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NON-BOREL MEASURES ON NON-SEPARABLE METRIC SPACES

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ABSTRACT. Our study of non-Borel measures on non-separable metric spaces, the Borel σ-algebra of which differs from the σ-algebra generated by the open balls, is motivated by the works of R. M. Dudley [4, 5] and D. Pollard [11]. Our aim is to show that the "non-separable" theory containing the general results from [4, 5] and [11] can be without difficulties developed by applying the "theory of additive set functions on abstract spaces" which was developed by A. D. Alexandroff [1] and other authors (see [8, 12—16]).

1. Introduction

The history concerning the non-Borel measures on a σ-algebra $\mathscr{B}_0(X)$ which is generated by the open balls of a nonseparable metric space X, d, the Borel σ-algebra $\mathscr{B}(X)$ of which is strongly larger than $\mathscr{B}_0(X)$, is described in detail in the introduction to [11]. We may only note that the problem arose in connection with the study of random elements with values in X, d which are not $\mathscr{B}(X)$ measurable, but which are measurable with respect to $\mathscr{B}_0(X)$ and converge in some sense to a $\mathscr{B}(X)$ measurable element.

The main results of the "non-separable" measure theory are related to the convergence of a sequence (P_n) of probability measures on $\mathscr{B}_0(X)$ to a probability measure P on $\mathscr{B}(X)$ with a separable support. The greatest problem in the study of such sequences was to define a suitable notion of convergence and then to derive by means of it some manageable convergence criterions.

The original "non-separable" theory was developed by R. M. Dudley [4, 5]. In [4] (P_n) was said to be convergent to P if and only if $\lim \int_* f \,\mathrm{d}P_n = \lim \int^* f \,\mathrm{d}P_n = \int f \,\mathrm{d}P$ For all bounded continuous functions f. Here $\int_*$ and $\int^*$ denote the lower and upper integrals of f, respectively.

Later Dudley observed in [5] that this kind of convergence is equivalent with the relation $\lim \int f \,\mathrm{d}P_n = \int f \,\mathrm{d}P$ for all bounded continuous $\mathscr{B}_0(X)$ measurable functions f.

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D. Pollard [11] decided to study (P_n) and P restricted to the support S of P. He showed that the Borel σ-algebra of S, where S is provided with the topology induced from X, d, is contained in $\mathscr{B}_0(X)$ and thus (P_n) with P can be considered as measures on a Borel σ-algebra of a separable metric space.

Our approach is based on the fact that the original topological measure theory was constructed by A. D. Alexandroff [1] for abstract spaces of functions and measures. The bounded continuous $\mathscr{B}_0(X)$ measurable functions on X form naturally an abstract space in the sense of [1] and Dudley's result from [5] suggests the possibility to obtain a stimulating theory by applying the abstract constructions from [1] and the later works (see i.e., the works of F. Topsøe [12—15]).

2. The Representation theorems

Let X, d be a metric space. By $\mathscr{G}(X)$, $\mathscr{F}(X)$ and $\mathscr{K}(X)$ we denote the classes of all open, closed and compact subsets of X, respectively. The subclasses of $\mathscr{G}(X)$ and $\mathscr{F}(X)$ which are in $\mathscr{B}_0(X)$ (the σ-algebra generated by the open balls) are denoted by $\mathscr{G}_0(X)$ and $\mathscr{F}_0(X)$, respectively.

$\mathscr{G}_0(X)$ is the smallest class of open sets which is closed with respect to the formation of countable unions, finite intersections and containing all the sets $\{x: d(x_0, x) < c\}$ and $\{x: d(x_0, x) > c\}$, where $x_0 \in X$, $c \in]0, \infty[$. $\mathscr{F}_0(X)$ consists of the complements to the sets in $\mathscr{G}_0(X)$ and both classes generate $\mathscr{B}_0(X)$.

