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PRECISE LOWER BOUND FOR THE NUMBER OF EDGES OF MINOR WEIGHT IN PLANAR MAPS

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ABSTRACT. Let $e_{i,j}$ be the number of those edges in a planar map which join i-vertices with j-vertices. For each simplicial 3-polytope, it is proved that $20e_{3,3} + 25e_{3,4} + 16e_{3,5} + 10e_{3,6} + 6\frac{2}{3}e_{3,7} + 5e_{3,8} + 2\frac{1}{2}e_{3,9} + 2e_{3,10} + 16\frac{2}{3}e_{4,4} + 11e_{4,5} + 5e_{4,6} + 1\frac{2}{3}e_{4,7} + 5\frac{1}{3}e_{5,5} + 2e_{5,6} \geq 120$; moreover, each coefficient is the best possible. Similar results are obtained for some other classes of planar maps, thus completely solving some problems raised by E. Jucovič in 1974.

1. Introduction and statement of results

Let $e_{i,j}$ be the number of edges joining the vertices of degree i with the vertices of degree j in a planar map under consideration. Kotzig [3] defined the *weight* of an edge to be the sum of degrees of its end-vertices and proved that in each 3-polytope there exists an edge of the weight at most 13; in other words, $\sum_{i+j\leq 13} e_{i,j} > 0$. (By the Steinitz Theorem [4], 3-polytopes are distinguished among all planar maps by the property that their graphs are 3-connected.) Grünbaum [5, p. 454] conjectured that the number of edges of the weight at most 13 is great enough, or, more specifically, that

$$\begin{gathered} 20e_{3,3} + 15e_{3,4} + 12e_{3,5} + 10e_{3,6} + 6\frac{2}{3}e_{3,7} + 5e_{3,8} + 3\frac{1}{3}e_{3,9} + 2e_{3,10} \\ +12e_{4,4} + 7e_{4,5} + 5e_{4,6} + 4e_{4,7} + 2\frac{2}{3}e_{4,8} + \frac{2}{3}e_{4,9} \\ +4e_{5,5} + 2e_{5,6} + \frac{1}{3}e_{5,7} \\ +12e_{6,6} \geq 120. \end{gathered} \tag{1}$$

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J u c o v i č [2] proved for simplicial polytopes the bound

$$\begin{aligned}20e_{3,3}+25e_{3,4}+16e_{3,5}+10e_{3,6}+6\frac{2}{3}e_{3,7}+5e_{3,8}+2\frac{1}{2}e_{3,9}+2e_{3,10}\\ +20e_{4,4}+11e_{4,5}+5e_{4,6}+5e_{4,7}+5e_{4,8}+3e_{4,9}\\ +8e_{5,5}+2e_{5,6}+2e_{5,7}+2e_{5,8}\geq 120.\end{aligned}\tag{2}$$

As it is easily seen, the coefficients $a_{i,j}$ at $e_{i,j}$ in (2) for $i \leq j \leq 5$ are worse than those in (1). However, J u c o v i č [2] gives examples of 3-polytopes which imply that $a_{3,k}$ for $k \geq 6$, as well as $a_{4,6}$ and $a_{5,6}$ are the best possible, while $a_{3,4} \geq 22$, $a_{3,5} \geq 15$, $a_{4,4} \geq 12$, and $a_{4,5} \geq 7$.

One of the purposes of the present paper is the following final version of the theorem by Jucovič:

THEOREM 1. *For each simplicial 3-polytope, there holds*

$$\begin{aligned}20e_{3,3}+25e_{3,4}+16e_{3,5}+10e_{3,6}+6\frac{2}{3}e_{3,7}+5e_{3,8}+2\frac{1}{2}e_{3,9}+2e_{3,10}\\ +16\frac{2}{3}e_{4,4}+11e_{4,5}+5e_{4,6}+1\frac{2}{3}e_{4,7}\\ +5\frac{1}{3}e_{5,5}+2e_{5,6}\geq 120;\end{aligned}\tag{3}$$

moreover, each coefficient of this inequality is the best possible.

