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Mathematica Slovaca, Vol. 41 (1991), No. 3, 225--230

Persistent URL: http://dml.cz/dmlcz/136528

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MINIMAL GENERICS OF SOME REGULAR VARIETIES

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ABSTRACT. Given a variety $\mathcal{K}$ denote by $\mathcal{K}_r$ the variety of all algebras satisfying all regular identities which hold in $\mathcal{K}$. Let $m_g(\mathcal{K})$ be the minimal cardinal of an algebra generating $\mathcal{K}$. We find under some assumptions that the sum $S(\mathfrak{A})$ of the direct system $\mathfrak{A}$ of pairwise disjoint minimal generics A_j of the non-trivial independent varieties K_j, $j = 1, 2, \dots, n$, is a minimal generic of the regular variety $\mathcal{K}_r$, where $\mathcal{K} = \mathcal{K}_1 \vee \mathcal{K}_2 \vee \dots \vee \mathcal{K}_r$, and $m_g(\mathcal{K}_r) = \sum_{j=1}^{n} m_g(\mathcal{K}_j)$.

In [10; Theorem 2] the following was proved (for the definitions see below).

Theorem A. *If $\mathcal{K}_1$ and $\mathcal{K}_2$ are two incomparable independent varieties, A_1 and A_2 are carrierwise disjoint minimal generics of $\mathcal{K}_1$ and $\mathcal{K}_2$ respectively, $m(\mathcal{K}_i) = |A_i| = m_g(\mathcal{K}_i)$ $(i = 1, 2)$, there exists a homomorphism h_1^2 of A_1 into A_2 and $\mathcal{K} = \mathcal{K}_1 \vee \mathcal{K}_2$ then $S(\mathfrak{A})$ is a minimal generic of $\mathcal{K}_r$ and $m_g(\mathcal{K}_r) = m_g(\mathcal{K}_1) + m_g(\mathcal{K}_2)$.*

The aim of the present paper is to generalize this theorem to the case of finitely many independent varieties (Theorem 1 below). The condition (1) in Theorem 1 is not suitable for the induction argument (see Remark 2 below) hence we give here a straight proof. Moreover we replace the condition on a homomorphism h_1^2 in Theorem A with the weaker condition (2).

In this paper we consider only algebras of a given type $\tau\colon F \to \mathbf{N}$, where F is a *set of fundamental operation symbols* and $\mathbf{N}$ is a *set of positive integers* (i.e. there are *no nullary symbols* in F). Further we assume that $\tau(F) - \{0, 1\} \neq \emptyset$.

An *identity* $\varphi = \psi$ is called *regular* (see [8]) if the sets of variables in φ and ψ coincide. A *variety* $\mathcal{K}$ is called *regular* if all identities in $\mathrm{Id}(\mathcal{K})$ are regular. $\mathcal{K}$ is called *non-regular* if a non-regular identity belongs to $\mathrm{Id}(\mathcal{K})$. Regular varieties were studied by many authors (see e.g. [9], [10], [8], [2], [7], [6]).

For a variety $\mathcal{K}$ (of algebras of type τ) we denote by $\mathcal{K}_r$ the variety of algebras of type τ defined by all regular identities from $\mathrm{Id}(\mathcal{K})$. Due to A. Tarski it is well known that for every variety $\mathcal{K}$ there exists an algebra A generating $\mathcal{K}$ by means of direct products, subalgebras and homomorphic

AMS Subject Classification (1985): Primary 08B99
Key words: Regular varieties, Minimal generics

images, i.e. $\mathcal{K} = HSP(A)$. Such algebras A are called *generics of* $\mathcal{K}$ (see [3]). A generic A of $\mathcal{K}$ will be called a *minimal generic of* $\mathcal{K}$ if for every generic B of $\mathcal{K}$ we have $|A| \leq |B|$ (where $|A|$ denotes $\operatorname{card} A$). Finding minimal generics of a variety $\mathcal{K}$ is important because the smaller a finite generic is, the easier it is to decide if a given identity $\varphi = \psi$ belongs to $\mathrm{Id}(\mathcal{K})$ or not.

