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Mathematica Slovaca, Vol. 42 (1992), No. 3, 257--263

Persistent URL: http://dml.cz/dmlcz/136555

Mathematica Slovaca
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Slovak Academy of Sciences

Math. Slovaca, **42** (1992), No. 3, 257–263

WEAK RIESZ GROUPS

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ABSTRACT. In this paper a modification of the Riesz decomposition property is investigated on directed *po*-groups. Namely, a lattice characterization of the set of all directed convex subgroups of a directed *po*-group with this decomposition property is described.

Riesz groups are directly partially ordered groups (*po*-groups, briefly) which have the well-known interpolation property (or the decomposition property, equivalently – see [8] and [3]). P e d e r s e n [6] proved that a weak variant of the Riesz decomposition property holds in C^*-algebras. In this paper a similar modification of the Riesz decomposition property is investigated on directed *po*-groups. Namely, a lattice characterization of the set of all directed convex subgroups (*o*-ideals, respectively) of a directed *po*-group with this decomposition property is described.

1. Decomposition on C^*-algebras

P e d e r s e n in [6] shows that the following decomposition property is true in a C^*-algebra A:

If $x, a, b \in A^+$, $0 \le x \le a+b$, then $u, v \in A$ exist such that $x = u^* \cdot u + v^* \cdot v$ and $u \cdot u^* \le a$, $v \cdot v^* \le b$.

In the case that u, v are normal we obtain the Riesz decomposition property. All unexplained facts concerning C^*-algebras can be found in D i x m i e r [1]. The set of all hermitian elements (positive elements) in a C^*-algebra A is denoted by A_h (A^+). Let us denote $|a| = (a^* \cdot a)^{\frac{1}{2}}$ for $a \in A$ (see [2], preceding Th. 2.4).

PROPOSITION 1.1. *If A is a C^*-algebra, then the following assertions are equivalent:*

1. *A has the Riesz decomposition property.*

A M S Subject Classification (1991): Primary 06F20.

K e y w o r d s: Decomposition property, C^*-algebra, Directed *po*-group, Directed convex subgroups.

2. *$a \wedge b = 0$ in A_h if and only if $a \wedge b = 0$ in A^+, for $a, b \in A$.*
3. *A is commutative.*

Proof.

$1 \Longrightarrow 2$: If $a \wedge b = 0$ in A^+ and $c \in A_h$ exists such that $c \leq a, b$, $c \not\leq 0$, then $c \parallel 0$, $c, 0 \leq a, b$ and the Riesz interpolation property implies an existence of $z \in A$ with $0, c \leq z \leq a$, a contradiction.

$2 \Longrightarrow 3$: If $a, b \in A$ and $|a| \wedge |b| = 0$ in A_h, then with regard to [10, 2.11] there holds $a \cdot b = 0$ and $|a| \wedge |b^*| = 0$ in A^+. We have $a^* \cdot b = 0 \Longrightarrow b^* \cdot a = 0 \Longrightarrow |b^*| \wedge |a^*| = 0$ in $A^+ \Longrightarrow |b^*| \wedge |a^*| = 0$ in $A_h \Longrightarrow b^* \cdot a^* = 0 \Longrightarrow a \cdot b = 0$. Further, $a \cdot b = 0 \Longrightarrow (a^*)^* \cdot b = 0$ and the previous consideration implies $a^* \cdot b = 0$. Finally, $a^* \cdot b = 0 \Longleftrightarrow a \cdot b = 0$ holds, for each $a, b \in A$ and according to [10, 2.13] A is commutative.

$3 \Longrightarrow 1$ is clear.

COROLLARY 1.2. *If A is non-commutative C^*-algebra, then A has the Pedersen decomposition property but not the Riesz decomposition property.*

PROPOSITION 1.3. *A C^*-algebra A has the following decomposition property:*

If $x, a, b \in A^+$, $0 \leq x \leq a + b$, then $k, l \in A$ exists such that $k, l \geq 0$, $x = k + l$ and $k \in \overline{AaA}$, $l \in \overline{AbA}$ ($\overline{AaA}$ is the closed ideal in A generated by a).