As remarked by J u c o v i č [2], for arbitrary 3-polytopes the proof of (2) in [2] does not work, and one may only prove $\sum_{i+j\leq 13} e_{i,j} \geq 3$.

A question naturally arises as to what the widest classes of planar maps are which allow a bound for the number of edges of minor weight similar to (1)–(3). Trivial examples of complete bipartite graphs $K_{1,n}$ and $K_{2,n}$ and of maps dual to $K_{1,n}$ and a cycle C_n show that the vertices and faces incident with less than three edges must be avoided; i.e., we should restrict ourselves to *normal* maps. Recently, the author extended [1] to all normal planar maps Kotzig's relation $\sum_{i+j\leq 13} e_{i,j} > 0$.

The main result of the present paper is

THEOREM 2. *For each normal planar map there holds*

$$\begin{aligned}40e_{3,3}+25e_{3,4}+16e_{3,5}+10e_{3,6}+6\frac{2}{3}e_{3,7}+5e_{3,8}+2\frac{1}{2}e_{3,9}+2e_{3,10}\\ +16\frac{2}{3}e_{4,4}+11e_{4,5}+5e_{4,6}+1\frac{2}{3}e_{4,7}\\ +5\frac{1}{3}e_{5,5}+2e_{5,6}\geq 120;\end{aligned}\tag{4}$$

moreover, each coefficient of this inequality is the best possible.

This is a further generalization of [1–3] and a complete solution to the problem raised by J u c o v i č in [2]. Other two problems posed in [2] were to find relations similar to our (3) for simplicial 3-polytopes without 3-vertices and also for those without 3- or 5-vertices. Complete answers to these questions follow easily from (4) and considerations in Sections 2:

THEOREM 3. *For each normal planar map without* 3*-vertices,*

$$16\frac{2}{3}e_{4,4} + 11e_{4,5} + 5e_{4,6} + 1\frac{2}{3}e_{4,7} + 5\frac{1}{3}e_{5,5} + 2e_{5,6} \geq 120;$$

moreover, each coefficient is the best possible even in the subclass of simplicial 3*-polytopes.*

THEOREM 4. *For each normal planar map without* 3*- or* 5*-vertices, there holds*

$$16\frac{2}{3}e_{4,4} + 5e_{4,6} + 1\frac{2}{3}e_{4,7} \geq 120;$$

with all the coefficients being the best possible even for the simplicial 3*-polytopes.*

Observe that the inequality (3) in Theorem 1 follows easily from Theorem 2: it suffices to note that $e_{3,3} = 0$ for each simplicial 3-polytope with five or more vertices, whereas $e_{3,3} = 6$ for complete graph K_4. Thus, to prove Theorems 1 and 2, it remains to prove (4) and to give optimal constructions for the coefficients in (3) and (4).

2. Unimprovability of the coefficients in Theorems 1 and 2

All $a_{i,j}$ but $a_{3,3}$ are evidently the same in (3) and (4). We must show that neither of $a_{i,j}$ in (3) or (4) may be decreased, keeping all the other $a_{i,j}$ constant, without violating the correspondent relation.

Remark that for $j \geq 6$ (we everywhere assume $i \leq j$), a stronger assertion is valid: each of $a_{i,j}$ is the minimal among all the relations of the type

$$\sum_{i \leq 5} a_{i,j}e_{i,j} \geq 120.$$

For $a_{4,7}$, this follows from Fig. 1: we here have $e_{4,7} = 72$, while all the other $e_{i,j}$ vanish; therefore, each of (3), (4) implies $a_{4,7} \geq 120/72 = 5/3$. For all $a_{3,k}$ with $6 \leq k \leq 10$, for $a_{4,6}$, and $a_{5,6}$, similar homogeneous optimal constructions were given by J u c o v i č [2].

