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ANTINEIGHBOURHOOD GRAPHS

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ABSTRACT. A graph H is called a neighbourhood graph if there exists a graph G in which the subgraph induced by the neighbours of each vertex is isomorphic to H. A graph H is said to be a j-antineighbourhood graph if there exists a graph G in which, for each vertex v of G, the subgraph induced by the vertices at distance at least $j+1$ from v is isomorphic to H. The classes of neighbourhood and j-antineighbourhood graphs are denoted by $\mathcal{N}$ and $\mathcal{A}_j$, respectively. It is shown that every graph belongs to $\mathcal{A}_j$ with $j \geq 2$, and that a graph belongs to $\mathcal{A}_0$ if and only if it is a vertex-deleted subgraph of a vertex-symmetric graph. Some examples and properties of graphs which belong to $\mathcal{A}_0$ are given. It is shown that a graph belongs to $\mathcal{A}_1$ if and only if its complement belongs to $\mathcal{N}$. Next, the block graphs which belong to $\mathcal{A}_1$ are determined. Finally, some results on cycles whose squares belong to $\mathcal{N}$ and to $\mathcal{A}_1$ are also included.

1. All graphs considered in this paper are finite, undirected, and with no loops or multiple edges. For a graph G, let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. For a vertex v of G, let $N_G(v)$ be the set of vertices (neighbours) adjacent to v in G and, more generally, $N_G(S) = \bigcup_{v \in S} N_G(v)$ for a subset S of $V(G)$. If X is a subset of $V(G)$, then $G - X$ denotes the subgraph of G induced by $V(G) - X$. We write $G - x$ instead of $G - \{x\}$ for $x \in V(G)$. For a vertex v of G, the neighbourhood graph $N(v, G)$ of the vertex v is the subgraph of G induced by the set $N_G(v)$. Denote by $\mathcal{N}$ the set of all graphs H with the property that there exists a graph G in which the neighbourhood graph of every vertex is isomorphic to H. The problem which graphs belong to $\mathcal{N}$ was raised by Zykov in [37]. Many papers on this subject have been published. Some of these papers investigated which graphs H are in $\mathcal{N}$ and some characterized, for a given graph $H \in \mathcal{N}$, all graphs G such that $N(v, G)$ is isomorphic to H for any vertex v of G. For example, [4] lists all trees with fewer than 10 vertices which belong to $\mathcal{N}$. Similarly, [18] presents all graphs on 6 or fewer vertices which are in $\mathcal{N}$. Many

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examples and characterizations of graphs from $\mathcal{N}$ in more restricted classes of graphs were obtained in [4, 5, 7-9, 11-14, 17, 18, 21, 34, 36]. The ideas and methods of group theory applied to Zykov's problem (and to related problems) gave many interesting and important results in [4, 5, 12, 13, 15-18, 22, 23, 30, 31]. Some generalizations and modifications of Zykov's problem were considered in [2, 3, 24-28, 32, 35].

We consider the next modification in which we wish to change somewhat the point of view. Let j be a non-negative integer. For a vertex v of a graph G, the j-antineighbourhood graph $A_j(v,G)$ of the vertex v is the subgraph of G induced by the set $\{u \in V(G)\colon d_G(u,v) \geq j+1\}$, where $d_G(x,y)$ denotes the distance between vertices x and y in G. Certainly, for a vertex v of G, $A_0(v,G)$ is the vertex-deleted subgraph $G-v$ of G. Similarly, $A_1(v,G)$ is obtained from G by removing the vertex v and all its neighbours. Let $\mathcal{A}_j$ denote the set of all graphs H with the property that there exists a graph G in which the j-antineighbourhood graph $A_j(v,G)$ of every vertex v of G is isomorphic to H. It is natural to ask about graphs which belong to the set $\mathcal{A}_j$ ($j = 0, 1, \ldots$). In Section 2, we characterize the graphs of $\mathcal{A}_0$ in terms of vertex-symmetric graphs. We also present some examples and structural properties of graphs from $\mathcal{A}_0$. The connection between the graphs from the set $\mathcal{A}_1$ and those which belong to $\mathcal{N}$ is given in Section 3. Then we consider the problem of characterizing block graphs which belong to $\mathcal{A}_1$. We have some results for cycles whose squares belong to $\mathcal{A}_1$ and $\mathcal{N}$, respectively. Finally, in Section 4, it is indicated that every graph H belongs to the class $\mathcal{A}_j$ for each integer $j \geq 2$.

In general, we follow the terminology and notation of Harary [19], and introduce new notation as it is required. Let $d_H(v)$, $\delta(H)$ and $\Delta(H)$ denote the degree of a vertex v in a graph H, the minimum degree and maximum degree of H, respectively. A graph H is regular if $\delta(H) = \Delta(H)$. A graph H is biregular if $\delta(H) < \Delta(H)$ and each vertex of H is of degree either $\delta(H)$ or $\Delta(H)$. For a graph H, let $(H)_\delta$ denote the graph obtained from H by adding a new vertex and joining it to all vertices of degree $\delta(H)$ in H. For example, we have $(K_n)_\delta \cong K_{n+1}$. The symbols $F \cup G$, $F + G$, $F[G]$ and $F \times G$ represent the union, join, lexicographic product and cartesian product of graphs F and G, respectively. By nG we denote the disjoint union of n copies of a graph G. A path, cycle, and complete graph with n vertices is denoted by P_n, C_n, and K_n, respectively. $K_{n_1,\ldots,n_p}$ denotes a complete p-partite graph with the vertex classes having n_1, $n_2, \ldots, n_p$ vertices, respectively. A wheel W_n on $n+1$ vertices is a graph isomorphic to $C_n + K_1$. The complement graph of a graph G is denoted by $\overline{G}$. By $\cong$ we denote an isomorphism of graphs.

A vertex v of a graph G is called a cut vertex of G if $G - v$ has more components than G. A connected graph with no cut vertex is called a block. A block of a graph G is a subgraph of G which is itself a block and which is maximal with respect to that property. A block H of G is called an end block of G if H has at most one cut vertex of G. A connected graph G is a block graph if every block of G is a complete graph. Note that if v is a non-cut vertex in a block graph G, then the vertices of $N_G(v) \cup \{v\}$ induce a block in G.

2. Before proceeding to a characterization of graphs which belong to $\mathcal{A}_0$, we recall some useful definitions and facts. In a graph H, two vertices v and u are said to be similar if there exists an automorphism α of H such that $\alpha(v) = u$. A graph H is said to be vertex-symmetric if every two vertices of H are similar. Two edges vu and tw of a graph H are similar if there exists an automorphism α of H such that $\{\alpha(v), \alpha(u)\} = \{t, w\}$. A graph is edge-symmetric if each pair of its edges is similar. A graph is symmetric if it is both vertex-symmetric and edge-symmetric. There is an important class of graphs known as circulants. Following Boesch and Tindell [6], for an integer $n \geq 3$ and a subset S of $\{1, 2, \ldots, \lfloor (n+1)/2 \rfloor\}$, the circulant graph $C_n(S)$ is a graph on n vertices $v_0, v_1, \ldots, v_{n-1}$, where each vertex v_i is adjacent to the vertices $v_{i \pm s}$ for $s \in S$ (the subscripts are taken modulo n). Certainly, $C_n(\emptyset) \cong \overline{K_n}$, $C_n(\{1\}) \cong C_n$, and $C_n(\{1, 2\})$ is isomorphic to the square C_n^2 of C_n. It is easy to observe that circulant graphs are vertex-symmetric. The converse is not true since, for example, $C_4 \times K_2$ is a vertex-symmetric graph which is not circulant. However, Turner [29] has proved that every vertex-symmetric graph of prime order is a circulant graph. For further results about vertex-symmetric, edge-symmetric, and symmetric graphs, the reader is referred to the book by Yap [33] and the paper [10]. Other papers on this subject can be found in the references of Yap [33, pp. 145–155].