In virtue of the Lindelöff theorem, if X, d is a separable metric space, then $\mathscr{G}_0(X)$ agrees with $\mathscr{G}(X)$ and $\mathscr{B}_0(X)$ agrees with the Borel σ-algebra $\mathscr{B}(X)$. However, generally it may not be so and throughout this paper we will automatically suppose, in order to avoid the trivialities, that $\mathscr{G}_0(X)$ is strongly smaller than $\mathscr{G}(X)$.

If we denote the space of all bounded continuous functions with the supremum norm, $\|\,.\,\|$ by $C(X)$ and its subspace, consisting of all the $\mathscr{B}_0(X)$ measurable functions by $C_0(X)$, then

Theorem 2.1.: a) *If $f, g \in C_0(X)$ and $c \in]-\infty, \infty[$, then fg, $f+g$ and cf are in $C_0(X)$, all the constant functions are in $C_0(X)$ and if $\left\|\frac{g}{f}\right\| < \infty$, then $\frac{g}{f} \in C_0(X)$.*

b) *$C_0(X)$ is norm complete*

c) *$C_0(X)$ consists just of those functions f, for which $\{x: f(x) \geqslant c\}$ and $\{x: f(x) \leqslant c\}$ are in $\mathscr{F}_0(X)$ for all $c \in]-\infty, \infty[$.*

The proof of 2.1. follows immediately from the properties of $C(X)$, the $\mathscr{B}_0(X)$ measurable functions and the definition of $C_0(X)$.

Theorem 2.2.: *$\mathscr{F}_0(X)$ agrees with the class of all the sets $\{x: f(x) \leq c\}$ and $\{x: f(x) \geqslant c\}$ where $f \in C_0(X)$ and $c \in]-\infty, \infty[$.*

Proof: Let $\mathscr{F}$ consist of all sets of the form $\{x: f(x) \geqslant c\}$ or $\{x: f(x) \leqslant c\}$ for some $f \in C_0(X)$, $c \in]-\infty, \infty[$. By 2.1. c) $\mathscr{F} \subset \mathscr{F}_0(X)$. By the definition of $\mathscr{G}_0(X)$ all the functions $c \wedge d(x_0, x)$, $x_0 \in X$, $c \in]-\infty, \infty[$ are in $C_0(X)$. Consequently $\mathscr{F}$ contains all the sets $\{x: c \wedge d(x_0, x) \leqslant c\}$ and $F = \{x: c \wedge d(x_0, x) \geqslant c\}$ for $x_0 \in X$. However, these sets generate $\mathscr{F}_0(X)$, whence $\mathscr{F}_0(X) \subset \mathscr{F}$.

Theorem 2.3.: a) *If $F \in \mathscr{F}_0(X)$, then there is $f \in C_0(X)$ such that $F = \{x: f(x) = 0\}$.*

b) *If $F_1, F_2 \in \mathscr{F}_0(X)$ are disjoint, then there is $f \in C_0(X)$ such that $F_1 = \{x: f(x) = 0\}$, $F_2 = \{x: f(x) = 1\}$.*

c) *$\mathscr{F}_0(X)$ contains all one point sets*

d) *$\mathscr{F}_0(X)$ contains all closed separable subsets of X, particularly the class $\mathscr{K}(X)$.*

Proof: a) By 2.2. to each $F \in \mathscr{F}_0(X)$ there is $g \in C_0(X)$ and $c \in]-\infty, \infty[$ for which $F = \{x: g(x) \leqslant c\}$. Now it suffices to put $f = (g - c)^+$.

b) $f = f_1^2/(f_1^2 + f_2^2)$, where $f_1, f_2 \in C_0(X)$ are those functions for which $F_1 = \{x: f_1(x) = 0\}$, $F_2 = \{x: f_2(x) = 0\}$.

c) follows from the fact that all the functions $1 \wedge d(x_0, x)$, $x_0 \in X$, are in $C_0(X)$.

d) If (x_i) is a dense subset in $F \in \mathscr{F}(X)$, then

$$f(x) := 1 \wedge d(x, F) = \inf_i 1 \wedge d(x, x_i)$$

is $\mathscr{B}_0(X)$ measurable, whence $F = \{x: f(x) = 0\} \in \mathscr{B}_0(X)$. By the definition $F \in \mathscr{F}_0(X)$.