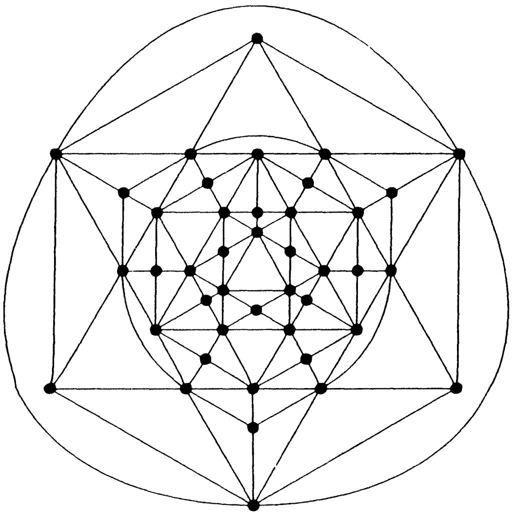

Figure 1.

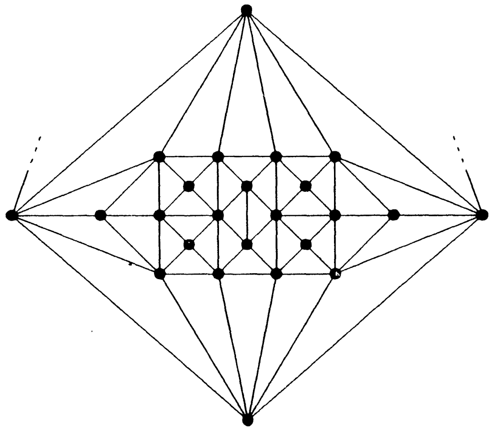

Figure 2.

From now on, let $j \leq 5$. The graph represented in Fig. 2 has $e_{4,6} = 20$, $e_{4,7} = 4$, $e_{5,5} = 1$, and $e_{5,6} = 4$, hence

$$a_{5,5} \geq 120 - 20 \times 5 - 4 \times 1\frac{2}{3} - 4 \times 2 = 5\frac{1}{3}.$$

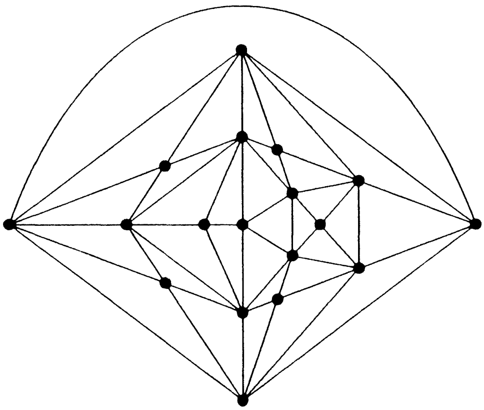

Figure 3.

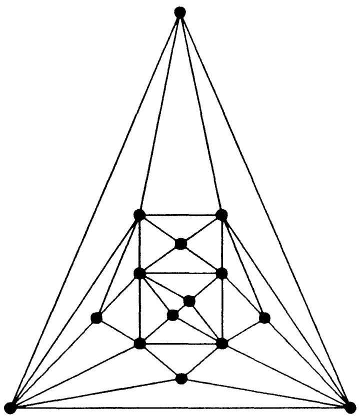

Figure 4.

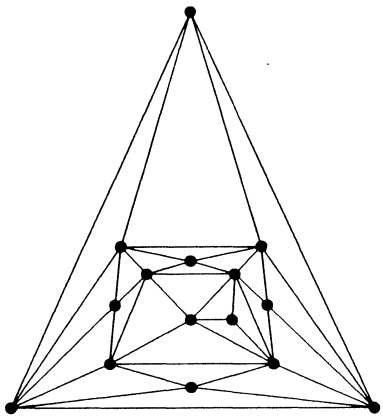

Figure 5.