We now state and prove a characterization of graphs which belong to $\mathcal{A}_0$ in terms of vertex-symmetric graphs. The proof is based in part on facts announced in [20].

THEOREM 1. *A graph H belongs to $\mathcal{A}_0$ if and only if $H = nK_1$ for some positive integer n or its supergraph $(H)_\delta$ is a vertex-symmetric graph.*

Proof. Certainly, $nK_1 \in \mathcal{A}_0$ since $(n+1)K_1 - v \cong nK_1$ for each vertex v of $(n+1)K_1$. Suppose now that $(H)_\delta$ is a vertex-symmetric graph and let w be a vertex such that $(H)_\delta - w = H$. Since $(H)_\delta$ is vertex-symmetric, for each vertex v of $(H)_\delta$ there exists an automorphism α of $(H)_\delta$ that maps w to v. Then α restricted to $(H)_\delta - w$ is an isomorphism between $(H)_\delta - w = H$ and $(H)_\delta - v$. Hence $H \in \mathcal{A}_0$.

To prove the converse we assume that $H \in \mathcal{A}_0$ and $H \neq nK_1$. Let G be a supergraph of H with $V(G) = V(H) \cup \{w\}$ and such that $G - v \cong H$ for each $v \in V(G)$. We prove that $G \cong (H)_\delta$ and G is vertex-symmetric.

First we show that G is regular. Let v and u be two vertices of G. Since the graphs $G - v$ and $G - u$ are isomorphic, they have the same number of edges. Hence, $|E(G)| - d_G(v) = |E(G-v)| = |E(G-u)| = |E(G)| - d_G(u)$ and, therefore, $d_G(v) = d_G(u)$. That establishes the regularity of G. Surely, $\delta(G-w) = \delta(G) - 1$ and $\{v \in V(G-w)\colon d_{G-w}(v) = \delta(G-w)\} = N_G(w)$. It now follows easily that G is isomorphic to $(G-w)_\delta$ and, therefore, to H.

In order to prove that G is vertex-symmetric, it suffices to show that for every vertex v of G there exists an automorphism α of G for which $\alpha(w) = v$. Let $\alpha^*\colon V(H) \to V(G-v)$ be an isomorphism between H and $G-v$. Since α^* maps the set $\{x \in V(H)\colon d_H(x) = \delta(H)\} = N_G(w)$ onto the set $\{y \in V(G-v)\colon d_{G-v}(y) = \delta(G-v)\} = N_G(v)$, the function $\alpha\colon V(G) \to V(G)$, where $\alpha(x) = \alpha^*(x)$ if $x \in V(H)$ and $\alpha(w) = v$, is the desired automorphism. □

The following two results follow easily from Theorem 1, and they are of help in deciding whether or not a given graph belongs to the family $\mathcal{A}_0$.

COROLLARY 1. *A graph H belongs to $\mathcal{A}_0$ if and only if it is a vertex-deleted subgraph of a vertex-symmetric graph.* □

COROLLARY 2. *If a graph H belongs to $\mathcal{A}_0$, then exactly one of the following statements is true:*

(i) *H is regular and $H = nK_1$ or $H = K_n$ for some positive integer n;*
(ii) *H is biregular, in which case (a) $\Delta(H) = \delta(H) + 1$ and (b) H has exactly $\delta(H) + 1$ vertices of degree $\delta(H)$.*

□

Note that the converse of Corollary 2 is not true. This can be seen with the aid of the graph H illustrated in Fig. 1. This graph satisfies the condition (ii) of Corollary 2, but it does not belong to $\mathcal{A}_0$ since its supergraph $(H)_\delta$ is not vertex-symmetric as it has some vertices that are contained in two triangles and others which are not.

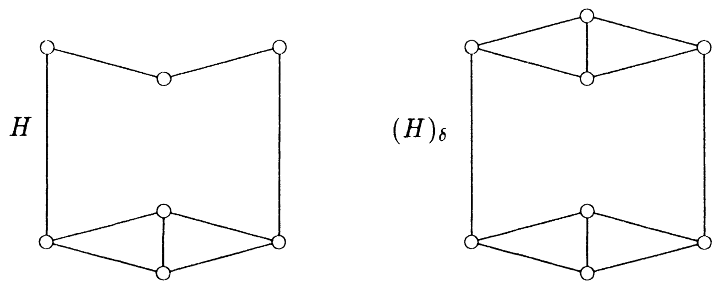

Figure 1.

COROLLARY 3. *A cycle C_n belongs to $\mathcal{A}_0$ if and only if $n = 3$.*

Proof. The result follows easily from Corollary 2. □

COROLLARY 4. *A wheel W_n belongs to $\mathcal{A}_0$ if and only if $n = 3$ or $n = 4$.*

Proof. The assertion is apparent for W_3 since $W_3 \cong K_4$. Since $(W_4)_\delta \cong C_4 + 2K_1$ is a vertex-symmetric graph, $W_4 \in \mathcal{A}_0$ by Theorem 1. Finally, Corollary 2 implies that $W_n \notin \mathcal{A}_0$ for $n \geq 5$ since in this case W_n is biregular and $\Delta(W_n) \geq \delta(W_n) + 2$. □

COROLLARY 5. *A block graph H belongs to $\mathcal{A}_0$ if and only if H is a complete graph or a path.*

Proof. According to Corollary 2, every complete graph belongs to $\mathcal{A}_0$. In particular, the path $P_1 = K_1 \in \mathcal{A}_0$. If $n \geq 2$, then $C_{n+1} - v \cong P_n$ for each $v \in V(C_{n+1})$ and thus $P_n \in \mathcal{A}_0$.

Conversely, assume that a block graph H belongs to $\mathcal{A}_0$ and H is not a complete graph. Let V_δ be the set of vertices of degree $\delta(H)$ in H. By Corollary 2, $|V_\delta| = \delta(H) + 1 = \Delta(H)$. Since H is not a complete graph, H has at least two end blocks and each of them has exactly $\delta(H)$ vertices of degree $\delta(H)$. It follows that $\delta(H) + 1 = |V_\delta| \geq 2\delta(H)$. Then $\delta(H) = 1 = \Delta(H) - 1$, so H is a path. □

THEOREM 2. *A graph H belongs to $\mathcal{A}_0$ if and only if its complement $\overline{H}$ belongs to $\mathcal{A}_0$.*

Proof. This follows from the fact that $\overline{G - v} = \overline{G} - v$ for every graph G and each vertex v of G. □

Let $\mathcal{A}_0^c$ be the subfamily of $\mathcal{A}_0$ consisting of all connected graphs which belong to $\mathcal{A}_0$. Since a graph or its complement graph is connected, in our effort to find all graphs of $\mathcal{A}_0$, Theorem 2 allows us to concentrate on the graphs of the family $\mathcal{A}_0^c$. However, for disconnected graphs we have a useful result.

THEOREM 3. *A disconnected graph H with $p \geq 2$ components belongs to $\mathcal{A}_0$ if and only if either $H \cong pK_1$ or $H \cong F \cup (p-1)(F)_\delta$ for some graph F from $\mathcal{A}_0^c$.*

Proof. Since the "if" part is apparent, we prove the "only if" part. Suppose that a graph H with $p \geq 2$ components belongs to $\mathcal{A}_0$ and $H \not\cong pK_1$. Since $(H)_\delta$ is vertex-symmetric (by Theorem 1), it does not have a cut vertex, so $(H)_\delta$ has also p components. Certainly, these components must be mutually isomorphic and vertex-symmetric. Thus, there exists a connected vertex-symmetric graph G ($\neq K_1$) such that $(H)_\delta \cong pG$. Consequently, $H \cong (H)_\delta - v \cong pG - u \cong$

$(G-u)\cup(p-1)(G-u)_\delta$ for every vertex v of $(H)_\delta$ and every vertex u of pG. Since G is vertex-symmetric, $G-u\in\mathcal{A}_0$ by Corollary 1. But G has no cut vertex, so $G-u$ is connected and it belongs to $\mathcal{A}_0^c$. Thus $H\cong F\cup(p-1)(F)_\delta$ for $F=G-u\in\mathcal{A}_0^c$. □

A similar result holds for graphs whose complements are disconnected.