For an algebra A we denote by $R(A)$ the *set of all regular identities of type* τ *from* $\mathrm{Id}(A)$. For a variety $\mathcal{K}$ let $R(\mathcal{K})$ be the *set of all regular identities from* $\mathrm{Id}(\mathcal{K})$.

The variety $\mathcal{K}$ is *strongly non-regular* (see [2]) if there exists a binary term $\varphi(x,y)$ containing the variable y such that the identity $\varphi(x,y) = x$ belongs to $\mathrm{Id}(\mathcal{K})$.

For a variety $\mathcal{K}$ of algebras of type τ let m' be the *cardinality of a free algebra with* $\aleph_0$ *free generators over* $\mathcal{K}$. We define the number $m(\mathcal{K})$ putting

$$m(\mathcal{K}) = 1 \quad \text{if } \mathcal{K} \text{ is trivial,}$$
$$m(\mathcal{K}) = \min\{m\colon 1 < m \leq m' \quad \text{and} \quad \exists_{A\in\mathcal{K}}(|A| = m)\} \quad \text{if } \mathcal{K} \text{ is nontrivial.}$$

Let $m_g(\mathcal{K})$ denote the *cardinality of a minimal generic of* $\mathcal{K}$. Obviously for every variety $\mathcal{K}$ we have $m_g(\mathcal{K}) \geq m(\mathcal{K})$. It is known (see [10] or cf. [6]) that if $\mathcal{K}$ is a non-regular variety of type τ, then $m_g(\mathcal{K}_r) \leq m_g(\mathcal{K}) + 1$.

Varieties $\mathcal{K}_1, \ldots, \mathcal{K}_n$ of the same type are said to be *independent* (for $n = 2$ see [4]) if there is an n-ary term p such that the identity $p(x_1, \ldots, x_n) = x_i$ holds in $\mathcal{K}_i$, $i = 1, 2, \ldots, n$. $\mathcal{K}_1 \vee \mathcal{K}_2 \vee \cdots \vee \mathcal{K}_n$ will denote the *smallest variety containing all* $\mathcal{K}_i$; $\mathcal{K}_1 \times \mathcal{K}_2 \times \cdots \times \mathcal{K}_n$ will denote the *class of all algebras* A *which are isomorphic to the direct product* $A_1 \times A_2 \times \cdots \times A_n$ *of algebras* $A_i \in \mathcal{K}_i$, $i = 1, 2, \ldots, n$.

The proof of the following Lemma can be found in [4], [5], [1].

Lemma 1. *If* $\mathcal{K}_1, \ldots, \mathcal{K}_n$ *are independent varieties, then* $\mathcal{K}_1 \wedge \mathcal{K}_2 \wedge \cdots \wedge \mathcal{K}_n$ *consists of one-element algebras only and each algebra* $A \in \mathcal{K}_1 \vee \mathcal{K}_2 \vee \cdots \vee \mathcal{K}_n$ *has, up to isomorphism a unique representation* $A \simeq A_1 \times A_2 \times \cdots \times A_n$ $A_i \in \mathcal{K}_i$, $i = 1, 2, \ldots, n$. *Hence* $\mathcal{K}_1 \vee \mathcal{K}_2 \vee \ldots \vee \mathcal{K}_n = \mathcal{K}_1 \times \mathcal{K}_2 \times \cdots \times \mathcal{K}_n$.

The next Lemma can be proved analogously to the Theorem 3 in [1].

Lemma 2. *Varieties* $\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_n$ *are independent if and only if for each* $i \in 1, 2 \ldots n$, $\mathcal{K}_i$ *and* $\mathcal{K}_i' = \bigvee(\mathcal{K}_j\colon j \neq i,\ j = 1, 2, \ldots, n)$ *are independent*

Remark 1. If $\mathcal{K}_1, \mathcal{K}_2$ are non-trivial independent varieties then they are incomparable (by Lemma 1) and the variety $\mathcal{K}_1 \vee \mathcal{K}_2$ is strongly non-regular, since if $p(x,y)$ is the term establishing the independence of $\mathcal{K}_1, \mathcal{K}_2$, then $\varphi(x,y) = p(p(x,y),x)$ is the desired term for strong non-regularity (i.e. $\varphi(x,y) = x$ in $\mathcal{K}_1 \vee \mathcal{K}_2$).