Proof. The Pedersen decomposition property implies an existence of $u, v \in A$ such that $x = |u|^2 + |v|^2$, $|u^*|^2 \leq a$, $|v^*|^2 \leq b$ hold. If $k = |u|^2$, $l = |v|^2$, then $k, l \geq 0$ and $x = k + l$. We have $|u|^4 = u^* \cdot u \cdot u^* \cdot u = u^* \cdot |u^*|^2 \cdot u \leq u^* \cdot a \cdot u \in \overline{AaA}$ and thus $k = |u|^2 \in \overline{AaA}$ holds (see [2, 2.2]). Similarly $l = |v|^2 \in \overline{AbA}$.

This decomposition property is a generalization of the Riesz decomposition property.

PROPOSITION 1.4. *A C^*-algebra A has the following interpolation property:*

If $x_1, x_2, y_1, y_2 \in A$, $x_i \leq y_j$ for $i, j \in \{1, 2\}$, then elements $z, h, k \in A$ exists such that $x_1 \leq z \leq h + y_1$, $x_2 - k \leq z \leq y_2$, $h, k \leq 0$, $h \in \langle x_1, y_1 \rangle$, $k \in \langle x_2, y_2 \rangle$, where $\langle x_i, y_i \rangle$ is the ideal in A generated by x_i, y_i ($i = 1, 2$).

Proof. We have $y_j - x_i \geq 0$ for $i, j \in \{1, 2\}$ and $y_2 - x_1 = (y_2 - x_2) + (x_2 - x_1) \leq (y_1 - x_1) + (y_2 - x_2)$. According to the Pedersen decomposition property there exist elements $u, v \in A$ such that $y_2 - x_1 = |u|^2 + |v|^2$ and $|u^*|^2 \leq y_1 - x_1$, $|v^*|^2 \leq y_2 - x_2$. If we put $z = |u|^2 + x_1$, then $z \geq x_1$, $y_1 \geq |u^*|^2 + x_1 = |u^*|^2 - |u|^2 + z$ and $z \leq h + y_1$ for $h = \big||u|^2 - |u^*|^2\big|$. Further, $y_2 = |u|^2 + |v|^2 + x_1 \geq |u|^2 + x_1 = z$, $x_2 \leq -|v^*|^2 + y_2 = -|v^*|^2 + |u|^2 + |v|^2 + x_1 = -|v^*|^2 + |v|^2 + z$ hold. Thus we have $x_2 - k \leq z$ for $k = \big|-|v|^2 + |v^*|^2\big|$. Finally,

$|u^*|^2 \in \langle x_1, y_1\rangle$, $|u|^4 = u^* \cdot |u^*|^2 \cdot u \in \langle x_1, y_1\rangle$ and [2, 2.2] implies $|u|^2 \in \langle x_1, y_1\rangle$, i.e., $h \in \langle x_1, y_1\rangle$. Similarly we prove that $k \in \langle x_2, y_2\rangle$.

2. Weak decomposition property on *po*-groups

Effros [2] in Theorem 2.8 describes a bijection between closed ideals of a C^*-algebra A and closed invariant order ideals in A. If I is an ideal in A, then $I \cap A_h$ is an o-ideal (i.e., a directed convex normal subgroup) in a directed po-group A_h. These considerations give the following generalization.

DEFINITION 2.1. *Let G be a directed po-group with the following property:*

If $x, a, b \in G^+$, $0 \le x \le a + b$, then elements $k, l \in G$ exist such that $k, l \ge 0$, $x = k + l$, $k \in \langle a\rangle$ and $l \in \langle b\rangle$, where $\langle a\rangle$ ($\langle b\rangle$, resp.) is a directed convex subgroup in G generated by a (b, resp.). Then we say that G is a weak Riesz group (or G has the weak decomposition property).

Weak Riesz groups fulfil a theorem similar to the theorem of Stormer [9] for C^*-algebras.

PROPOSITION 2.2. *Let G be a directed po-group. Then G is a weak Riesz group if and only if $I^+ + J^+ = (I + J)^+$ holds for arbitrary directed convex subgroups I, J in G.*

Proof.