COROLLARY 6. *If H is a graph whose complement has $p\ge 2$ components, then H belongs to $\mathcal{A}_0$ if and only if either $H\cong K_p$ or $H\cong\overline{F}+K_{p-1}\left[\overline{(F)_\delta}\right]$ for some graph $F\in\mathcal{A}_0^c$.*

P r o o f. Since the complement graph $\overline{H}$ of H has $p\ge 2$ components, Theorems 2 and 3 imply that H belongs to $\mathcal{A}_0$ if and only if either $\overline{H}\cong pK_1$ or $\overline{H}\cong F\cup(p-1)(F)_\delta$ for some $F\in\mathcal{A}_0^c$. But this is equivalent to saying that either $H\cong\overline{pK_1}\cong K_p$ or $H\cong\overline{F\cup(p-1)(F)_\delta}\cong\overline{F}+\overline{(p-1)(F)_\delta}\cong\overline{F}+K_{p-1}\left[\overline{(F)_\delta}\right]$ for some $F\in\mathcal{A}_0^c$. □

COROLLARY 7. *A complete p-partite graph $K_{n_1,\dots,n_p}$ with $p\ge 2$ and $n_1\le n_2\le\dots\le n_p$ belongs to $\mathcal{A}_0$ if and only if either $n_1=\dots=n_p=1$ or $n_1+1=n_2=\dots=n_p$ for any positive integer n_1.*

P r o o f. The result is immediate if $n_1=n_2=\dots=n_p=1$. If $n_1+1=n_2=\dots=n_p$ with $p\ge 2$ and $n_1\ge 1$, then $K_{n_1,\dots,n_p}\cong\overline{K_{n_1}}+K_{p-1}\left[\overline{(K_{n_1})_\delta}\right]$ belongs to $\mathcal{A}_0$ by Corollaries 2 and 6. Suppose now that $K_{n_1,\dots,n_p}\in\mathcal{A}_0$ for some integers $n_1\le n_2\le\dots\le n_p$, where $p\ge 2$ and $n_p>1$. Since $\overline{K_{n_1,\dots,n_p}}=K_{n_1}\cup\dots\cup K_{n_p}$ belongs to $\mathcal{A}_0$ (by Theorem 2), it follows from Theorem 3 that we must have $\overline{K_{n_1,\dots,n_p}}\cong K_{n_1}\cup(p-1)(K_{n_1})_\delta\cong K_{n_1}\cup(p-1)K_{n_1+1}$. Hence $n_1+1=n_2=\dots=n_p$. □

To conclude this section, we describe the line and total graphs which belong to $\mathcal{A}_0$.

THEOREM 4. *If G is a graph, then its line graph $L(G)$ belongs to $\mathcal{A}_0$ if and only if there exists an edge-symmetric graph H such that $L(G)\cong L(H-e)$ for some edge e of H.*

P r o o f. Assume that the line graph $L(G)$ belongs to $\mathcal{A}_0$. According to Theorem 1 and Corollary 2, $\big(L(G)\big)_\delta$ is vertex-symmetric, or $L(G)\cong\overline{K_n}$, or $L(G)\cong K_n$ for some positive integer n. Certainly, if $L(G)\cong\overline{K_n}$ ($L(G)\cong K_n$, resp.), then the graph $H=(n+1)K_2$ ($H=K_{1,n+1}$, resp.) has the desired properties. Thus assume that $\big(L(G)\big)_\delta$ is a vertex-symmetric graph.

First we claim that $\big(L(G)\big)_\delta$ is a line graph, that is, $\big(L(G)\big)_\delta$ is isomorphic to the line graph $L(F)$ of some graph F. Assume the contrary. Then by Beineke's theorem [1] (see [19, p. 74]), at least one of the nine forbidden graphs $G_1, \ldots, G_9$ shown in Fig. 8.3 of [19] is an induced subgraph of $(L(G))_\delta$. Moreover, since each vertex-deleted subgraph $(L(G))_\delta - x$ of $(L(G))_\delta$ is a line graph (as $(L(G))_\delta - x \cong L(G)$), $(L(G))_\delta$ is isomorphic to one of the graphs $G_1, \ldots, G_9$. This contradicts the fact that $(L(G))_\delta$ is vertex-symmetric; therefore we must reject the assumption that $(L(G))_\delta$ is not a line graph. Consequently, there exists a graph F such that $(L(G))_\delta \cong L(F)$. Certainly, $L(F)$ is vertex-symmetric and $L(G) \cong L(F-e)$ for each edge e of F. Now, if we replace each component which is isomorphic to $K_{1,3}$ in F (if any) by a component isomorphic to K_3, the line graph of the resulting graph H is isomorphic to $L(F)$ and H does not contain both K_3 and $K_{1,3}$ as components. Then Theorem 6 of [10], which states that a graph which does not contain both K_3 and $K_{1,3}$ as components is edge-symmetric if and only if its line graph is vertex-symmetric, implies that H is edge-symmetric. Moreover, $L(G) \cong L(H-e)$ for each edge e of H.

Conversely, assume that H is an edge-symmetric graph such that $L(G) \cong L(H-e)$ for some edge e of H. It follows from the edge symmetry of H that $H - e \cong H - f$ for each edge f of H (see Theorem 5 in [10]). Thus $L(G) \cong L(H-f) = L(H) - f$ for each vertex $f \in V(L(H)) = E(H)$, so $L(G) \in \mathcal{A}_0$. □

THEOREM 5. *If G is a graph, then its total graph $T(G)$ belongs to $\mathcal{A}_0$ if and only if $G \cong K_2$ or $G \cong \overline{K_n}$ for some positive integer n.*

P r o o f. Certainly, $T(K_2) \cong K_3$ and $T(\overline{K_n}) \cong \overline{K_n}$ $(n \geq 1)$ belong to $\mathcal{A}_0$. Conversely, assume that G is a graph such that $T(G) \in \mathcal{A}_0$. Combining this with Corollary 2 we conclude that $T(G)$ is a regular graph; for if $T(G)$ were not regular, then G would not be regular and, therefore, we would have $\Delta(G) \geq \delta(G) + 1$ and so $\Delta(T(G)) = 2\Delta(G) \geq 2\delta(G) + 2 = \delta(T(G)) + 2$, which is impossible. Corollary 2 now yields that $T(G) \cong K_n$ or $T(G) \cong \overline{K_n}$ for some positive integer n. This clearly forces that $G \cong K_2$ or $G \cong \overline{K_n}$ for some positive integer n. □

3. We collect here some results on the classes $\mathcal{A}_1$ and $\mathcal{N}$. We begin by observing the relationship between these two classes.

THEOREM 6. *A graph H belongs to $\mathcal{N}$ if and only if its complement $\overline{H}$ belongs to $\mathcal{A}_1$.*

P r o o f. It is easy to observe that $\overline{N(v,G)} = A_1(v,\overline{G})$ for every graph G and each vertex v of G. This implies the result. □

The above theorem implies that all the elements of $\mathcal{A}_1$ are determined by the elements of the class $\mathcal{N}$, and vice versa. In particular, the following result follows immediately from Theorem 6 and corresponding results of [8, 14, 21, 34].

