulers.

THEOREM 3.2. *There is no perfect biruler of size $S > 5$.*

Proof. The base row of a perfect biruler of size S contains (let $S > 5$)

(i) the numbers from 1 to $S/2$ twice each for even S, and

(ii) the numbers from 1 to $(S-1)/2$ twice each and the number $(S+1)/2$ once for odd S.

If S is even, possible neighbours of 1 are only the 2 numbers $S/2$. Thus the difference $1+S/2$ occurs twice, which implies that the 2 numbers $S/2$ are also the only possible neighbours of 2. If $S=6$, this implies the following base row $1,3,2,2,3,1$. In the second row the number 4 occurs three times, which is contradictory to the definition of a biruler. If $S>6$, there is a contradiction since the numbers 2 need further neighbours, which do not exist.

If S is odd the neighbours of 1 are $(S-1)/2$ and $(S+1)/2$. Again the other neighbour of $(S+1)/2$ is the number 2. Since $S>5$, the numbers 2 are not neighbours of each other. Two further neighbours of 2 are needed which , however, do not exist.

3.2 Golomb birulers.

Golomb birulers are defined and all small examples are investigated.

DEFINITION 3.3. *A biruler of size S with smallest possible length is called Golomb biruler.*

A perfect ruler of size 6 (and length 12) does not exist (compare Theorem 3.2). Hence the following biruler of length 13 is a Golomb biruler.

$$\begin{array}{ccccccccccc}
1 & & 2 & & 4 & & 1 & & 3 & & 2 \\
 & 3 & & 6 & & 5 & & 4 & & 5 & \\
 & & 7 & & 7 & & 8 & & 6 & & \\
 & & & 8 & & 10 & & 10 & & & \\
 & & & & 11 & & 12 & & & & \\
 & & & & & 13 & & & & &
\end{array}$$

A computer search for all possible Golomb birulers of size 6 yields the result that there are exactly 2 non-equivalent Golomb birulers; the second one looks as follows:

$$\begin{array}{ccccccccccc}
1 & & 3 & & 2 & & 4 & & 1 & & 2 \\
 & 4 & & 5 & & 6 & & 5 & & 3 & \\
 & & 6 & & 9 & & 7 & & 7 & & \\
 & & & 10 & & 10 & & 9 & & & \\
 & & & & 11 & & 12 & & & & \\
 & & & & & 13 & & & & &
\end{array}$$

The computer search is continued for all Golomb birulers of size at most 8. One of 3 known birulers of size 9 with length 29 is the following (Probably no such biruler with shorter length exists):

1		1		4		5		3		4		3		6		2
	2		5		9		8		7		7		9		8	
		6		10		12		12		10		13		11		
			11		13		16		15		16		15			
				14		17		19		21		18				
					18		20		25		23					
						21		26		27						
							27		28							
								29								

The following table contains all the known results about perfect birulers and Golomb birulers in a similar manner as the table concerning rulers in 2.2.

S	L(S)	L(S)-L(S-1)	Number	Solutions
1	1		1	1
2	2	1	1	11
3	4	2	2	121
				112
4	6	2	1	1221
5	9	3	1	12231
6	13	4	2	124132
				132412
7	18	5	3	1144332
				1242531
				1254231
8	23	5	4	11544332
				12425531
				13252631
				13425512

A comparison with the corresponding table for rulers shows that

(i) the numbers $L(S)$ behave quite "reasonably", at least for these small values of S, and

(ii) the number of non-equivalent Golomb birulers of a certain size is not 1 for the bigger known values which is different from the situation of rulers.

Whether these properties are true in general or not will depend on future results on bigger values of S. The computing time which was spent for $S \leq 8$ was quite small. However, a further search for birulers with bigger S will soon need much more time. That is the reason why I decided to stop at this point and hope for an improvement of the used methods in the future.

3.3 2-Difference triangle sets.

A quite natural generalization of birulers will be introduced in the following. Some small examples are exhibited in order to indicate problems rather than to give complete solutions.

DEFINITION 3.4. *A* 2*-difference triangle set or a* 2-(I,J)-DTS *is a set of* I *birulers of size* J *such that no number occurs more than twice in the* I *triangles which contain* $IJ(J+1)/2$ *elements. If the largest element (called the length of the* 2-DTS) *is the smallest possible, the* 2-DTS *is called optimal.*

LEMMA 3.5. *The following* 2-$(2,4)$-DTS *is optimal.*

$$
\begin{array}{ccccccc}
1 & & 3 & & 4 & & 2 \\
& 4 & & 7 & & 6 & \\
& & 8 & & 9 & & \\
& & & 10 & & &
\end{array}
\qquad\qquad
\begin{array}{ccccccc}
1 & & 5 & & 3 & & 2 \\
& 6 & & 8 & & 5 & \\
& & 9 & & 10 & & \\
& & & 11 & & &
\end{array}
$$

P r o o f . The length of a 2-$(2,4)$-DTS is at least 10. However, such a 2-DTS cannot exist since there is no perfect ruler of size 4. Hence length 11 is smallest possible and the above 2-DTS is optimal.