$\Longrightarrow$: Clearly $I^+ + J^+ \subseteq (I+J)^+$ and if $x \in (I+J)^+$, then $0 \le x \le a+b$ for suitable elements $a \in I^+$ and $b \in J^+$. Thus $k, l \in G$ exist such that $k, l \ge 0$, $x = k + l$, $k \in \langle a\rangle$, $l \in \langle b\rangle$ and it implies $x \in I^+ + J^+$.

$\Longleftarrow$: If $x, a, b \in G^+$, $0 \le x \le a + b$ then there holds $x \in \langle a + b\rangle^+ = \langle a\rangle^+ + \langle b\rangle^+$. Finally, $k \in \langle a\rangle^+$ and $l \in \langle b\rangle^+$ exists such that $x = k + l$.

PROPOSITION 2.3. *If G is a weak Riesz group, $0 \le x \le y_1 + y_2 + \cdots + y_n$ for $x, y_1, y_2, \ldots, y_n \in G^+$, then elements $x_1, x_2, \ldots, x_n \in G^+$ exist such that $x = x_1 + x_2 + \cdots + x_n$ and $x_i \in \langle y_i\rangle$ for $i = 1, 2, \ldots, n$.*

Proof can be done by induction.

PROPOSITION 2.4. *If G is a weak Riesz group, then a sum of directed convex subgroups in G is again a directed convex subgroup in G.*

Proof. If $\{X_i\colon\ i \in I\}$ is a set of directed convex subgroups in G and $X = \sum X_i$, then X_i is generated by X_i^+ for $i \in I$ and thus X is generated by a subset in G^+, i.e., X is directed. If $0 \le y \le x$, $x \in X$, $y \in G$, then $x \le \sum\limits_{i \in K} x_i$ for suitable $x_i \in X_i^+$ and $K \subseteq I$ finite. With regard to 2.3 we have $y = \sum_{i \in K} y_i$ for $y_i \in \langle x_i \rangle^+ \subseteq X_i$ $(i \in K)$. Finally, X is a directed convex subgroup in G.

PROPOSITION 2.5. *Let G be a weak Riesz group. Then there holds:*

1. *If H is a directed convex subgroup in G, then H is a weak Riesz group.*
2. *If H is an o-ideal in G, then G/H is a weak Riesz group.*

Proof.

1. If $x, a, b \in H^+$, $0 \leq x \leq a + b$, then elements $k, l \in G^+$ exist such that $x = k + l$, $k \in \langle a \rangle$, $l \in \langle b \rangle$ and it implies that $k, l \in H$.

2. If $H \leq x + H \leq (a + H) + (b + H)$ for $x, a, b \in G^+$, then elements $c, d \in H^+$ exist such that $0 \leq x + c \leq a + b + d$. Further, there exist $k, l \in G^+$ such that $x + c = k + l$, $k \in \langle a \rangle$, $l \in \langle b + d \rangle$. Thus $x + H = (k + H) + (l + H)$, $k + H, l + H \in G/H^+$ and $k + H \in \langle a + H \rangle$, $l + H \in \langle b + H \rangle$ hold.

PROPOSITION 2.6.

1. *If G is a weak Riesz group, then G has the following interpolation property: If $x_1, x_2, y_1, y_2 \in G$, $x_i \leq y_j$ $(i, j \in \{1,2\})$, then elements $z_1, z_2 \in G$ exist such that $x_1 \leq z_1 \leq y_2$, $x_2 \leq z_2 \leq y_1$ and $z_1, z_2 \in (\langle y_1 - x_1 \rangle + \{x_1\}) \cap (\langle y_2 - x_2 \rangle + \{y_2\})$.*

2. *Let G be a commutative po-group. Then G is a weak Riesz group if and only if G has the following property:*
If $x_1, x_2, y_1, y_2 \in G^+$, $x_1 + x_2 = y_1 + y_2$, then elements $z_{ij} \in G^+$ exist such that $x_i = z_{i1} + z_{i2}$, $y_j = z_{1j} + z_{2j}$ and $z_{ij} \in \langle y_j \rangle$ for $i, j \in \{1,2\}$.

Proof.