LEMMA 1.

(1) [21] *For every positive integer* n, K_n *and* $\overline{K_n}$ *belong to* $\mathcal{A}_1$;
(2) [34] *A cycle* C_n *belongs to* $\mathcal{A}_1$ *if and only if* $n = 3, 4, 5$, *or* 6;
(3) [8] *The complement* $\overline{C_n}$ *of a cycle* C_n *belongs to* $\mathcal{A}_1$ *for every* $n \geq 3$;
(4) [8] *The complement* $\overline{P_n}$ *of a path* P_n *belongs to* $\mathcal{A}_1$ *if and only if* $n \neq 3$;
(5) [14] *The union* $K_{n_1} \cup \cdots \cup K_{n_p}$ *of complete graphs* $K_{n_1}, \dots, K_{n_p}$ *belongs to* $\mathcal{A}_1$ *if and only if* $n_1 = \cdots = n_p$;
(6) [21] *A graph* H *with* n *isolated vertices belongs to* $\mathcal{A}_1$ *if and only if either* $H \cong \overline{K_n}$ *or* $H \cong \overline{K_n} \cup F\big[\overline{K_{n+1}}\big]$ *where* F *is some graph without isolated vertices which belongs to* $\mathcal{A}_1$;
(7) [21] *If graphs* H_1 *and* H_2 *belong to* $\mathcal{A}_1$, *then their join* $H_1 + H_2$ *belongs to* $\mathcal{A}_1$;
(8) [21] *If* H *is a graph in which no vertex is adjacent to all other vertices of* H, *then* H *belongs to* $\mathcal{A}_1$ *if and only if* $H + K_n$ *belongs to* $\mathcal{A}_1$ *for a positive integer* n.

□

The parts (1) and (7) of Lemma 1 imply that every complete p-partite graph $K_{n_1,\dots,n_p} \cong \overline{K_{n_1}} + \overline{K_{n_2}} + \cdots + \overline{K_{n_p}}$ belongs to $\mathcal{A}_1$. It follows from the parts (1), (2), and (8) that a wheel $W_n = C_n + K_1$ belongs to $\mathcal{A}_1$ if and only if $n = 3, 4, 5$, or 6. Many other examples, properties, and structural characterizations of graphs which belong (or do not belong) to $\mathcal{A}_1$ can be obtained from the results of [4, 5, 7-9, 11, 13-15, 17, 18, 21, 23, 36].

In [34], Zelinka proved that every path belongs to $\mathcal{A}_1$. Our intent now is to characterize the block graphs which belong to $\mathcal{A}_1$. The following two definitions and two lemmas will be relevant in the sequel.

A block graph G is a regular windmill graph if it is isomorphic to $nK_p + K_1$ for some positive integers n and p. Note that a regular windmill graph $nK_p + K_1$ is a tree if and only if it is a star $K_{1,n} \cong nK_1 + K_1$. For an integer $n \geq 4$, we denote by M_n the graph obtained by taking $n-2$ disjoint copies of K_n and a new vertex v_0, and then joining the vertex v_0 to exactly one vertex in each copy of K_n. Note that M_n is a graph of maximum degree n, minimum degree $n-2$, and it has exactly one vertex of minimum degree and this vertex is a center of M_n. Figure 2 shows M_4.

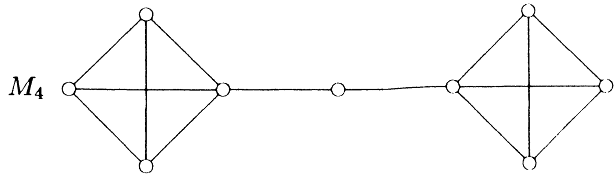

Figure 2.

LEMMA 2. *Let G be a block graph of diameter d, maximum degree Δ, and minimum degree δ. If $d \geq 3$ and $\Delta \geq \delta + 2$, then either in G there exist nonadjacent vertices x and y such that $|d_G(x) - d_G(y)| \geq 2$ or $G \cong M_\Delta$.*

Proof. Let v and u be vertices of degree Δ and δ in G, respectively. Note that v and every other vertex of degree Δ is a cut vertex in G. Let v' be any farthest vertex from v in G. Since $d \geq 3$, the vertices v and v' are not adjacent. In addition, v' is not a cut vertex and it belongs to an end block of G. Let B be the end block that contains v', and let v'' be the only cut vertex adjacent to v'. It follows that $\Delta \geq d_G(v'') \geq d_G(v') + 1$ and $d_G(x) = d_G(v')$ for each $x \in V(B) - \{v''\}$. We distinguish two cases depending on the difference $\Delta - \delta$.

Case: $\Delta > \delta+2$. In this case it is straightforward to see that if $d_G(v') \leq \Delta-2$ ($d_G(v') = \Delta - 1$, resp.), then the vertices v and v' (u and v', resp.) have the required property.

Case: $\Delta = \delta+2$. Assume that $|d_G(x)-d_G(y)| \leq 1$ for every two nonadjacent vertices x and y of G. We need only show that G is isomorphic to M_Δ. Our assumption implies that every vertex of degree $\Delta-2$ is adjacent to every vertex of degree Δ. It follows that $d_G(v') = \Delta - 1$, so $d_G(v'') = \Delta$. Consequently, $N_G(v'') - V(B) = \{u\}$ and, therefore, u is a unique vertex of degree $\Delta - 2$ in G. Assume that $N_G(u) = \{v_1, \ldots, v_{\Delta-2}\}$. We claim that each vertex of $N_G(u)$ is a cut vertex. For if not, let v_i be a counterexample. Then the set $N_G(v_i) \cup \{v_i\}$ induces a block of order $d_G(v_i) + 1 \geq \Delta$ in G. But then we have $d_G(x) \geq \Delta-1$ for each $x \in N_G(v_i) \cup \{v_i\}$, which is impossible since $u \in N_G(v_i)$ and $d_G(u) = \Delta - 2$. This implies the desired claim. In order to complete the proof, it suffices to show that each vertex of $N_G(u)$ belongs to an end block of order Δ. For this purpose, let X_i be the set of all vertices x of G for which the shortest $x-u$ path passes through the vertex v_i ($i \in \{1, \ldots, \Delta-2\}$). We now claim that X_i induces a block of order Δ in G. To prove this, let v_i' be a vertex of X_i which is farthest from u. Since v_i is a cut vertex, v_i' is not adjacent to u and, therefore, $d_G(v_i') = \Delta - 1$. Furthermore, the choice of v_i' implies that v_i' is a non-cut vertex and it belongs to some end block of G, say B_i is an end block such that $v_i' \in V(B_i)$. Since $d_G(v_i') = \Delta - 1$, B_i is an end block of order Δ. Note that the unique cut vertex which belongs to B_i is of degree Δ and therefore it must be the vertex v_i. Consequently, v_i is a vertex of degree Δ and B_i is an end block induced by X_i. Hence $\bigcup_{i=1}^{\Delta-2} V(B_i) = \bigcup_{i=1}^{\Delta-2} X_i = V(G) - \{u\}$. Finally, since no two vertices of $\{v_1, \dots, v_{\Delta-2}\}$ are adjacent (as the vertices of $N_G(v_i) = V(B_i) \cup \{u\} - \{v_i\}$ ($i = 1, \dots, \Delta-2$) are of degree $\Delta - 1$ and $\Delta - 2$), it follows from the above that G is isomorphic to M_Δ. □

