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NEIGHBOURHOODS IN LINE GRAPHS

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ABSTRACT. Graphs in which subgraphs induced by the neighbourhoods of the vertices are all isomorphic or belong to some special class are investigated. It is shown that for a given graph H all graphs with every neighbourhood isomorphic to H are line graphs or none of them are. A new characterization of line graphs is also obtained.

1. Introduction

In this paper only finite undirected graphs without loops and multiple edges are considered. We follow the notation of Behzad, Chartrand and Lesniak-Foster [2] with the exception of the following concepts. By the *neighbourhood* $N_G(v)$ [or $N(v)$] of a nonisolated vertex v of a graph G we mean the subgraph of G induced by vertices adjacent to v. A graph without isolated vertices is said to be a *locally-H graph* if for all its vertices v we have $N(v)$ isomorphic to a given graph H. We write $G \angle H$ if the graph G is an induced subgraph of a graph H.

The aim of this paper is to study the collection of neighbourhoods in line graphs. If somebody is interested only in the degree sequences in line graphs, we refer him to Bauer [1].

There are several well-known characterizations of line graphs based on the Krausz partition (see [8]), even triangles [10] and nine forbidden induced subgraphs [3]. Recently it was proved that a connected graph with at least nine vertices is a line graph if and only if it does not contain any of the seven given graphs as an induced subgraph. Moreover, the number seven cannot be reduced even if the number of vertices is increased [10]. Here we give a new characterization expressed in terms of neighbourhoods together with five forbidden induced subgraphs.

Zelinka [12] has investigated polytopal (= 3-connected and planar) locally linear graphs. He has proved that such a graph G on a given number n

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of vertices has the maximal possible number of edges ($= 2n$) if and only if G is a line graph of a polytopal triangle-free cubic graph. This evokes the feeling that certain graphs which can be easily described by their neighbourhoods are line graphs of a simply characterized class of graphs.

We present two neighbourhood conditions sufficient for a graph to be a line graph and also a neighbourhood characterization. We will prove that for a given graph H either all locally-H graphs are line graphs or none of them. Moreover, we determine which case occurs. Locally-H graphs, where H is an induced subgraph of the cartesian product $K_2 \times K_n$, are also studied.

2. Characterizations of line graphs

Let K denote the complete infinite graph with countable many vertices, and let Θ be the set of all finite induced subgraphs of $K_2 \times K$. If G is in Θ, then there exist integers $0 \le t \le r \le s$, with $s \ge 1$, such that r of its vertices lie in one copy of K, s in the other, and there are t edges between vertices in the two sets. Such a graph G we denote by $K_r \overset{t}{\bigcup} K_s$.

Note that both $K_0 \overset{0}{\bigcup} K_2$ and $K_1 \overset{1}{\bigcup} K_1$ are complete graphs on two vertices, but whenever a graph G lies in $\Theta - \{K_2\}$, then the integers $0 \le t \le r \le s$ are uniquely determined.

The following characterization of neighbourhoods in line graphs follows directly from the definition.

LEMMA 1. *If $e = uv$ is an edge of a graph G, t is the number of triangles containing e, $\deg(u) \le \deg(v)$ and $\deg(v) \ge 2$, then $N_{L(G)}(e)$ is isomorphic to $K_{\deg(u)-1} \overset{t}{\bigcup} K_{\deg(v)-1}$.*

By $F(H_1, H_2, \dots, H_k)$ we mean the set of all graphs which do not contain any of the graphs $H_1, H_2, \dots, H_k$ as an induced subgraph. B e i n e k e in [3], (see also [2, p. 192]) gave the following characterization of line graphs.

THEOREM 2. *A graph G is a line graph if and only if $G \in F(G_1, G_2, \dots, G_9)$ (see Fig. 1).*

A graph G is said to be a *locally-Θ graph* if $N(v) \in \Theta$ for each nonisolated vertex v. The next theorem characterizes these graphs.

THEOREM 3. *Let G be a graph. Then*

(1) *G is locally-Θ if and only if $G \in F(G_6, \dots, G_9)$.*
(2) *G is a line graph if and only if G is locally-Θ and $G \in F(G_1, \dots, G_5)$.*

Proof. It is sufficient to prove (1). Since $G_6 = K_1 + \overline{K}_3$, $G_7 = K_1 + K_{1,1,2}$, $G_8 = K_1 + (K_1 + 2K_2)$, and $G_9 = K_1 + C_5$, we show

$$\Theta = F(\overline{K}_3,\, K_{1,1,2},\, K_1 + 2K_2,\, C_5\}\,.$$

Clearly, $\Theta \subseteq F(\overline{K}_3,\, K_{1,1,2},\, K_1 + 2K_2,\, C_5\}$. Now assume

$$G \in F(\overline{K}_3,\, K_{1,1,2},\, K_1 + 2K_2,\, C_5\}$$

and distinguish two cases:

Case 1. Let G be triangle-free. Since G is also $\overline{K}_3$-free, it is a path, a union of two paths, C_4 or C_5. C_5 is forbidden and the others are in Θ.