Similarly,

Fig. 3: $e_{4,5} = 1$, $e_{4,6} = 17$, $e_{4,7} = 6$, $e_{5,6} = 7$, hence

$$a_{4,5} \geq 120 - 17 \times 5 - 6 \times 1\frac{2}{3} - 7 \times 2 = 11;$$

Fig. 4: $e_{4,4} = 1$, $e_{4,6} = 18$, $e_{4,7} = 8$, hence

$$a_{4,4} \geq 120 - 18 \times 5 - 8 \times 1\frac{2}{3} = 16\frac{2}{3};$$

Fig. 5: $e_{3,5} = 1$, $e_{3,7} = 2$, $e_{4,6} = 16$, $e_{4,7} = 4$, $e_{5,6} = 2$, hence

$$a_{3,5} \geq 120 - 2 \times 6\frac{2}{3} - 16 \times 5 - 4 \times 1\frac{2}{3} - 2 \times 2 = 16;$$

Fig. 6: $e_{3,4} = 1$, $e_{3,8} = 15$, $e_{3,9} = 8$, hence

$$a_{3,4} \geq 120 - 15 \times 5 - 8 \times 2\frac{1}{2} = 25.$$

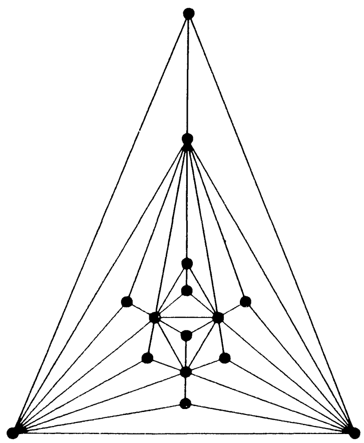

Figure 6.

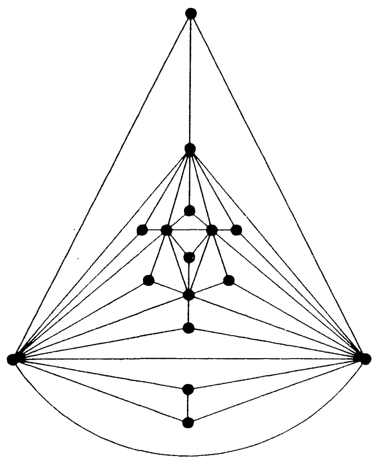

Figure 7.

As already mentioned, the relation $a_{3,3} \geq 20$ for (3) follows from K_4, which has $e_{3,3} = 6$. Examine the map in Fig. 7. It fails to be a polytope since it contains a two-vertex separating set and has $e_{3,3} = 1$, $e_{3,8} = 16$. Hence

$$a_{3,3} \geq 120 - 16 \times 5 = 40 \qquad \text{for} \quad (4).$$

3. Completing the proof of Theorem 2

The validity of (4) remains to be proved. Denote the left part of (4) by $\sum$. Suppose, there exists a normal planar map M for which $\sum < 120$. First, we shall construct, starting from M, a simplicial normal map M^* for which also $\sum < 120$.

Suppose that our M is not simplicial. A ***diagonal*** is defined to be such an edge, d, that $M+d$ is also planar and normal. The end-vertices of the considered diagonal are hereafter denoted by x and y. Let $s(v)$ be the degree of a vertex v, i.e., the number of edges incident with v (loops are counted twice). We shall assume below that $s(x) \leq s(y)$. A diagonal is called ***ckecked*** if $s(x) + s(y) \leq 9$ (in M).

If there exists a non-checked diagonal, d, for M, then $\sum < 120$ for $M + d$ as well, since $a_{i,j}$ in (4) are non-increasing in each subscript.

Let all the diagonals for M be checked. For any diagonal, d, of a face f, denote by d' a diagonal which cuts from f a 3-face with x in its boundary. (To define d' more specifically, introduce the left- and the right-hand clock-wise neighbours, e_1 and e_2, of d in the set of those edges incident with x, and say that d' joins the second end-vertices of e_1, e_2 by "approximating" the chain $[e_1 x e_2]$.) A similar diagonal, defined by y, is denoted by d''.

We are looking for such a diagonal whose adding to M does not increase $\sum$.