LEMMA 3. *For every integer $n \geq 4$, M_n does not belong to $\mathcal{A}_1$.*

P r o o f. Assume to the contrary that $M_n \in \mathcal{A}_1$ for some $n \geq 4$. Let G be a graph such that $A_1(x, G) \cong M_n$ for each vertex x of G. Fix an arbitrary vertex x_0 of G and consider the graph $A_1(x_0, G)$. Since $A_1(x_0, G)$ is isomorphic to M_n, let v_0 be the unique vertex of degree $n - 2$ in $A_1(x_0, G)$, and let $v_1, \dots, v_{n-2}$ be the neighbours of v_0 in $A_1(x_0, G)$. For each $i \in \{1, \dots, n-2\}$, let B_i be the end block of $A_1(x_0, G)$ that contains the vertex v_i. Certainly, B_i is a block of order n. We now choose any vertex $v \in V(B_1) - \{v_1\}$ and consider the graph $A_1(v, G)$. Since $A_1(v, G)$ is isomorphic to M_n and $V(A_1(v, G)) \cap (V(G) - N_G(x_0)) = \{x_0, v_0\} \cup \bigcup_{i=2}^{n-2} V(B_i)$, exactly $n - 1$ vertices of $A_1(v, G)$ belong to $N_G(x_0)$, say $V(A_1(v, G)) \cap N_G(x_0) = \{x_1, \dots, x_{n-1}\}$. It is a simple matter to observe that v_0 must be the unique vertex of degree $n - 2$ in $A_1(v, G)$, the vertices $x_0, x_1, \dots, x_{n-1}$ form an end block in $A_1(v, G)$ (we denote it by B_{n-1}), exactly one vertex of $\{v_0\} \cup \bigcup_{i=2}^{n-2} V(B_i)$ is adjacent to exactly one of the vertices $x_0, x_1, \dots, x_{n-1}$, and precisely v_0 must be adjacent to exactly one of the vertices $x_1, \dots, x_{n-1}$, say v_0 is adjacent to x_{n-1}. Next consider the graph $A_1(v_0, G)$. In this graph we have $V(A_1(v_0, G)) = \bigcup_{i=1}^{n-1} V(B_i) - \{v_1, \dots, v_{n-2}, x_{n-1}\}$. Furthermore, every vertex of $A_1(v_0, G)$ belongs to a complete subgraph of order at least $n - 1 \geq 3$ in $A_1(v_0, G)$. This contradicts the fact that $A_1(v_0, G)$ is isomorphic to M_n because the center of M_n only belongs to complete subgraphs of order at most two in M_n. Consequently, M_n does not belong to $\mathcal{A}_1$. □

We now state and prove the main theorem of this section.

THEOREM 7. *If G is a block graph, then G belongs to $\mathcal{A}_1$ if and only if G is a path or a regular windmill graph.*

P r o o f. Assume that G is a block graph of diameter d, maximum degree Δ, and minimum degree δ. The result is clear if $d \leq 1$. Thus assume that $d \geq 2$

and consider two cases: $d = 2$ or $d \geq 3$.

Case: $d = 2$. In this case $G = (K_{n_1} \cup \cdots \cup K_{n_p}) + K_1$ for some positive integers $n_1 \leq \cdots \leq n_p = n$ and $p \geq 2$. It follows from the parts (5) and (8) of Lemma 1 that $G \in \mathcal{A}_1$ if and only if $n_1 = \cdots = n_p = n$, that is, if and only if G is a regular windmill graph, $G = pK_n + K_1$.

Case: $d \geq 3$. It is clear that if G is a path, $G = P_{d+1}$, then $G \in \mathcal{A}_1$ since $A_1(v, C_{d+4}) \cong P_{d+1}$ for each vertex $v \in V(C_{d+4})$. It remains to prove that G does not belong to $\mathcal{A}_1$ if it is not a path. Suppose to the contrary that G belongs to $\mathcal{A}_1$ and G is not a path. Then, let H be a graph such that $A_1(x, H) \cong G$ for each vertex x of H. Let r be the maximum degree in H. Our assumptions on G imply that $\Delta > \delta \geq 1$. In addition, it is easy to observe that H is a regular graph of degree r and $r \geq \Delta \geq 3$. Now let x_0 be an arbitrary vertex of H and consider the graph $A_1(x_0, H) \cong G$. We distinguish two subcases: $\Delta \geq \delta + 2$ or $\Delta = \delta + 1$.

Subcase: $\Delta \geq \delta + 2$. Using Lemma 3 we need only consider the case $G \not\cong M_\Delta$. In this case Lemma 2 implies that there exist two nonadjacent vertices v and u in $A_1(x_0, H)$ such that for their degrees $d_v = d_{A_1(x_0,H)}(v)$ and $d_u = d_{A_1(x_0,H)}(u)$ we have $|d_v - d_u| \geq 2$, say $d_v \geq d_u + 2$. We now consider the graphs $A_1(v, H)$ and $A_1(u, H)$ each of which is isomorphic to G. The graphs $A_1(v, H)$ and $A_1(u, H)$ have d_v and d_u vertices in $N_H(x_0)$, respectively. This implies that $|N_H(v) \cap N_H(x_0)| = r - d_v$ and $|N_H(u) \cap N_H(x_0)| = r - d_u$. Since $d_v \geq d_u + 2$, we can find two different vertices x and y such that $\{x, y\} \subseteq (N_H(u) - N_H(v)) \cap N_H(x_0)$. Certainly, the vertices x, y, x_0, and u belong to the block graph $A_1(v, H)$ and, therefore, they are mutually adjacent in $A_1(v, H)$ and in H. Hence, $u \in N_H(x_0)$ and, therefore, $u \notin V(A_1(x_0, H))$, a contradiction.

Subcase: $\Delta = \delta + 1$. In this case every vertex of the graph $A_1(x_0, H) \cong G$ is of degree Δ or $\Delta - 1$, and each vertex of degree Δ is a cut vertex in $A_1(x_0, H)$. In addition, each end block of $A_1(x_0, H)$ is of order Δ. Moreover, if B is an end block in $A_1(x_0, H)$ and z is a unique cut vertex of $A_1(x_0, H)$ that belongs to B, then z is a vertex of degree Δ and, therefore, there exists exactly one other block B' in $A_1(x_0, H)$ that contains z, and B' is of order two. This implies that if a vertex x belongs to an end block in $A_1(x_0, H)$ then it belongs to exactly one such block and we denote this block by $B_e(x)$. We now choose a vertex v_0 in $A_1(x_0, H)$ such that $d_{A_1(x_0,H)}(v_0, v) = d$ for some vertex v of $A_1(x_0, H)$. Let V_d be the set of all vertices at distance d from v_0 in $A_1(x_0, H)$. We shall use the notation $V_d' = N_{A_1(x_0,H)}(V_d) - V_d$ and $V_d'' = N_{A_1(x_0,H)}(V_d') - V_d$. It is clear that the sets V_d, V_d', V_d'' are nonempty and mutually disjoint. Moreover, each vertex x of $V_d \cup \{v_0\}$ is a non-cut vertex and it belongs to some end block of $A_1(x_0, H)$, each vertex x' of V_d' is a cut vertex and it belongs to some end block of $A_1(x_0, H)$, each vertex x'' of V_d'' is a cut vertex, and for each vertex

x' of V_d' there exists exactly one vertex x'' in V_d'' such that together they form a block in $A_1(x_0,H)$.