Case 2. Let J be a maximal clique in G, $K_1 \neq J \neq K_2$. If $G = J$, then $G \in \Theta$. Otherwise, since G is $K_{1,1,2}$-free, each edge lies in just one maximal clique (see also [5, p. 6]). Moreover, G is $(K_1 + 2K_2)$-free, hence each clique which shares a common vertex with J is K_2 and edges joining J with vertices outside J induce a matching. We prove that the graph $G - J$ is complete, hence $G \in \Theta$. If u and v are not adjacent vertices of $G - J$, then they are adjacent to at most two vertices of J. Another vertex of J, u and v induce $\overline{K}_3$, a contradiction. □

Next we give *a* "neighbourhoods characterization" of line graphs of triangle-free graphs.

THEOREM 4. *The following statements are equivalent for a graph G without isolated vertices.*

(1) *G is a line graph of a triangle-free graph.*
(2) $G \in F(K_{1,3},\, K_{1,1,2})$.
(3) $N(v) \in F(\overline{K}_3,\, K_{1,2})$ *for each vertex* v.
(4) *The neighbourhood of each vertex is either a complete graph or the union of two complete graphs.*

Proof.

(1) $\iff$ (2) is proved in [4, p. 284].
(2) $\iff$ (3) clearly.
(1) $\Longrightarrow$ (4) according to Lemma 1.
(4) $\Longrightarrow$ (3) can be directly verified. □

Now we give two restrictions on G which allow us to omit the condition $G \in F(G_1, \dots, G_5)$ in Theorem 3.

THEOREM 5. *Let a graph G without isolated vertices satisfy at least one of the two following conditions:*

(C1): *Each edge joining vertices with connected neighbourhoods lies in at most one triangle.*

(C2): *The neighbourhood of no vertex belongs to the set*

$$f := \left\{K_r \bigcup^{t} K_s \mid r = 2 \quad or \quad s = 2\right\} - \left\{K_2, P_3, C_4\right\}.$$

Then G is a line graph if and only if G is a locally-Θ graph.

P r o o f. Let G be a graph satisfying (C1) or (C2). If G is a line graph, then G is locally-Θ by Theorem 3.

Now assume that G is a locally-Θ graph. It is sufficient to prove $G \in F(G_1, \dots, G_5)$. We do it indirectly. So let $G_i \angle G$ for some $i \le 5$. But in each case, the edge v_2v_3 is contained in two triangles, while each of the vertices v_2 and v_3 has a connected but not complete neighbourhood within G_i. Given the structure of the set Θ, the latter observation implies that the graphs $N_G(v_2)$, $N_G(v_3)$ are also connected - which contradicts (C1).

Hence (C2) holds. Denote S_i the set of all vertices from $V(G) - V(G_i)$ which are adjacent to all the vertices v_1, v_2 and v_3 (see Fig.1). First we will verify that we can restrict ourselves to the cases in which S_i is empty.

If $i \le 3$ and $x \in S_i$, then $\langle x, v_1, v_3\rangle$ is in one copy K' of $K \subseteq N(v_2)$ and v_4 is in the other, since $v_4 \notin N(v_1)$. But, if $v_4 \in N(x)$, then it contradicts to v_2, $v_3 \in N(v_4)$. Hence $v_4 \notin N(x)$. Now, by symmetry, we can interchange the indices of v_3 and v_4 in the case when there is no vertex y adjacent to all v_2, v_3 and v_4. In the other case, the triangles xv_1v_2 and v_2v_4y lie in $N(v_3)$, but since $v_4 \notin N(x)$, $N(v_3) \notin \Theta$, a contradiction. Thus we can assume $S_i = \emptyset$.

If $i \ge 4$ and $x \in S_i$, then $\langle V(N_{G_i}(v_2)) \cup \{x\}\rangle \notin \Theta$ (In $\langle V(N_{G_i}(v_2)) \cup \{x\}\rangle$, the vertices v_1, v_3, x lie in one copy K' of K, and since $v_6 \notin N(v_4)$ and $v_4 \notin N(v_1)$, v_6 and v_4 lie in the other copy of K, but $v_6 \notin N(v_4)$, hence $N_G(v_2) \notin \Theta$.) and so $N_G(v_2) \notin \Theta$, which is a contradiction.