The bounded linear functional L on $C_0(X)$ is said to be τ-smooth (σ-smooth) if for each (countable) decreasing net (sequence) $(f_\alpha) \subset C_0(X)$ with $\lim f_\alpha(x) = 0$ for all $x \in X$ $\lim L(f_\alpha) = 0$. The same functional is said to be tight if for each net $(f_\alpha) \subset C_0(X)$ with $\|f_\alpha\| \leqslant 1$ and $\lim f_\alpha = 0$ uniformly on the compact sets $\lim L(f_\alpha) = 0$. Replacing $C_0(X)$ with $C(X)$ we obtain the definition of σ-smooth, τ-smooth and tight functionals on $C(X)$.

Let $L(X)$ and $L_0(X)$ be the spaces of all bounded linear functionals on $C(X)$ and $C_0(X)$, respectively. If we denote the subspaces of the σ-smooth, τ-smooth and tight functionals in $L(X)$ ($L_0(X)$) by $L(X, \sigma)$, $L(X, \tau)$ and $L(X, \mathrm{t})$ ($L_0(X, \sigma)$, $L_0(X, \tau)$ and $L_0(X, \mathrm{t})$), respectively, then

Theorem 2.4.: If $\Theta \in \{\sigma, \tau, \mathrm{t}\}$ and $L \in L_0(X)$, then

$$L \in L_0(X, \Theta) \Leftrightarrow L^+ \text{ and } L^- \in L_0(X, \Theta) \Leftrightarrow |L| \in L_0(X, \Theta)$$

where

$$L^+(f) = \sup_{0 \leqslant h \leqslant f} L(f),\ L^-(f) = -\inf_{0 \leqslant h \leqslant f} L(h),\ |L| = \sup_{0 \leqslant |h| \leqslant f} L(h)$$

for all $f \in C_0(X)$. The h are taken in $C_0(X)$.

Proof: The fact that L^+, L^- and $|L|$ are well-defined (non-negative) functionals in $L_0(X)$ is proved in [1; ch. II, theorem 2]. Particularly, it is verified that $L = L^+ - L^-$. The proofs of the equivalences agree with the proofs of the theorems 7, 8 and 9 in [16; part I]. However, we have to work with $C_0(X)$ instead of $C(X)$.

Of course, 2.4. stays to be true if we consider $L(X)$ instead of $L_0(X)$ and $L(X, \Theta)$ instead of $L_0(X, \Theta)$ for $\Theta \in \{\sigma, \tau, \mathrm{t}\}$.

Let $M(X)$ be the space of all bounded Borel measures on $\mathscr{B}(X)$ and $M_0(X)$ be the space of all bounded $\mathscr{F}_0(X)$ regular measures on $\mathscr{B}_0(X)$ (m is said to be $\mathscr{F}_0(X)$ regular if $mE = \lim_{F \in D(E)} mF$ for all $E \in \mathscr{B}_0(X)$, where $D(E) = \{F\colon F \subseteq E, F \in \mathscr{F}_0(X)\}$ is directed by the inclusion). In [1, ch. II] it is proved that.

Theorem 2.5.: *To each $L \in L_0(X)$ there is just one $m \in M_0(X)$ with the property $L(f) = m(f)$ for all $f \in C_0(X)$. Moreover $L^+(f) = m^+(f)$, $L^-(f) = m^-(f)$ and $|L|(f) = |m|(f)$ for all $f \in C_0(X)$, where*

$$m^+E = \sup_{F \subseteq E} mF, \quad m^-E \doteq -\inf_{F \subseteq E} mF, \quad |m|E = \sup_{F \subseteq E} |mF|$$

for all $E \in \mathscr{B}_0(X)$. The F are taken in $\mathscr{F}_0(X)$.