Now we recall the definition of a direct system of algebras (see [3; Chap.3]).

(i) *I is a directed poset (partially ordered set) whose ordering relation is denoted by* $\leq$.

(ii) *For each* $i \in I$ *an algebra* $A_i = (A_i\,;\ (f_t^{(i)})_{t\in T})$ *is given, all algebras* A_i *being of the same type.*

(iii) *For each pair* i, j *of elements of* I *with* $i \leq j$ *a homomorphism* $h_i^j\colon A_i \to A_j$ *is given. The resulting set of homomorphisms satisfies the following conditions :*

  (a) $i \leq j \leq k$ *implies* $h_j^k \circ h_i^j = h_i^k$ *and*

  (b) h_i^i *is the identity map for each* $i \in I$.

The system $\left(I, (A_i)_{i\in I}, (h_i^j)_{i\leq j;\, i,j\in I}\right)$ is called a *direct system of algebras* A_i, $i \in I$.

Let $\mathfrak{A} = \left(I, (A_i)_{i\in I}, (h_i^j)_{i\leq j;\, i,j\in I}\right)$ be a direct system of similar algebras, without nullary fundamental operations, indexed by a poset I with the *least upper bound property*. Let $(f_t)_{t\in T}$ be the set of fundamental operations of the algebras in $\mathfrak{A}$. The *sum of the direct system* $\mathfrak{A}$ (see [8]) is an algebra $S(\mathfrak{A}) = (\mathbf{A};\ (f_t)_{t\in T})$, where $\mathbf{A}$ is a disjoint sum of the carriers $\mathbf{A}_i$ $(i \in I)$ and the fundamental operations f_t are defined by $f_t(a_1, a_2, \ldots, a_n) = f_t\left(h_{i_1}^k(a_1), \ldots \ldots, h_{i_n}^k(a_n)\right)$, where $a_j \in \mathbf{A}_{i_j}$ and k is the least upper bound of $i_1, i_2, \ldots, i_n$.

Theorem 1. *Let* $\mathcal{K}_1, \ldots, \mathcal{K}_n$ *be non-trivial independent varieties, and* $A_1, \ldots, A_n$ *be pairwise disjoint minimal generics of* $\mathcal{K}_1, \ldots, \mathcal{K}_n$ *respectively. Let the following conditions hold:*

(1) $\qquad m(\mathcal{K}_j) = |A_j| = m_g(\mathcal{K}_j)$ *for* $j = 1, 2, \ldots, n$,

(2) $\qquad$ *the algebras* $A_1, A_2, \ldots, A_n$ *form a direct system* $\mathfrak{A}$ *in which* $(I, \leq)$ *is a semilattice,* $I = \{1, 2, \ldots, n\}$.

Then $S(\mathfrak{A})$ *is a minimal generic of* $\mathcal{K}_r$, *where* $\mathcal{K} = \mathcal{K}_1 \vee \mathcal{K}_2 \vee \ldots \vee \mathcal{K}_n$ *and* $m_g(\mathcal{K}_r) = m_g(\mathcal{K}_1) + m_g(\mathcal{K}_2) + \cdots + m_g(\mathcal{K}_n)$.

P r o o f. $S(\mathfrak{A})$ is a generic of $\mathcal{K}_r$ since by [8; Theorem 1] we have $\mathrm{Id}(S(\mathfrak{A})) = R(A_1) \cap R(A_2) \cap \cdots \cap R(A_n) = R(K_1) \cap R(K_2) \cap \cdots \cap R(K_n) = R(\mathcal{K}) = \mathrm{Id}(\mathcal{K}_r)$. Obviously $|S(\mathfrak{A})| = |A_1| + |A_2| + \cdots + |A_n| = m_g(\mathcal{K}_1) + m_g(\mathcal{K}_2) + \cdots + m_g(\mathcal{K}_n)$.