We now consider the graph $A_1(v_0,H) \cong G$. In this graph we have $V(A_1(v_0,H)) = \Big(V(A_1(x_0,H)) - V(B_e(v_0))\Big) \cup \{x_0\} \cup (N_H(x_0) - N_H(v_0))$. Therefore the set $N_H(x_0)-N_H(v_0)$ has $\Delta-1$ vertices, say $N_H(x_0)-N_H(v_0) = \{x_1,\dots,x_{\Delta-1}\}$. Since $A_1(v_0,H)$ is connected and $N_{A_1(v_0,H)}(x_0) = \{x_1,\dots,x_{\Delta-1}\}$, there exists at least one edge joining a vertex of $\{x_1,\dots, x_{\Delta-1}\}$ to a vertex of $V(A_1(x_0,H)) - V(B_e(v_0))$. We may assume that $x_{\Delta-1}$ is adjacent to a vertex $x^* \in V(A_1(x_0,H)) - V(B_e(v_0))$. Then no other vertex of $\{x_1,\dots,x_{\Delta-2}\}$ is adjacent to a vertex of $A_1(x_0,H) - V(B_e(v_0))$. For if a vertex x_i ($i \in \{1,\dots,\Delta-2\}$) were adjacent to a vertex $x' \in V(A_1(x_0,H)) - V(B_e(v_0))$, then a $x^* - x'$ path (in $A_1(x_0,H)$) together with the edges $x'x_i$, x_ix_0, $x_0x_{\Delta-1}$, $x_{\Delta-1}x^*$ would form a cycle in $A_1(v_0,H)$. Consequently, the vertices x_0 and x^* would be in the same block of the block graph $A_1(v_0,H)$ and, therefore, they would be adjacent in $A_1(v_0,H)$ and in H, which is impossible since $x^* \in V(A_1(x_0,H)) = V(H) - (N_H(x_0) \cup \{x_0\})$. We therefore henceforth assume that $x_{\Delta-1}$ is a unique vertex of $\{x_1,\dots,x_{\Delta-1}\}$ which is adjacent to a vertex of $A_1(x_0,H) - V(B_e(v_0))$ in $A_1(v_0,H)$. We now show that the vertices $x_0,x_1,\dots,x_{\Delta-1}$ form an end block in $A_1(v_0,H)$. To prove this, it would suffice to show that every vertex x_i ($1 \le i \le \Delta-2$) is adjacent to the vertex $x_{\Delta-1}$ (since $A_1(v_0,H)$ is a block graph and $N_{A_1(v_0,H)}(x_0) = \{x_1,\dots,x_{\Delta-1}\}$). Suppose to the contrary that some vertex $x_{i_0} \in \{x_1,\dots,x_{\Delta-2}\}$ is not adjacent to $x_{\Delta-1}$. Then $N_{A_1(v_0,H)}(x_{i_0}) \subseteq \{x_0,\dots,x_{\Delta-2}\} - \{x_{i_0}\}$ and therefore $d_{A_1(v_0,H)}(x_{i_0}) < \Delta - 1$, a contradiction. This proves that the vertices $x_0,\dots,x_{\Delta-1}$ form an end block of order Δ in $A_1(v_0,H)$. Moreover, since $\{x^*,x_0,\dots,x_{\Delta-2}\} \subseteq N_{A_1(v_0,H)}(x_{\Delta-1})$, $d_{A_1(v_0,H)}(x_{\Delta-1}) = \Delta$ and, therefore, there exists exactly one vertex $x^* \in V(A_1(x_0,H)) - V(B_e(v_0))$ adjacent to $x_{\Delta-1}$ in $A_1(v_0,H)$.

Next we show that x^* is a vertex of degree $\Delta-1$ in $A_1(x_0,H)$. Suppose to the contrary that $d_{A_1(x_0,H)}(x^*) = \Delta$. Then consider the graph $A_1(x^*,H)$. Clearly, v_0 is a vertex of $A_1(x^*,H)$. In addition, exactly $\Delta+1$ vertices of $A_1(x^*,H)$ belong to the set $N_H(x_0)\cup\{x_0\}$ which is the disjoint union of the sets $\{x_0,\dots,x_{\Delta-1}\}$ and $N_H(x_0) \cap N_H(v_0)$. By the above we have $V(A_1(x^*,H)) \cap \{x_0,\dots,x_{\Delta-1}\} = \{x_0,\dots,x_{\Delta-2}\}$. Hence, there exist two vertices y and y' such that $V(A_1(x^*,H)) \cap N_H(x_0) \cap N_H(v_0) = \{y,y'\}$. But then x_0, y, y', and v_0 belong to the same block of $A_1(x^*,H)$. Thus x_0 is adjacent to v_0 in $A_1(x^*,H)$ and in H, a contradiction. This shows that $d_{A_1(x_0,H)}(x^*) = \Delta - 1$.

It follows from the above that $x^* \notin V_d'$ (since each vertex of V_d' is of degree

Δ in $A_1(x_0, H)$). We now show that $x^* \notin V_d''$. For $d = 3$, this is evident. Thus assume that $d \geq 4$ and suppose to the contrary that $x^* \in V_d''$. Since $d_{A_1(x_0,H)}(x^*) = \Delta - 1$, there exists a vertex $y \in N_H(x_0) \cap N_H(v_0)$ such that $V(A_1(x^*, H)) \cap (N_H(x_0) \cup \{x_0\}) = \{y, x_0, \ldots, x_{\Delta-2}\}$. Now let us observe that if v belongs to $N_{A_1(x_0,H)}(x^*) \cap V_d'$, then $N_{A_1(x^*,H)}(x) \subseteq V(B_e(v)) \cup \{y\} - \{v\}$ and $d_{A_1(x^*,H)}(x) \geq \Delta - 1$ for each $x \in V(B_e(v)) - \{v\}$. Hence, $N_{A_1(x^*,H)}(x) = V(B_e(v)) \cup \{y\} - \{v\}$ for each $x \in V(B_e(v)) - \{v\}$. Consequently, $V(B_e(v)) \cup \{x_0, v_0\} - \{v\} \subseteq N_{A_1(x^*,H)}(y)$ and $d_{A_1(x^*,H)}(y) \geq \Delta + 1$, a contradiction. This shows that $x^* \notin V_d''$.

Next we show that $x^* \in V_d$. This is clear if $d = 3$. Assume that $d \geq 4$ and suppose to the contrary that $x^* \notin V_d$ and, therefore, $x^* \notin V_d \cup V_d' \cup V_d''$. Choose any vertex $v \in V_d'$ and consider the graph $A_1(v, H)$. It is easy to observe that there exists a vertex $y \in N_H(x_0) \cap N_H(v_0)$ such that $V(A_1(v, H)) = V\Big(A_1(v, A_1(x_0, H))\Big) \cup \{y, x_0, \ldots, x_{\Delta-1}\}$. On the other hand, our assumption on x^* implies that the vertices v_0 and x^* are in the same component of $A_1(v, A_1(x_0, H))$. But now an $x^* - v_0$ path in $A_1(v, A_1(x_0, H))$ and the edges $v_0 y$, $y x_0$, $x_0 x_{\Delta-1}$, $x_{\Delta-1} x^*$ form a cycle in the block graph $A_1(v, H)$. This implies that the vertices v_0 and x_0 are adjacent in $A_1(v, H)$ and in H, which is impossible. It follows that $x^* \in V_d$..

Let $P = (v_0, v_1, \ldots, v_d = x^*)$ be the shortest $v_0 - x^*$ path in $A_1(x_0, H)$. Let B_i denote the block of $A_1(x_0, H)$ that contains the vertices v_{i-1} and v_i of P ($i = 1, \ldots, d$). Note that B_1 and B_d are end blocks of $A_1(x_0, H)$. As a matter of fact, B_1 and B_2 are the only end blocks of $A_1(x_0, H)$. For if not, let B be another end block of $A_1(x_0, H)$, and let $z \in V(B)$ be a vertex of degree $\Delta - 1$ in $A_1(x_0, H)$. Clearly, B is disjoint with the blocks $B_1, \ldots, B_d$, and, therefore, we have $d_{A_1(x_0,H)-V(B)}(v_0, x^*) = d_{A_1(x_0,H)}(v_0, x^*) = d$. On the other hand, observe that $A_1(z, H)$ consists of $A_1(x_0, H) - V(B)$, the block induced by the vertices $x_0, \ldots, x_{\Delta-1}$, and the edge $x^* x_{\Delta-1}$. This combined with the above forces $d_{A_1(z,H)}(v_0, x_0) = d + 2$, which is impossible since $A_1(z, H) \cong G$ is a graph of diameter d. This proves that B_1 and B_d are the only end blocks of $A_1(x_0, H)$. Consequently, B_1, $B_2, \ldots, B_d$ are the only blocks of $A_1(x_0, H)$. Moreover, d is an odd integer, and B_1, $B_3, \ldots, B_d$ are blocks of order Δ, while $B_2, B_4, \ldots, B_{d-1}$ are blocks of order two.