It is well known that 2.5. holds if we replace $L_0(X)$, $M_0(X)$ and $C_0(X)$ by $L(X)$, $M(X)$ and $C(X)$, respectively. Using the theorem 2.4. and the methods in [12] we can easily prove.

Theorem 2.6.: *The functional $L \in L_0(X)$ is σ-smooth, τ-smooth or tight if and only if the measure $m \in M_0(X)$ from* 2.5. *representing L on $C_0(X)$ is σ-smooth, τ-smooth or tight, respectively.*

We note that $m \in M_0(X)$ is said to be τ-smooth (σ-smooth) if for each (countable) decreasing net (sequence) $(F_\alpha) \in \mathscr{F}_0(X)$ with $\bigcap F_\alpha = \emptyset$ $\lim mF_\alpha = 0$ and m is said to be tight if to each $\varepsilon > 0$ there is $K \in \mathscr{K}(X)$ with $|m|\, X - K < \varepsilon$.

So we have stated that there is a one to one correspondence between $L_0(X)$, $M_0(X)$ and $L_0(X, \Theta)$, $M_0(X, \Theta)$ for $\Theta \in \{\sigma, \tau, \mathrm{t}\}$, where $M_0(X, \sigma)$, $M_0(X, \tau)$ and $M_0(X, \mathrm{t})$ denotes the subspaces of all σ-smooth, τ-smooth and tight measures in $M_0(X)$. An analogous result can be formulated for the spaces $M(X, \sigma)$, $M(X, \tau)$ and $M(X, \mathrm{t})$ of all σ-smooth, τ-smooth and tight Borel measures, respectively.

Some other analogies with the Borel measures are true for $M_0(X)$ as well. The bounded measure m on $\mathscr{B}_0(X)$ (not necessarily regular) is said to be σ-additive if $mE = \sum_{n=1}^{\infty} mE_n$ whenever $E = \bigcup_{n=1}^{\infty} E_n \in \mathscr{B}_0(X)$ and the E_n are pairwise disjoint.

If m is σ-additive then $\lim mE_n = m \bigcup_{n=1}^{\infty} E_n$ for each increasing sequence $(E_n) \subset \mathscr{B}_0(X)$.

Theorem 2.7.: *Each bounded σ-additive measure on $\mathscr{B}_0(X)$ is $\mathscr{F}_0(X)$ regular, i.e. belongs to $M_0(X, \sigma)$.*

Proof: Each σ-additive measure on $\mathscr{B}_0(X)$ is a decomposition of two non-negative σ-additive measures on $\mathscr{B}_0(X)$ (see the Hahn and Jordan decomposition [8; sec. 9, theorem A]). The pavings $\mathscr{G}_0(X)$ and $\mathscr{F}_0(X)$ form on X a completely normal space, thus each $G \in \mathscr{G}_0(X)(F \in \mathscr{F}_0(X))$ can be written in the form $G = \bigcup_{n=1}^{\infty} F_n$ $\left(F = \bigcap_{n=1}^{\infty} G_n\right)$, where $(F_n) \subset \mathscr{F}_0(X)$ form an increasing $((G_n) \subset \mathscr{G}_0(X)$ a decreasing) sequence of sets. Thus if m is non-negative and σ-additive, then $mG = \lim mF_n = \sup\{mF: F \subseteq G, F \in \mathscr{F}_0(X)\}$, $mF = \lim mG_n = \inf\{mG: G \subseteq F, G \in \mathscr{G}_0(X)\}$, i.e. m is inner regular on $\mathscr{G}_0(X)$ and outer regular on $\mathscr{F}_0(X)$. Consequently m is regular on $\mathscr{B}_0(X)$ [8, sec. 52].