Let B be a generic of $\mathcal{K}_r$. According to Lemma 2 and Remark 1, $\mathcal{K}$ is a strongly non-regular variety (since $\mathcal{K}_i$ and $\mathcal{K}_i'$ are independent and $\mathcal{K}_i \vee \mathcal{K}_i' = \mathcal{K}$), hence by [9; Theorem 1] B is the sum of a direct system of algebras $C_i \in \mathcal{K}$, $i \in I$. Since B is a generic of $\mathcal{K}_r$, there must be $|I| \geq 2$. By Lemma 1 for each

$i \in I \quad C_i \cong C_i^1 \times C_i^2 \times \cdots \times C_i^n\,,\ C_i^j \in \mathcal{K}_j\,,\ j = 1,2,\ldots,n$. Hence

$$|C_i| = |C_i^1| \cdot |C_i^2| \cdot \cdots \cdot |C_i^n| \quad \text{and} \tag{3}$$

$$|B| = \sum_{i \in I} |C_i|\,. \tag{4}$$

We assert that

(5) to any $l \in \{1,2,\ldots,n\}$ there is $i(l) \in I$ such that $|C^l_{i(l)}| > 1$.

Suppose that there is $l \in \{1,2,\ldots,n\}$ such that $|C_i^l| = 1$ for each $i \in I$. Then for each $i \in I$ $C_i \in \bigvee(\mathcal{K}_j : j \neq l,\ j \in \{1,2,\ldots,n\}) = \mathcal{K}'_l \subseteq \mathcal{K}$, hence $B \in (\mathcal{K}'_l)_r$ and $\mathcal{K}_r = HSP(B) \subseteq (\mathcal{K}'_l)_r \subseteq \mathcal{K}_r$. By Lemma 2 $\mathcal{K}'_l$, $\mathcal{K}_l$ are non-trivial independent varieties, $\mathcal{K} = \mathcal{K}'_l \vee \mathcal{K}_l$ is strongly non-regular and $\mathcal{K}'_l \neq \mathcal{K}$ (see Remark 1). So by [2] $(\mathcal{K}'_l)_r \neq \mathcal{K}_r$ - a contradiction. Thus (5) holds.

Now we choose for each $l \in \{1,2,\ldots,n\}$ an $i(l) \in I$ such $|C^l_{i(l)}| > 1$. According to (3) $|C_{i(l)}| \geq |C^l_{i(l)}|$. Using (4) we get

$$|B| \geq \sum_{l=1}^{n} |C_{i(l)}| \geq \sum_{l=1}^{n} |C^l_{i(l)}| \geq \sum_{l=1}^{n} |A_l| = |S(\mathfrak{A})|.$$

E x a m p l e 1. Let $\mathcal{K}_{p(i)}$ (where $p(i)$, $i = 1,2,\ldots,n$, are distinct primes) denote the equational classes of Abelian groups with exactly one binary fundamental operation (and without nullary operations) satisfying $x^{p(i)} = y^{p(i)}$. It is easy to check that $\mathcal{K}_{p(i)}$, $i = 1,2,\ldots,n$ are independent. For each i the variety $\mathcal{K}_{p(i)}$ is equationally complete and cyclic group $A_{p(i)}$ of order $p(i)$ is a minimal generic of $\mathcal{K}_{p(i)}$ $\left(m\left(\mathcal{K}_{p(i)}\right) = m_g\left(\mathcal{K}_{p(i)}\right) = p(i)\right)$. The groups $A_{p(i)}$, $i = 1,2,\ldots,n$ form a suitable direct system (since we can take for the poset I an n-element chain $1 \leq 2 \leq \cdots \leq n$ and the trivial homomorphisms h_i^j $(i,j \in \{1,2,\ldots,n\},\ i \leq j)$ given by the rule $h_i^i = \mathrm{id}_{A_{p(i)}}$ and $h_i^j(x) = b^{p(j)}$ $(i < j$, $x \in A_{p(i)}$, $b \in A_{p(j)})$. Hence by Theorem 1 $m_g\left(\mathcal{K}_{p(1)} \vee \mathcal{K}_{p(2)} \vee \ldots \vee \mathcal{K}_{p(n)}\right) = p(1) + p(2) + \cdots + p(n) = m_g\left(\mathcal{K}_{p(1)}\right) + m_g\left(\mathcal{K}_{p(2)}\right) + \cdots + m_g\left(\mathcal{K}_{p(n)}\right)$.