But, if $S_i = \emptyset$, we have $N_G(v_1) \angle K \times K_2$ and the vertex v_5 is joined to none of the adjacent vertices v_2 and v_3. Hence v_2 and v_3 lie in one copy K' of the infinite graph K. But $S_i = \emptyset$ implies that just the two vertices v_2 and v_3 of $N_G(v_1)$ lie in K'. Moreover, none of the graphs K_2, P_3 or C_4 has $N_{G_i}(v_1)$ as an induced subgraph. Hence $N_G(v_1)$ lies in f, which contradicts (C2). This completes the proof. □

3. Locally-Θ graphs with constant neighbourhoods

In this section locally-H graphs with $H \angle K \times K_2$ and line graphs with constant neighbourhoods as a special case of them are studied.

Let H be a graph. By $Loc(H)$ we mean the class of all connected locally-H graphs. As G is a locally-H graph if and only if all its components belong to $Loc(H)$, we can restrict ourselves to the study of the class $Loc(H)$. Next let $Loc_L(H)$ be the class of all line graphs which are contained in $Loc(H)$.

If numbers $r < s$ are given and if every edge in a connected graph G joins the vertex of degree r with a vertex of degree s, then G is called an (r,s)-*biregular bipartite graph.*

Further, for integers $0 \le t \le r \le s$, we denote by $\lambda(t,r,s)$ the class of all such connected graphs G that each edge of G lies in exactly t triangles and for two adjacent vertices u and v we have $\{\deg(u), \deg(v)\} = \{r+1, s+1\}$.

LEMMA 6. *A connected graph G belongs to the class $\lambda(t,r,s)$ if and only if either $t=0$, $r \ne s$ and G is an $(r+1,s+1)$-biregular bipartite graph or $r=s$, $t(r+1)$ is even and G is an $(r+1)$-regular graph with t-regular neighbourhoods.*

Proof. First assume $r \ne s$. Then $G \in \lambda(t,r,s)$ if and only if G is a bipartite graph with the parts which consist of the vertices with the same degree (because every edge joins vertices with a different degree). This means G has no triangle, so $t=0$.

Now let $r=s$ and $1 \le t$. If $G \in \lambda(t,r,r)$, then note that the neighbourhood of any vertex is a t-regular graph with $r+1$ vertices. Hence $t(r+1)$ is even. □

Denote $l := \{K_r \overset{t}{\bigcup} K_s \in \Theta \mid t=0 \ \text{ or } \ (r=s \ \text{ and } \ t(r+1) \ \text{ is even})\}$. The following theorem states that if one locally-H graph is a line graph, then $Loc(H) = Loc_L(H)$ and $H \in l$.

THEOREM 7. *Let H be a graph and G be a locally-H graph. Then*

(1) *G is a line graph if and only if $H \in l$.*
(2) *$H \cong K_r \cup K_s$ with $(0 \le r < s)$ if and only if G is a line graph of an $(r+1,s+1)$-biregular bipartite graph.*
(3) *$H \cong K_r \overset{t}{\bigcup} K_r$, $(t(r+1)$ is even and $r \ge t \ge 0)$ if and only if G is a line graph of an $(r+1)$-regular graph with t-regular neighbourhoods.*

Proof. First assume G is a line graph. Then Lemma 1 yields that $Loc_L(K_r \overset{t}{\bigcup} K_s)$ is the class of line graphs of graphs $G \in \lambda(t,r,s) - K_2$. This, together with Lemma 6, means that $H \in l$ and (2) and (3) hold.

Let now H be in l. Then either $H \notin f$ or H is in $f \cap l = \{K_i \cup K_2,\ i \ge 1\}$ and from Theorem 5 it follows that G is a line graph, hence (1), (2) and (3) hold. □

Brouwer, Cohen and Neumaier [5, p. 1 1–12] have proved that for a connected k-regular graph G with t-regular neighbourhoods we have:

1. If $t \geq k - 1/2 - \sqrt{k}$, then G is complete multipartite.
2. If $t \geq k + 1/2 - \sqrt{2k+2}$, then G is the complement of a strongly regular graph.
3. If $t \geq \frac{2}{3}k - 1$, then G has at most $2t - 2$ vertices.