Suppose first that there exists a diagonal, d_0, having $s(x) = 4$. When d_0 is added in M, $\sum$ may increase by at most $a_{5,5} = 5\frac{1}{3}$ due to increasing either $e_{5,5}$, or $e_{5,6}$ by 1 (recall that $s(x) \leq s(y)$). On the other hand, at least one of the end-vertices of d_0 has in M the degree at most 4 as all the diagonals are checked. It follows that d_0 being added in, either some $(3,4)$-edge turns into a $(3,5)$-edge, or some $(4,4)$-edge becomes a $(4,5)$-edge. This results in decreasing $\sum$ by at least $a_{4,4} - a_{4,5} = 5\frac{1}{3}$. Therefore, adding d_0 in M does not increase $\sum$ in the end.

Let us assume from now on that each diagonal is incident with a 3-vertex; that is, $s(x) = 3$. A diagonal with the minimal $s(y)$ is denoted by d_*.

If $s(y) = 3$, then adding d_* in M increases $\sum$ by $a_{4,4} = 16\frac{2}{3}$ due to increasing $e_{4,4}$ by 1, but decreases $\sum$ by at least $4(a_{3,3} - a_{3,4}) = 20$ since four $(3,3)$-edges incident with d_* turn into $(3,4)$-edges. This results in a total decreasing of $\sum$.

If $4 \leq s(y) \leq 6$, then similarly, $\sum$ increases by at most $a_{4,5} = 11$, while it decreases by 15 due to turning a $(3,3)$-edge incident with x into a $(3,4)$-edge, i.e., decreases totally.

Thus any non-simplicial normal map may be augmented by a diagonal so that $\sum$ does not increase. Hence, in a finite number of steps, a simplicial normal planar map, M^*, will be constructed which has $\sum < 120$.

The Euler formula $|V|-|E|+|F|=2$ for M^* may be rewritten as follows:

$$\sum_{v\in V}\big(s(v)-6\big) = -12.$$

We shall say that each vertex v has a *charge* $H(v)=s(v)-6$.
Thus

$$\sum_{v\in V} H(v) = -12.$$

Let also $H(e)=0$ for each edge e, and initially all the edges are not labelled. Redistribute the charges of the vertices and edges (keeping the sum of the charges constant) and label some of the edges according to the following Rules 1 – 4:

RULE 1. *If $s(w)\le 6$, then the vertex w transfers to each incident edge a charge $\big(s(w)-6\big)/s(w)$.*

RULE 2. *Let an edge $e=(u,v)$ be in the boundary of a face $[uvw]$. Then e transfers to w:*

$1/3$ *if* $s(u)=s(v)=4$ *and* $s(w)>7$;
$1/15$ *if* $s(u)=s(v)=5$ *and* $s(w)\ge 7$.

RULE 3. *Let an edge $e=(u,v)$, where $s(u)\le s(v)\le 5$, share a face, f, with a vertex w, where $s(w)\ge 7$. Let also $s(u)\le s(v)$; we denote by e' that boundary edge of f which joins w with v. Then e' is declared to be labelled, and e transfers to e' a charge:*

1 *if* $s(u)=s(v)=3$;
$1/2$ *if* $s(u)=3$, $s(v)=4$;
$1/3$ *if* $s(u)=s(v)=4$, $s(w)=7$;
$1/5$ *if* $3\le s(u)\le 4$, $s(v)=5$.

(*Observe that some edge may be labelled twice.*)

RULE 4. *If $s(w)\ge 7$, $s(v)\le 5$, and an edge $e=(w,v)$ is non-labelled, then w transfers to e a charge*

$$\min\Big\{\big(s(w)-6\big)/\lfloor s(w)/2\rfloor, \big(6-s(v)\big)/s(v)\Big\}.$$

The *resulting charge* of any vertex or edge is denoted by the function H^*
From (5),

$$\sum_{v\in V} H^*(v)+\sum_{e\in E} H^*(e) = -12. \tag{6}$$

Almost all the remainder of our proof consist in verifying $H^*(w) \geq 0$ for each vertex w. This will easily yield a concluding contradiction.