1. We have $0 \leq y_2 - x_1 = (y_2 - x_2) + (x_2 - x_1) \leq (y_2 - x_2) + (y_1 - x_1)$ and thus there exist elements $k, l \in G$ such that $k, l \geq 0$, $y_2 - x_1 = k + l$, $k \in \langle y_2 - x_2 \rangle$, $l \in \langle y_1 - x_1 \rangle$. For $z_1 = l + x_1 = -k + y_2$ there holds $y_2 \geq z_1 \geq x_1$, $z_1 \in \langle y_1 - x_1 \rangle + \{x_1\}$, $z_1 \in \langle y_2 - x_2 \rangle + \{y_2\}$. Similarly we can prove existence of an element z_2 of required properties.

2. $\Longrightarrow$: We have $0 \leq x_1 \leq y_1 + y_2$ and thus there exist $z_{11}, z_{12} \in G^+$ such that $x_1 = z_{11} + z_{12}$, $z_{11} \in \langle y_1 \rangle$, $z_{12} \in \langle y_2 \rangle$. For $z_{2j} = -z_{1j} + y_j$ there holds $y_j = z_{1j} + z_{2j}$ $(j = 1,2)$ and $x_1 + x_2 = y_1 + y_2 = z_{11} + z_{21} + z_{12} + z_{22} = x_1 + z_{21} + z_{22}$. It implies $x_2 = z_{21} + z_{22}$, where $z_{21} \in \langle y_1 \rangle$, $z_{22} \in \langle y_2 \rangle$.

$\Longleftarrow$: If $x, a, b \in G^+$, $0 \leq x \leq a + b$, then we have $a + b = (a + b - x) + x$ and thus there exist $z_{21}, z_{22} \in G^+$ such that $x = x_2 = z_{21} + z_{22}$, $z_{21} \in \langle a \rangle$, $z_{22} \in \langle b \rangle$.

DEFINITION. *Let G be a directed po-group with the following property:*

If $x, a, b \in G^+$, $0 \le x \le a + b$, then elements $k, l \in G$ exist such that $k, l \ge 0$, $x = k + l$, $k \le a$, $l \in \langle b \rangle$. Then we say that G is a semiweak Riesz group (an sw-Riesz group).

PROPOSITION 2.7.

1. *An sw-Riesz group G has the interpolation property from Proposition 2.6 and $z_1 \ge x_2$.*

2. *If G is an sw-Riesz group, then a meet of two directed convex subgroups in G is again a directed convex subgroup in G.*

Proof.

1. If we repeat the proof of Prop. 2.6, 1., then we receive that G has the interpolation property and $k \le y_2 - x_2$, i.e., $z_1 = -k + y_2 \ge x_2$.

2. If A, B are directed convex subgroups in G, then $A \cap B$ is a convex subgroup in G. If $x \in A \cap B$, then $p \in A$, $q \in B$ exist such that $0, x \le p, q$. There exist elements $z_1, z_2 \in (\langle q - x \rangle + \{x\}) \cap \langle p \rangle$ such that $x \le z_1 \le p$, $0 \le z_2 \le q$, $z_1 \ge 0$. Finally, we have $z_1, z_2 \in A \cap B$, $z_1 + z_2 \ge 0$, $z_1 + z_2 \ge z_1 \ge x$, $z_1 + z_2 \in A \cap B$. $A \cap B$ is a directed subgroup in G.

3. Lattice characterization

The lattice of all convex l-subgroups of a lattice-ordered group G was investigated by M. Jakubíková [4]. This lattice is a complete distributive lattice which is a complete sublattice of the lattice of all subgroups of G. This result was generalized by J. Rachůnek [7] for the case of Riesz groups. Let us investigate a similar situation for sw-Riesz groups.

THEOREM 3.1. *If G is an sw-Riesz group, then the set $C(G)$ of all directed convex subgroups in G is a locale.*

Remark. Let us recall that a *locale* is a complete lattice L in which the infinite distributive law $a \wedge \bigvee S = \bigvee\{a \wedge S\colon s \in S\}$ holds for all $a \in L$ and $S \subseteq L$. The important examples of locales are lattices of all open sets of topological spaces. All unexplained facts concerning locales can be found in Johnstone [5].