R e m a r k 3.6. The above 2-DTS indicates a construction method which can be used in general to obtain "good" 2-DTS. Take a Golomb ruler of size S. Since it is not perfect for large S some numbers do not occur in the triangle. E.g. the second triangle does not contain 4. This improves the chance to find a suitable partner triangle which need not be a ruler. The first triangle contains 4 twice.

The strategy described above is used to construct the following 2-$(2,5)$-DTS of length 17:

$$
\begin{array}{ccccccccc}
1 & & 3 & & 6 & & 5 & & 2 \\
& 4 & & 9 & & 11 & & 7 & \\
& & 10 & & 14 & & 13 & & \\
& & & 15 & & 16 & & & \\
& & & & 17 & & & &
\end{array}
\qquad\qquad
\begin{array}{ccccccccc}
1 & & 4 & & 3 & & 6 & & 2 \\
& 5 & & 7 & & 9 & & 8 & \\
& & 8 & & 13 & & 11 & & \\
& & & 14 & & 15 & & & \\
& & & & 16 & & & &
\end{array}
$$

A Golomb ruler of size 5 not containing 8 is combined with a triangle of length 16 which contains 8 twice. However, it is not yet decided whether this 2-DTS is optimal since it is possible that there is also a 2-$(2,5)$-DTS of length 16.

4. 2-configurations

4.1 Application of birulers.

Golomb rulers and optimal difference triangle sets have been used to construct configurations (see [4], [5]). In the same way birulers can be used to obtain 2-configurations. The definition of a λ-configuration can be found in [6] and is repeated below for the special case of symmetric 2-configurations. Some of the results on the existence of 2-configurations are cited below, for further details see [6].

DEFINITION 4.1. *A symmetric 2-configuration* $(v_k)_2$ *is a finite incidence structure consisting of a set of* v *points and a set of* v *subsets (called lines) of this set such that*

(i) *there are* k *points on each line and* k *lines through each point,*
(ii) *two different points are connected by at most* 2 *lines and two different lines intersect each other in at most* 2 *points.*

The construction of 2-configurations from birulers is described in [6] and explained below by giving an example. Take the perfect biruler of size 5.

$$\begin{array}{ccccccccc} 1 & & 2 & & 2 & & 3 & & 1 \\ & 3 & & 4 & & 5 & & 4 & \\ & & 5 & & 7 & & 6 & & \\ & & & 8 & & 8 & & & \\ & & & & 9 & & & & \end{array}$$

The first line ℓ_0 of the 2-configuration contains 6 points, point 0 and the first entries of each row of the above triangle. Hence $\ell_0 = \{0,1,3,5,8,9\}$. Now construct ℓ_i by adding i (mod 18) to all elements of ℓ_0:

$$\ell_1 = \{1,2,4,6,9,10\}, \ldots, \ell_{17} = \{17,0,2,4,7,8\}.$$

The 18 lines $\ell_0, \ldots, \ell_{17}$ form a 2-configuration $(18_6)_2$.

The same procedure is possible for all $v \geq 18$. By adding i (mod v) to ℓ_0 we obtain a 2-configuration $(v_6)_2$ with a cyclic automorphism of order v.

However, for $v = 17$ this procedure is not successful, since $-8 \equiv 9 \pmod{17}$. This shows that the existence of a suitable biruler is sufficient for the existence of the corresponding 2-configuration. That it is not at all necessary will be seen in the following subsection.

4.2 Existence of 2-configurations.

The investigation of 2-configurations was already started during the 19th century. A historical survey can be found in [7].

There are exactly 3 non-isomorphic 2-configurations $(16_6)_2$, one of which was constructed by E. Kummer in 1864. None of the 3 mentioned 2-configurations has an automorphism of order 16. Hence none of them can be constructed by the "biruler method".

While this Kummer-2-configuration is a quite well-known combinatorial structure, the existence of a 2-configuration $(17_6)_2$ has been in doubt until recently; it was constructed in [6].

In general, the existence of a 2-configuration $(v_k)_2$ implies $v \geq k(k-1)/2+1$. For $k \leq 6$ the reverse implication is also true (for details see [6]). However, there is no 2-configuration $(22_7)_2$. This was proved by Hussain (1946) for the first time and can be deduced from a general theorem, the Bruck-Ryser-Chowla Theorem (1950).

The until now obtained birulers of size 6 do not help to construct a 2-configuration $(23_7)_2$. No other method has been used successfully either to obtain such a structure. Since neither non-existence proof is available, its existence is still in doubt.

For $v \geq 24$ the Golomb birulers of size 6 help to construct an example of a cyclic 2-configuration $(v_7)_2$. By analyzing all suitable birulers (not only the optimal ones) it will even be possible to construct all non-isomorphic cyclic 2-configurations with given parameters, e.g. all cyclic 2-configurations $(18_6)_2$ and $(24_7)_2$.

I hope that these combinatorial applications as well as the first results on birulers convince the reader that this research should be continued. I also recall that it should be investigated if and how birulers can be used in those non-mathematical fields of application mentioned earlier and explained in [2].

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