If $s(w) \leq 6$, then Rule 1 implies $H^*(w) = 0$. Let $s(w) \geq 7$. We need a few definitions. A face will be called ***labelled*** if it is incident with at least one labelled edge. The number of non-labelled faces of the type $[uvw]$, where $s(u) = s(v) = 4$, will be denoted by l_4. The number l_5 is defined similarly. The edges $e = (u, v)$ incident with at least one non-labelled face $[uvw]$, where $s(u) = s(v) = 4$, will be called ***special*** and their number denoted by L_4. Define the number L_5 similarly. It is evident that $l_4 \leq L_4 \leq 2l_4$ and $l_5 \leq L_5 \leq 2l_5$. Observe that due to Rules 1 - 4 w cannot transfer to two consecutive (i.e., sharing a face $[uvw]$) edges $e_1 = (w, u)$ and $e_2 = (w, v)$ at once, except for the cases when both e_1 and e_2 are either 4-special or 5-special. Denote by k the number of non-special edges which are transferred to by the vertex w; by the just above remark, $k \leq \lfloor s(w)/2 \rfloor$. It is also easily seen that $L_4 + L_5 \leq s(w) - 2k - 1$ with the only exception of $s(w) = 2k$.

Case 1: $s(w) = 7$. Recall that according to the item 3 of Rule 3, we have here $l_4 = L_4 = 0$.

$\underline{k = 3}$: Clearly, $L_5 = 0$; therefore

$$H^*(w) \geq s(w) - 6 - 3\min\Big\{(s(w) - 6)/\lfloor s(w)/2 \rfloor, (6 - s(v))/s(v)\Big\}$$
$$= 1 - 3\min\Big\{1/3, (6 - s(v))/s(v)\Big\} \geq 1 - 3 \times 1/3 = 0\,.$$

$\underline{k = 2}$: At most $2/3$ were transferred to non-special edges, but as one easily sees, $l_5 \leq 1$, i.e., $L_5 \leq 2$. If $l_5 = 0$, then

$$H^*(w) \geq 1 - 2/3 = 1/3\,.$$

Else if $l_5 = 1$, then

$$H^*(w) \geq 1/3 + 1/15 - 2 \times 1/5 = 0\,.$$

$\underline{k = 1}$: At least $2/3$ belong to special edges, therefore it is nothing to prove if $L_5 \leq 3$ (since $2/3 > 3 \times 1/5$). But on the other hand, one may easily verify that $L_5 \leq 4$. However, $l_5 = 3$ when $L_5 = 4$, so that

$$H^*(w) \geq 2/3 + 3 \times 1/15 - 4 \times 1/5 > 0\,.$$

$\underline{k = 0}$: It is nothing to prove if $L_5 \leq 5$; if $L_6 = 6$, we have $l_5 = 5$, and

$$H^*(w) \geq 1 + 5 \times 1/15 - 6 \times 1/5 > 0\,.$$

Let finally $L_5 = 7$, then $l_5 = 7$ and

$$H^*(w) \geq 1 + 7 \times 1/15 - 7 \times 1/5 > 0\,.$$

Case 2: $s(w) = 8$.

$\underline{3 \leq k \leq 4}$: Evidently, $L_4 + L_5 = 0$, that is

$$H^*(w) \geq 2 - 4 \times 1/2 = 0\,.$$

$\underline{k = 2}$: Now $L_4 + L_5 \geq 3$, but since each edge receives at most $1/2$ from w, the only possibility to be considered is $L_4 + L_5 = 3$. But then either $L_4 = 3$ or $L_5 = 3$; respectively, either $l_4 = 2$ or $l_5 = 2$. We have

$$H^*(w) \geq 1 + 2 \times 1/3 - 3 \times 1/2 > 0$$

in the first case and

$$H^*(w) \geq 1 + 2 \times 1/15 - 3 \times 1/5 > 0$$

otherwise.