The proof may now be completed. We distinguish two cases: $d > 3$ or $d = 3$. First, if $d > 3$, then we consider the graph $A_1(v_{d-2}, H)$. Certainly, there exists $y \in N_H(v_0) \cap N_H(x_0)$ such that $V\big(A_1(v_{d-2}, H)\big) = V\big(A_1(x_0, H)\big) \cup \{y, x_0, \ldots, x_{\Delta-1}\} - V(B_{d-2}) - \{v_{d-1}\}$. Note that the set $V(B_d) - \{v_{d-1}, v_d\}$ is nonempty and each its vertex x is adjacent to the vertex y in $A_1(v_{d-2}, H)$ (otherwise $d_{A_1(v_{d-2},H)} < \Delta - 1$). This implies that the vertices of $V(B_d) \cup$

$\{y, x_0, \ldots, x_{\Delta-1}\} - \{v_{d-1}\}$ are in the same block of $A_1(v_{d-2}, H)$. Hence, the vertex x_0 is adjacent to the vertex $x^* = v_d$ in $A_1(v_{d-2}, H)$ and in H, a contradiction. If $d = 3$, then we consider the graphs $A_1(v_0, H)$, $A_1(v_3, H)$, $A_1(v_3', H)$ for $v_3' \in V(B_3) - \{v_2, v_3\}$, and $A_1(v_1, H)$. Note that there exists a vertex $t \in N_H(v_0) \cap N_H(x_0)$ such that $V(A_1(v_3, H)) = V(B_1) \cup \{t, x_0, \ldots, x_{\Delta-2}\}$, $N_{A_1(v_3,H)}(t) = \{v_0, x_0, \ldots, x_{\Delta-2}\}$, and, therefore,

$$N_{A_1(v_3,H)}(v_1) \cap \{t, x_0, \ldots, x_{\Delta-2}\} = \emptyset .$$

On the other hand, since $V(A_1(v_3', H)) = V(B_1) \cup \{x_0, \ldots, x_{\Delta-1}\}$, the vertex t must be adjacent to the vertex v_3'. Finally, there exists a vertex $z \in N_H(x_0) - \{x_0, \ldots, x_{\Delta-2}\} - \{t\}$ such that $V(A_1(v_1, H)) = \{t, z, x_0, \ldots, x_{\Delta-2}\} \cup V(B_3) - \{v_2\}$. Since $\{v_3', x_0, \ldots, x_{\Delta-2}\} \subseteq N_{A_1(v_1,H)}(t)$ and $\{t, z, x_1, \ldots, x_{\Delta-2}\} \subseteq N_{A_1(v_1,H)}(x_0)$, the vertices t and x_0 are of degree Δ in $A_1(v_1, H)$ and they do not form a bridge in $A_1(v_1, H)$ (as x_1 is their common neighbour). This implies that $A_1(v_1, H)$ is not isomorphic to G, which is a final contradiction. □

COROLLARY 8. *If G is a tree, then $G \in \mathcal{A}_1$ if and only if G is a path or a star.* □

In the next theorem we determine the cycles whose squares belong to the family $\mathcal{N}$.

THEOREM 8. *If C_n is a cycle of length n, then its square C_n^2 belongs to $\mathcal{N}$ if and only if $n = 3, 4, 5$, or 6.*

Proof. It is clear that $C_3^2 \cong K_3$, $C_4^2 \cong K_4$, $C_5^2 \cong K_5$ belong to $\mathcal{N}$. Moreover, C_6^2 belongs to $\mathcal{N}$ since $N(v, C_6^2 + \overline{K_2}) \cong C_6^2$ for each vertex v of $C_6^2 + \overline{K_2}$. Thus, it remains to prove that C_n^2 does not belong to $\mathcal{N}$ if $n \geq 7$. Suppose to the contrary that $C_n^2 \in \mathcal{N}$ for some $n \geq 7$. Let G be a graph such that $N(x, G) \cong C_n^2$ for each $x \in V(G)$. Fix an arbitrary vertex x_0 of G. Since $N(x_0, G)$ is isomorphic to C_n^2, we may assume that $v_0, v_1, \ldots, v_{n-1}$ are the vertices of $N(x_0, G)$, and $v_i v_{i+1}$ and $v_i v_{i+2}$ are the edges of $N(x_0, G)$ (all indices are taken modulo n). We next consider the graph $N(v_2, G)$ which is also isomorphic to C_n^2. Assume that $u_0, u_1, \ldots, u_{n-1}$ are the vertices of $N(v_2, G)$, and let $u_i u_{i+1}$ and $u_i u_{i+2}$ be the edges of $N(v_2, G)$. Since x_0 and the vertices v_0, v_1, v_3, v_4 of $N(x_0, G)$ belong to $N(v_2, G)$, without loss of generality we may assume that $u_0 = v_0$, $u_1 = v_1$, $u_2 = x_0$, $u_3 = v_3$, and $u_4 = v_4$ (see Figure 3). But now the graph $N(v_3, G)$ contains a subgraph with the edges $x_0 v_1$, $x_0 v_2$, $x_0 v_4$, $x_0 v_5$, $v_1 v_2$, $v_2 v_4$, $v_2 u_5$, $v_4 v_5$, and $v_4 u_5$, which is impossible in C_n^2 for $n \geq 7$. This contradicts $N(v_3, G) \cong C_n^2$, and our theorem follows. □