Proof of 3.1. Let $A_i \in C(G)$ be for $i \in I$ and let $\left[\bigcup_{i\in I} A_i\right]$ denote a subgroup generated by $\bigcup_{i\in I} A_i$. Then each element $x \in \left[\bigcup_{i\in I} A_i\right]$ has the form $x = \sum_{i\in K} a_i$ for suitable elements $a_i \in A_i$ and a finite subset $K \subseteq I$. If $g \in G$,

$0 \leq g \leq x$ then 2.3 implies an existence of elements $g_i \in G^+$ such that $g = \sum_{i\in K} g_i$ and $g_i \in \left[\bigcup_{i\in I} A_i\right]$ for $i \in K$. $\left[\bigcup_{i\in I} A_i\right]$ is convex and let us prove that it is also directed. If $x = \sum_{i\in K} a_i$, $y = \sum_{i\in L} b_i$ are two elements from $\left[\sum_{i\in I} A_i\right]$, then from the directness of A_i it implies that there exist $z_i \in A_i$, $z_i \geq a_i, 0$ for all $i \in K$ and $z_i \geq b_i, 0$ for all $i \in L$. We have $x, y \leq \sum_{i\in K\cup L} z_i \in \left[\sum_{i\in I} A_i\right]$. Joins of $A_i \in C(G)$ are also subgroups generated by $\bigcup A_i$ and finite meets are meets of sets (see 2.7).

Now, let us verify the corresponding distributive law: If A, $B_i \in C(G)$ for $i \in I$, then $A \cap \bigvee_{i\in I} B_i \supseteq \bigvee_{i\in I}(A \cap B_i)$ clearly. If $a \in A \cap \bigvee_{i\in I} B_i$, then there exists an element $\bar{a} \in A \cap \bigvee_{i\in I} B_i$ such that $\bar{a} \geq a$, $O \geq -\bar{a}$. We have $\bar{a} = \sum_{i\in K} \bar{b}_i$ for suitable elements $\bar{b}_i \in B_i^+$ and $i \in K$, $K \subseteq I$ finite (see 2.3) and thus $\bar{b}_i \in A \cap B_i$ for $i \in K$. Finally, $\bar{a} \in \bigvee_{i\in I}(A \cap B_i)$ and from the convexity $a \in \bigvee_{i\in I}(A \cap B_i)$ holds.

COROLLARY 3.2. *If G is an sw-Riesz group, then the set $I(G)$ of all o-ideals in G is a locale with respect to arbitrary sums and finite meets.*

Proof follows from 3.1 and 2.4.

Recall that a *locale* L is regular when $l = \bigvee(x \in L\colon x^* \vee l = 1)$ holds for each $l \in L$, where $x^* = \bigvee(y \in L\colon y \wedge x = 0)$.

PROPOSITION 3.3. *Let G be an sw-Riesz group. Then a locale $I(G)$ is regular if and only if each principal o-ideal in G is a direct summand in G.*

Proof.

$\Longrightarrow$: $\langle g\rangle = \sum_{i\in I}\{X_i\colon X_i^* + \langle g\rangle = G \text{ for } i \in I\}$ holds for each $g \in G^+$. Since $g = \sum_{i\in K} x_i$ for suitable $x_i \in X_i$ and finite set $K \subseteq I$ there holds $\langle g\rangle = \sum_{i\in K} X_i$. Distributivity of $I(G)$ implies $G = \bigcap_{i\in K}(X_i^* + \langle g\rangle) = \bigcap_{i\in K} X_i^* + \langle g\rangle$ and $\bigcap_{i\in K} X_i^* = \left(\sum_{i\in K} X_i\right)^* = \langle g\rangle^*$.

$\Longleftarrow$: Clearly, $A = \sum_{a\in A^+}\left(\langle a\rangle\colon \langle a\rangle^+ + \langle a\rangle = G\right)$ holds for each $A \in I(G)$ and $\bigvee(X \in I(G)\colon X^* \vee A = G) \subseteq A$ because $X = G \wedge X = (X^* \vee A) \wedge X = (X^* \wedge X) \vee (A \wedge X) = A \wedge X$. Finally, $I(G)$ is regular.

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Received September 22, 1989

Revised July 17, 1991

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