$\underline{k = 1}$: Now $L_4 + L_5 \leq 5$; since at least $3/2$ remains for non-special edges, the only subcases to be considered are $L_4 + L_5 = 4$ and $L_4 + L_5 = 5$. In the first of them, $l_4 + l_5 \geq 2$; therefore if $l_5 \geq 1$, then

$$H^*(w) \geq 2 - 2 \times 1/5 - 3 \times 1/2 > 0,$$

else if $l_5 = 0$, then

$$H^*(w) \geq 2 + 2 \times 1/3 - 5 \times 1/2 > 0\,.$$

In the second subcase, $l_4 + l_5 = 4$; moreover, either $l_4 = 0$ and then

$$H^*(w) \geq 2 + 4 \times 1/15 - 1/2 - 5 \times 1/5 > 0\,,$$

or $l_5 = 0$ and then

$$H^*(w) \geq 2 + 4 \times 1/3 - 5/2 > 0\,.$$

$\underline{k = 0}$: One should analyse the situations when $L_4 + L_5 \geq 5$. If $L_4 + L_5 = 5$, then $l_4 + l_5 \geq 3$; therefore for $l_4 \geq 2$ we have

$$H^*(w) \geq 2 + 2 \times 1/3 - 5/2 > 0\,,$$

while for $l_4 \leq 1$ there holds $L_4 \leq 2$, which yields

$$H^*(w) \geq 2 - 2 \times 1/2 - 3 \times 1/5 > 0\,.$$

If $L_4 + L_5 = 6$, then obviously, $l_4 + l_5 \geq 4$, so that for $l_4 \geq 3$ we have

$$H^*(w) \geq 2 + 3 \times 1/3 - 6 \times 1/2 = 0;$$

for $l_4 = 2$ one has $L_4 \leq 4$, which implies

$$H^*(w) \geq 2 + 2 \times 1/3 - 4 \times 1/2 - 2 \times 1/5 > 0,$$

and for $l_4 \leq 1$ we have $L_4 \leq 2$ with an immediate consequence of

$$H^*(w) \geq 2 - 2 \times 1/2 - 4 \times 1/5 > 0.$$

Let finally $L_4 + L_5 \geq 7$; then as it is easily seen, either $L_4 = 0$ or $L_5 = 0$, and we have either

$$H^*(w) \geq 2 - 8 \times 1/15 > 0 \quad \text{or} \quad H^*(w) \geq 2 + 6 \times 1/3 - 8 \times 1/2 = 0,$$

respectively.

Case 3: $s(w) = 9$.

$\underline{k = 4}$: $H^*(w) \geq 3 - 4 \times 3/4 = 0$.

$\underline{k = 3}$: Now $L_4 + L_5 \leq 2$; if $l_4 = 0$, then

$$H^*(w) \geq 3 - 3 \times 3/4 - 2 \times 1/5 > 0,$$

otherwise if $l_4 = 1$, then

$$H^*(w) \geq 3 + 1/3 - 3 \times 3/4 - 2 \times 1/2 > 0.$$

$\underline{k = 2}$: Now $L_4 + L_5 \leq 4$, but since the special edges receive at least $3 - 2 \times 3/4 = 3/2$ totally while each of them receives at most $1/2$, the only subcase to be analyzed is $L_4 + L_5 = 4$. However, we then have also either $l_4 = 0$ which implies

$$H^*(w) \geq 3/2 - 4 \times 1/15 > 0,$$

or $l_4 = 3$ with

$$H^*(w) \geq 3/2 + 3 \times 1/3 - 4 \times 1/2 > 0.$$

$\underline{k = 1}$: One should analyse the subcases $L_4 + L_5 = 5$ and $L_4 + L_5 = 6$. In the first subcase, if $l_4 \geq 1$, then

$$H^*(w) \geq 3 + 1/3 - 3/4 - 5 \times 1/2 > 0,$$

otherwise if $l_4 = 0$, then

$$H^*(w) \geq 3 - 3/4 - 5 \times 1/5 > 0.$$

In the second subcase, either $l_4 = 0$ and then

$$H^*(w) \geq 3 - 3/4 - 6 \times 1/5 > 0,$$

or $l_4 = 5$ and then

$$H^*(w) \geq 3 + 5 \times 1/3 - 3/4 - 6 \times 1/2 > 0.$$

$\underline{k = 0}$: We should consider the subcases $L_4 + L_5 = 7$ and $L_4 + L_5 \geq 8$. In the first subcase, if $l_4 \geq 2$, then