E x a m p l e 2. The following example shows that the condition (1) in Theorem 1 is not necessary. Nevertheless the condition (2) is essential. Take the independent varieties $\mathcal{K}_{p(1)}$, $\mathcal{K}_{p(2)}$, $\mathcal{K}_{p(3)}$ described in Example 1. By Lemma 2 $\mathcal{K}'_{p(3)} = \mathcal{K}_{p(1)} \vee \mathcal{K}_{p(2)}$ and $\mathcal{K}_{p(3)}$ are independent. According to Example 1 we get that $m_g\left(\mathcal{K}_{p(1)} \vee \mathcal{K}_{p(2)} \vee \mathcal{K}_{p(3)}\right) = m_g\left(\mathcal{K}_{p(1)}\right) + m_g\left(\mathcal{K}_{p(2)}\right) + m_g\left(\mathcal{K}_{p(3)}\right) = m_g\left(\mathcal{K}'_{p(3)}\right) + m_g\left(\mathcal{K}_{p(3)}\right)$ (since $m_g\left(\mathcal{K}_{p(1)} \vee \mathcal{K}_{p(2)}\right) = m_g\left(\mathcal{K}_{p(1)}\right) + m_g\left(\mathcal{K}_{p(2)}\right)$). Nevertheless $m_g\left(\mathcal{K}_{p(1)} \vee \mathcal{K}_{p(2)}\right) > m\left(\mathcal{K}_{p(1)} \vee \mathcal{K}_{p(2)}\right)$ (since $|A_{p(1)} \times A_{p(2)}| > |A_{p(i)}|$, $i = 1,2$).

R e m a r k 2. One would think that Theorem 1 can be obtained by induction using Lemma 2 and Theorem A. But the trouble is that the condition (1) of Theorem 1 is not transferable from the varieties $\mathcal{K}_1$ and $\mathcal{K}_2$ to the variety $\mathcal{K}_1 \vee \mathcal{K}_2$ as the Example 2 shows.

E x a m p l e 3. Minimal generics of independent varieties need not form a direct system. Consider e.g. two independent varieties $\mathcal{K}_1$, $\mathcal{K}_2$ of the type (2,1,1). Suppose $\mathcal{K}_i$ $(i = 1, 2)$ is generated by a two-element algebra $A_i = (\{a_i, b_i\};\ f^i, g^i, h^i)$. Let $f^1(x,y) = x$, $f^2(x,y) = y$. Assume that

$$
\begin{aligned}
g^1(a_1) &= a_1, & h^1(a_1) &= b_1, \\
g^1(b_1) &= b_1, & h^1(b_1) &= a_1, \\
g^2(a_2) &= b_2, & h^2(a_2) &= a_2, \\
g^2(b_2) &= a_2, & h^2(b_2) &= b_2.
\end{aligned}
$$

There are no homomorphisms between the algebras A_i, $i = 1, 2$.

R e m a r k 3. Theorem 1 gives a better estimation for $m_g(\mathcal{K}_r)$ (in special cases) than that given by the relation $m_g(\mathcal{K}_r) \leq m_g(\mathcal{K}) + 1$ mentioned in the introduction. E.g. a minimal generic of the variety $\mathcal{K}_3 \vee \mathcal{K}_5 \vee \mathcal{K}_7$ from Example 1 is the cyclic group of order 105, however, by Theorem 1 $m_g((\mathcal{K}_3 \vee \mathcal{K}_5 \vee \mathcal{K}_7)_r) = 3 + 5 + 7 = 15$.

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Received August 7, 1989

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