From 2.7. it follows that studying the probability measures on $\mathscr{B}_0(X)$ we can restrict our attention to the regular ones and the theory related to the regular σ-smooth measures contains those from [4, 5] and [11].

Theorem 2.8.: *To each $m \in M_0(X, \tau)$ there is a separable set S, $S \in \mathscr{B}_0(X)$, which is a countable union of totally bounded sets from $\mathscr{F}_0(X)$ and $mS = mX$.*

Proof: In virtue of 2.6. it suffices to consider only the non-negative measures. Let $m \in M_0(X, \tau)$ be non-negative. The sets $N(x_0, n) = \left\{x: d(x_0, x) < \frac{1}{n}\right\}$, where $x_0 \in X$, n is a natural number, are in $\mathscr{G}_0(X)$. Their finite unions and intersections form a direction with respect to the inclusion and filter up to X. Due to the τ-smoothness to a given $\varepsilon > 0$ there are $x_1^n, \ldots, x_{k_n}^n$ such that $mX < m \bigcup_{i=1}^{k_n} N(x_i, n) + \varepsilon/2^n$. Put $F_\varepsilon = \bigcap_{n=1}^{\infty} \left(\bigcup_{i=1}^{k_n} \bar{N}(x_i^n, n)\right)$, where $\bar{N}(x_i^n, n) = \left\{x: d(x_i^n, x) \leqslant \frac{1}{n}\right\}$. $F_\varepsilon \in \mathscr{F}_0(X)$, it is totally bounded and therefore separable.

Moreover $mX < mF_\varepsilon + \varepsilon$. Now $S = \bigcup_{\varepsilon} F_\varepsilon$, ε-rational, has the desired properties.

Corollary 2.9.: *Each $m \in M_0(X, \tau)$ has a separable support in $\mathscr{F}_0(X)$.*
Proof: It is a simple consequence of 2.8. and 2.3. d).

Corollary 2.10.: *If X, d is a complete metric space, then $M_0(X, t) = M_0(X, \tau)$.*

3. The weak topology $w(M_0(X), C_0(X))$.

Now we are ready to state the main properties of the weak topology $w(M_0(X), C_0(X))$. The well-known Alexandroff and Portmanteau theorem [1; ch. IV, § 16, theorem 2] and [13, part II, theorem 8.1.] says:

Theorem 3.1.: *Let m be a measure in $M_0^+(X)$ and (m_α) a net in $M_0^+(X)$. Then the following conditions are equivalent*:

i) $\limsup m_\alpha(f) \leqslant m(f)$
for all bounded $\mathscr{B}_0(X)$ measurable u.s.c functions,

ii) $\liminf m_\alpha(f) \geqslant m(f)$
for all bounded $\mathscr{B}_0(X)$ measurable l.s.c functions,

iii) $\limsup m_\alpha F \leqslant mF$ *and* $\lim m_\alpha X = mX$ *for all* $F \in \mathscr{F}_0(X)$,

iv) $\liminf m_\alpha G \geqslant mG$ *and* $\lim m_\alpha X = mX$ *for all* $G \in \mathscr{G}_0(X)$,

v) (m_α) *converges to m in* $w(M_0(X), C_0(X))$.

Proof: Clearly i) $\Leftrightarrow$ ii) and i) with ii) implies v).

Since $C_0(X)$ is a complete system of functions and $\mathscr{F}_0(X)$ forms from X a completely normal space, for each $F \in \mathscr{F}_0(X)$ we can write $mF = \inf\{m(f): 1 \geqslant f \geqslant \chi_F, f \in C_0(X)\}$ (see i.e. [1, ch. II, § 7, theorem 1]. Consequently to each $\varepsilon > 0$ and fixed $F \in \mathscr{F}_0(X)$ there is $f \in C_0(X)$, $1 \geqslant f \geqslant \chi_F$ with $m(f) < mF + \varepsilon$.