$$H^*(w) \geq 3 + 2 \times 1/3 - 7 \times 1/2 > 0,$$

or if $l_4 \leq 1$, then $L_4 \leq 2$ and

$$H^*(w) \geq 3 - 2 \times 1/2 - 5 \times 1/5 > 0.$$

In the second subcase, either $l_4 = 0$ and then

$$H^*(w) \geq 3 - 9 \times 1/5 > 0,$$

or $l_5 = 0$ and

$$H^*(w) \geq 3 + 7 \times 1/3 - 9 \times 1/2 > 0.$$

Case 4: $s(w) = 10$.

$\underline{4 \leq k \leq 5}$: $L_4 + L_5 = 0$ and

$$H^*(w) \geq 4 - 5 \times 4/5 = 0.$$

$\underline{k = 3}$: $L_4 + L_5 \leq 3$ and

$$H^*(w) \geq 4 - 3 \times 4/5 - 3 \times 1/2 > 0.$$

$\underline{k = 2}$: $L_4 + L_5 \leq 5$, but since $4 - 2 \times 4/5 > 4 \times 1/2$, there remains the possibility $L_4 + L_5 = 5$ to be considered. Then $l_4 + l_5 = 4$ and

$$H^*(w) \geq 4 + 4 \times 1/15 - 2 \times 4/5 - 5 \times 1/2 > 0.$$

$\underline{k = 1}$: Similarly, $L_4 + L_5 = 7$, $l_4 + l_5 = 6$, and

$$H^*(w) \geq 4 + 6 \times 1/15 - 4/5 - 7 \times 1/2 > 0.$$

$\underline{k = 0}$: We have $L_4 + L_5 \geq 9$. If $l_5 = 0$, then

$$H^*(w) \geq 4 + 8 \times 1/3 - 10 \times 1/2 > 0,$$

otherwise if $l_4 = 0$, then

$$H^*(w) \geq 4 - 10 \times 1/5 > 0.$$

Case 5: $s(w) \geq 11$. If $s(w) = 2k$, then

$$H^*(w) \geq 2k - 6 - k \geq 0.$$

Otherwise, since $L_4 + L_5 \leq s(w) - 2k - 1$, one has

$$\begin{aligned} H^*(w) &\geq s(w) - 6 - k - \big(s(w) - 2k - 1\big)/2 \\ &= s(w) - 6 - k - s(w)/2 + k + 1/2 = \big(s(w) - 11\big)/2 \geq 0. \end{aligned}$$

We have thus proved that $H^*(w) \geq 0$ for each vertex w. Now (6) implies

$$\sum_{e\in E} H^*(e) \leq -12.$$

Define E' as the set of those (i,j)-edges, $i \leq j$, from E where either $i = 3$, $j \leq 10$, or $i = 4$, $j \leq 7$, or else $i = 5$, $j \leq 6$. In other words, E' is just the set of those edges which participate in (4). Observe that due to Rules 1–4, the resulting charges of (i,j)-edges from $E - E'$ (and of no other edges) are non-negative, since all of them have $(j-6)/\lfloor j/6 \rfloor \geq (6-i)/i$. It follows that

$$\sum_{e\in E'} H^*(e) \leq -12. \tag{7}$$

But according to Rules 1–4, the resulting charge of each (i,j)-edge from E' is precisely $-a_{i,j}/10$, where $a_{i,j}$ is the coefficient at $e_{i,j}$ in (4). Consequently, multiplying each term of (7) by -10, we obtain (4). Thus, (4) holds for our map M^*, which contradicts the property $\sum < 120$ of M^* proved earlier.

This complete the proof of Theorem 2.

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The author is indebted to the referee for pointing out that Theorem 4 improves Theorem 2.10 on p. 40 of E. Jucovič's book "Convex 3-polytopes" (in Slovak), Veda, Bratislava 1981, as well as for other remarks.

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