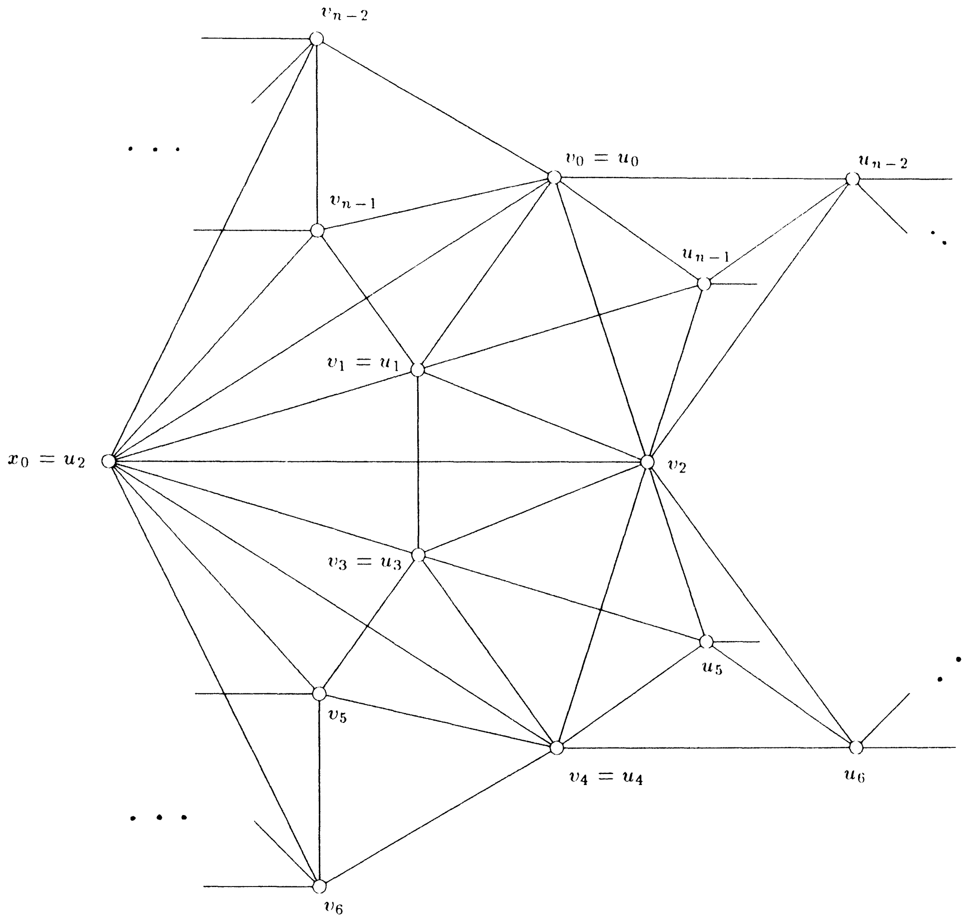

Figure 3.

Finally, we have the following partial result on cycles whose squares belong to $\mathcal{A}_1$.

THEOREM 9. *If C_n is a cycle of length n, then its square C_n^2 belongs to $\mathcal{A}_1$ for $n \in \{3,4,5,6,7\}$, and C_n^2 does not belong to $\mathcal{A}_1$ for $n \geq 10$.*

Proof. Certainly, $C_3^2 \cong K_3$, $C_4^2 \cong K_4$, $C_5^2 \cong K_5$, $C_6^2 \cong K_{2,2,2} \cong \overline{K_2} + \overline{K_2} + \overline{K_2}$, and $C_7^2 \cong \overline{C_7}$ belong to $\mathcal{A}_1$ by Lemma 1. We now claim that C_n^2 does not belong to $\mathcal{A}_1$ if $n \geq 10$. Suppose to the contrary that $C_n^2 \in \mathcal{A}_1$ for some $n \geq 10$. Then there exists a graph G such that $A_1(x, G) \cong C_n^2$ for each $x \in V(G)$. Take any vertex x_0 of G and consider the graph $A_1(x_0, G)$. Assume that

$\{v_0, v_1, \dots, v_{n-1}\}$ is the vertex set and $\{v_iv_{i+1}, v_iv_{i+2}: i = 0, 1, \dots, n-1\}$ is the edge set of $A_1(x_0, G)$ (all indices are taken modulo n). Now, the graph $A_1(v_2, G) \cong C_n^2$ contains the vertices $v_5, v_6, \dots, v_{n-1}$ of $A_1(x_0, G)$, the vertex x_0, and four vertices of $N_G(x_0)$. Assume that $V(A_1(v_2, G)) = \{u_0, u_1, \dots, u_{n-1}\}$, $V(A_1(v_2, G)) \cap N_G(x_0) = \{u_0, u_1, u_3, u_4\}$, $u_2 = x_0$, and $u_5 = v_5, \dots, u_{n-1} = v_{n-1}$. Then, without loss of generality we may assume that $\{u_iu_{i+1}, u_iu_{i+2}: i = 0, 1, \dots, n-1\}$ is the edge set of $A_1(v_2, G)$. But now, since $n \geq 10$, the vertices v_{n-2}, v_{n-1}, v_0, v_1, u_0, u_1 belong to the graph $A_1(v_5, G)$ and $\{v_{n-2}, v_0, v_1, u_0, u_1\} \subseteq N_{A_1(v_5,G)}(v_{n-1})$. This implies that v_{n-1} is a vertex of degree at least 5 in $A_1(v_5, G)$ and, therefore, $A_1(v_5, G)$ is not isomorphic to C_n^2, a contradiction. This proves that C_n^2 does not belong to $\mathcal{A}_1$ if $n \geq 10$. □

There are a number of questions raised by the results presented in this section. We present some of them. Indeed, the following simple question is still unresolved by us. Do the graphs C_8^2 and C_9^2 belong to $\mathcal{A}_1$? Since the square C_n^2 of C_n is a circulant graph, Theorems 8 and 9 also raise the more general question: characterize those circulant graphs which belong to $\mathcal{N}$ and those which belong to $\mathcal{A}_1$.

Hall [17] has proved that the Petersen graph belongs to $\mathcal{N}$. It is natural to ask which generalized Petersen graphs (see [33, p. 2] for the definition) belong to $\mathcal{N}$ and which of them belong to $\mathcal{A}_1$. In particular, which products $C_n \times K_2$ belong to $\mathcal{N}$ ($\mathcal{A}_1$, resp.)? Note that $C_3 \times K_2$ belongs to $\mathcal{N}$, since $N(x, G) \cong C_3 \times K_2$ for each vertex x of the graph G shown in Figure 4. Similarly, $C_3 \times K_2$ belongs to $\mathcal{A}_1$, since $C_3 \times K_2$ is isomorphic to $\overline{C_6}$, and $\overline{C_6}$ belongs to $\mathcal{A}_1$ by Lemma 1. Buset [11] has shown that $C_4 \times K_2$ belongs to $\mathcal{N}$.

A sunflower of order $n \geq 3$ is a graph S_n with the vertices $v_0, v_1, \dots, v_{n-1}$, $u_0, u_1, \dots, u_{n-1}$ and the edges v_iv_{i+1}, u_iv_i, and u_iv_{i+1} (i is taken modulo n). It is easy to check that $S_3 \notin \mathcal{N}$. On the other hand, Figure 5 exhibits a graph F of order 16 (the opposite sides of the square are to be identified) such that $N(x, F) \cong S_4$ for each $x \in V(F)$. This implies that $S_4 \in \mathcal{N}$. For which n does there exist a finite graph G such that $N(x, G) \cong S_n$ for each $x \in V(G)$? Which sunflowers belong to $\mathcal{A}_1$?

Finally, find the paths P_n for which the square P_n^2 belongs to $\mathcal{N}$ and those for which the square P_n^2 belongs to $\mathcal{A}_1$.

4. In this section we have only one result.

Theorem 9 Every graph H belongs to $\mathcal{A}_j$ for $j \geq 2$.

P r o o f. It is easy to see that for the lexicographic product $C_{2j+2}[H]$ of the cycle C_{2j+2} of length $2j+2$ and the graph H we have $A_j(v, C_{2j+2}[H]) \cong H$ for each vertex v of $C_{2j+2}[H]$. This implies that $H \in \mathcal{A}_j$. □

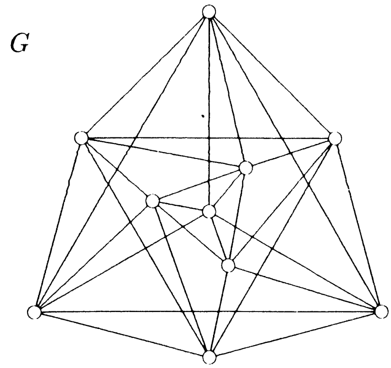

Figure 4.

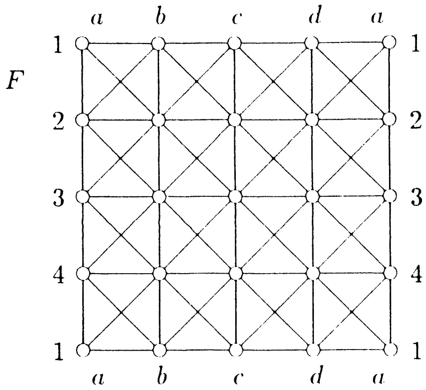

Figure 5.

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