$$mF + \varepsilon > m(f) = \lim m_\alpha(f) \geqslant \lim m_\alpha F,$$

which proves the implication v) $\Rightarrow$ ii). Clearly ii) $\Rightarrow$ iii). The proof of iii) $\Rightarrow$ i) is based on the fact that if $0 \leqslant f \leqslant 1$ and f is $\mathscr{F}_0(X)$ measurable and lower semicontinuous, then for each n the function $s_n(x) = \frac{1}{n} \sum_{i=1}^{n} \chi_{\{x: f(x) \geqslant \frac{i}{n}\}}$ satisfies the relations $\limsup m_\alpha(s_n) \leqslant m(s_n)$ and $s_n \leqslant f \leqslant \frac{1}{n} + s_n$. More in detail see [13, theorem 8.1.].

I hope the reader has noticed that l.s.c. and u.s.c. are the abbreviations of lower semicontinuous and upper semicontinuous.

Remark 3.2.: Theorem 3.1. holds as well for $M(X)$, $C(X)$.

Theorem 3.3.: The spaces $M^+(X, \tau)$ and $M_0^+(X, \tau)$ provided with the topologies $w(M(X), C(X))$ and $w(M_0(X), C_0(X))$, respectively, are homeomorphic.

Proof: Each $m_0 \in M_0^+(X, \tau)$ has a unique extension to a measure $m \in M^+(X, \tau)$ [13, theorem 5.1.]. This proves the one to one correspondence between the members of $M^+(X, \tau)$ and $M_0^+(X, \tau)$.

Let $(m_\alpha) \in M^+(X, \tau)$ be a net, m be a measure in $M^+(X, \tau)$. By $m_{0\alpha}$ and m_0 we will denote the restrictions of m_α and m from $\mathscr{B}(X)$ to $\mathscr{B}_0(X)$, respectively.

Clearly if (m_α) converges to m in $w(M(X), C(X))$, then $(m_{0\alpha})$ converges to m_0 in $w(M_0(X), C_0(X))$. Conversely, if $(m_{0\alpha})$ converges to m_0 in $w(M_0(X), C_0(X))$, then for each $G_0 \in \mathscr{G}_0(X)$ $\liminf m_\alpha G_0 \geqslant mG_0$ (see 3.1. iv)). $\mathscr{G}_0(X)$ contains the base for $\mathscr{B}(X)$. Thus to a given $G \in \mathscr{G}(X)$ there is a net $(G_0) \subset \mathscr{G}_0(X)$ filtering up to G. By the τ-smoothness of m to each $\varepsilon > 0$ there is $G_0 \in (G_0)$ with $mG < mG_0 + \varepsilon$ and

$$mG - \varepsilon < mG_0 \leqslant \liminf m_\alpha G_0 \leqslant \liminf m_\alpha G.$$

This proves that $mG \leqslant \liminf m_\alpha G$ for each $G \in \mathscr{G}(X)$. The rest of the proof follows from remark 3.2.

Theorem 3.4.: *Let $(m_\alpha) \subset M_0(X)$ be a net. If m is a measure in $M_0^+(X, \tau)$ and if $\bar{m}$ is the τ-smooth extension of m from $\mathscr{B}_0(X)$ to $\mathscr{B}(X)$, then the conditions*

i) (m_α) *converges to* $\bar{m}$ in $w(M_0(X), C_0(X))$,
ii) $\lim m_\alpha^*(f) = \lim m_{*\alpha}(f) = \bar{m}(f)$ for all $f \in C(X)$,
iii) $\limsup m_\alpha^*(f) \leqslant \bar{m}(f)$ *for all bounded u.s.c.* f,
iv) $\liminf m_{*\alpha}(f) \geqslant \bar{m}(f)$ *for all bounded l.s.c.* f,

where $m_\alpha^*(m_{*\alpha})$ denotes the upper (lower) integral of m_α, are equivalent.