As a consequence we obtain:

COROLLARY 8. *Let G be a connected locally-$(K_r \overset{t}{\bigcup} K_r)$ graph for $1 \leq t \leq r$. Then*

1. *If $t \geq r + 1/2 - \sqrt{r+1}$, then G is the line graph of the $(r+1)$-regular complete $\left(1 + \frac{r+1}{r-t+1}\right)$-partite graph and $(r+1) \equiv 0 \pmod{r - t + 1}$.*
2. *If $t \geq r + 3/2 - \sqrt{2r+4}$, then G is the line graph of the complement of a strongly regular graph.*
3. *If $t \geq (2r-1)/3$, then G has at most $(t-1)(r+1)$ vertices.*

Finally, we prove that for a graph $H \angle K \times K_2$ there is a non-line locally-H graph if and only if either $H \cong P_4$ or $H \cong K_2 \overset{2}{\bigcup} K_s$ and $s \geq 3$.

THEOREM 9. *Let a graph H be an induced subgraph of $K \times K_2$. Then*

(1) *All locally-H graphs are line graphs if H is in l.*
(2) *There are infinitely many connected locally-P_4 graphs.*
(3) *There are locally-$(K_2 \overset{2}{\bigcup} K_s)$ graphs for all $s \geq 3$.*
(4) *There is no locally-H graph otherwise.*

Proof.

(1): It follows from Theorem 8.

(2): It is stated by Doyen, Hubaut, Reynaert in [6].

(3): Let $V(K_{s+1}) = \{v_1, \dots, v_{s+1}\}$ for $s \geq 3$, $V(K_2) = \{0, 1\}$ and let the graph J_s be obtained by adding the edges $(v_i, 0)(v_{i+1}, 1)$ for $i = 1, \dots, s+1$ ($v_{s+2} = v_1$) to the cartesian product $K_{s+1} \times K_2$. It is easy to verify that J_s is a locally-$(K_2 \overset{2}{\bigcup} K_s)$ graph.

(4): Denote $f_2 := \left\{K_2 \overset{2}{\bigcup} K_s;\ \ s \geq 3\right\}$. Let $H \notin l \cup f_2 \cup \{P_4\}$. Then Theorem 7 gives $Loc_L(H) = \emptyset$. Denote $f_1 := \left\{K_2 \overset{1}{\bigcup} K_s;\ \ s \geq 3\right\}$. Hence we have $f = f \cap l \ \cup \ f \cap f_1 \ \cup \ f \cap f_2 \ \cup \ \{P_4\}$. Now we distinguish two cases.

1. Let $H \notin f$. Then Theorems 5 and 7 give $Loc(H) = Loc_L(H) = \emptyset$.

2. Let H be in $f - (l \cup f_2 \cup \{P_4\}) = f_1$. Suppose that there is a locally-$(K_2 \overset{1}{\bigcup} K_s)$ graph G for $s \geq 3$. Let v_1 be the vertex with maximal degree in $N(v)$, where v is a fixed vertex, and let y be the vertex from $N(v)$ adjacent to the endvertex in $N(v)$. Finally, let y' be the only neighbour of v_1 not adjacent to v (see Fig. 3). From $N(v_1) \cong K_2 \overset{1}{\bigcup} K_s$ it is clear that $y'y \in E$. It is easy to verify that $\deg_{N(y)}(v) = \deg_{N(v)}(y) = 2$, $\deg_{N(y)}(v_1) = \deg_{N(v_1)}(y) = 2$. So two adjacent vertices v and v_1 have degree 2 in $N(y)$. This gives $s = 3$ and, moreover, the vertex with maximal degree in $N(y)$ is adjacent to all of v, v_1 and y. But Fig. 3 shows that no such vertex exists. This completes the proof. □

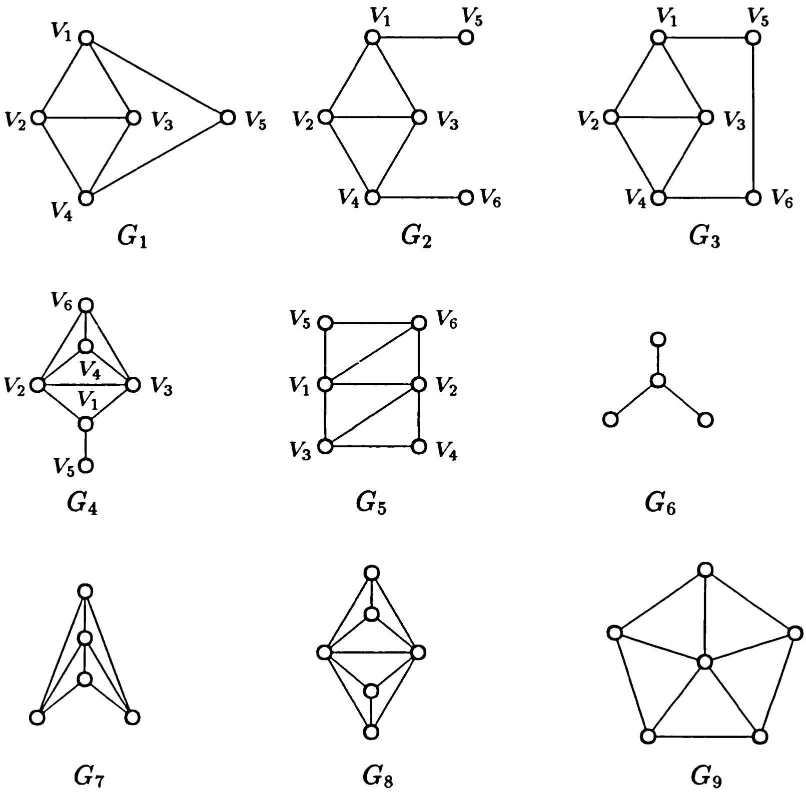

Figure 1. All nine forbidden induced subgraphs $G_1, \ldots, G_9$ for line graphs.

4. Open problems

We conclude with two problems of line graphs that are not locally-H graphs.

Clearly, there exist many connected graphs with the same neighbour multiset $\{N(v) \mid v \in V(G)\}$, particularly all r-regular triangle-free graphs on n vertices have the same neighbour multisets. Theorem 7 implies that a neighbour multiset is sufficient for determining whether a given graph is a line graph, in the case that all neighbours are isomorphic. But it is an open problem in general.

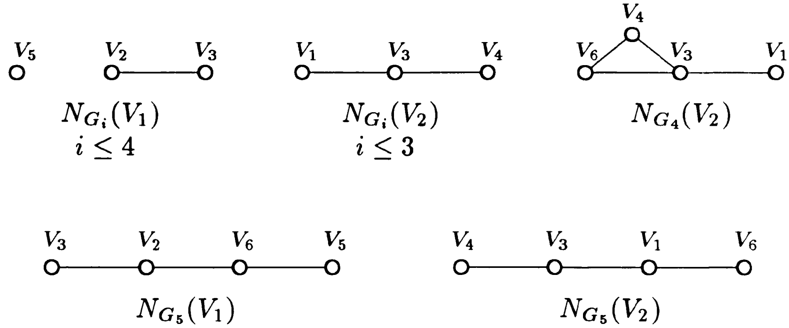

Figure 2. Some important neighbourhoods in the graphs $G_1, \dots, G_9$.

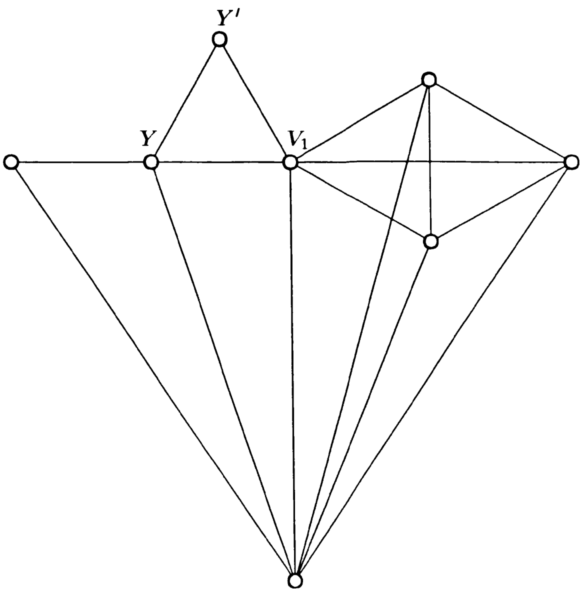

Figure 3. A subgraph which any locally-$(K_2 \overset{1}{\bigcup} K_4)$ graph should contain.

Problem 1. Are there graphs G and H with the vertex-set $\{v_1, \dots, v_n\}$

such that $N_G(v_i) \cong N_H(v_i)$ for $i \leq n$ and where G is a line graph, but H is not?

P r o b l e m 2. A graph is said to be a *graph with nonisomorphic neighbourhoods* if for all distinct $u, v \in V(G)$ we have $N(u) \ncong N(v)$. S e d l á č e k in [9] has found for every $n \geq 6$ a connected graph on n vertices with nonisomorphic neighbourhoods. But none of them was a line graph. For which $n \geq 6$ is there a line graph on n vertices with nonisomorphic neighbourhoods?

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