Proof: ii) $\Rightarrow$ i) is trivial. If i) holds and if f_0 is a bounded $\mathscr{B}_0(X)$ measurable l.s.c. function, then by 3.1. ii) $m(f_0) \leqslant \liminf m_\alpha(f_0)$. If f is bounded and l.s.c., $0 < f < 1$, then there is an increasing net $(f_{0\beta})$ of bounded $\mathscr{B}_0(X)$ measurable l.s.c. functions with $0 < f_{0\beta} < 1$ and $\lim f_{0\beta} = f$. By the τ-smoothness of m $m(f) = \lim m(f_{0\beta})$. Whence to each $\varepsilon > 0$ there is β such that $\bar{m}(f) < \bar{m}(f_{0\beta}) + \varepsilon$ and

$$\bar{m}(f) - \varepsilon < m(f_{0\beta}) \leqslant \liminf m_\alpha(f_{0\beta}) \leqslant \liminf m_{*\alpha}(f),$$

which shows that $\bar{m}(f) \leqslant \liminf m_{*\alpha}(f)$. This relation can be extended to each l.s.c. function. i) $\Rightarrow$ iv) is proved. The implications iv) $\Rightarrow$ iii) $\Rightarrow$ ii) are now obvious.

3.3. states that there is no need to develop any "non-separable" theory for $M_0^+(X, \tau)$ provided with the topology $w(M_0(X), C_0(X))$. 3.4. i) $\Rightarrow$ ii) was proved by Dudley [5] and later, by methods more similar to ours, by Pollard [11].

Theorem 3.5.: *If $(m_\alpha) \subset M_0(X)$ is a net converging to $m \in M_0(X, \tau)$ in $w(M_0(X), C_0(X))$, then the conditions* i)—iv) *are equivalent to the relations*

v) $\lim m_\alpha E = \bar{m} E$ *for each $\bar{m}$ continuity set $E \in \mathscr{B}_0(X)$*
vi) $\lim m_\alpha^*(f) = \lim m_{*\alpha}(f) = \bar{m}(f)$ *for each bounded $\bar{m}$ continuity function.*

The proof of 3.5. uses the arguments analogous to those in the proof of [13, part II, theorem 8.1.] (compare also with [11, theorem 2]).

4. Completness and Compactness in $w(M_0(X), C_0(X))$

The theorem 4 from A. D. Alexandroff in [1; ch. V, § 19] can be interpreted by the following:

Theorem 4.1.: *If $(m_n) \subset M_0(X, \sigma)$ is a sequence of measures converging to $m \in M_0(X)$ in the weak topology $w(M_0(X), C_0(X))$, then $m \in M_0(X, \sigma)$.*

Using the arguments of V. S. Varadarjan [13, part II, § 6] we can state.

Theorem 4.2.: *If $(m_n) \subset M_0(X, \tau)$ is a sequence converging weakly to $m \in M_0(X)$, then $m \in M_0(X, \tau)$.*

Proof: The m_n are concentrated on closed separable subsets in $\mathscr{F}_0(X)$ (corollary 2.9.), thus by 2.3.d there is a separable $F \in \mathscr{F}_0(X)$ such that all the m_n are concentrated on F. Since $|m|\, X - F \leqslant \liminf |m_n|\, X - F = 0$, m has a separable support. By 4.1. m is σ-smooth, whence, it must be τ-smooth.

Corollary 4.3.: *If X, d is a complete metric space and $(m_n) \subset M_0(X, \mathrm{t})$ is weakly convergent in $M_0(X)$, then it converges in $M_0(X, \mathrm{t})$.*

4.1.—4.3. state that $w(M_0(X), C_0(X))$ is sequentially complete under the same conditions as $w(M(X), C(X))$, although as regards some equivalence of these topologies, we are sure only if the measures are non-negative and τ-smooth. This situation is made clear in

Theorem 4.4.: *Let $(m_\alpha) \subset M(X)$ be a net, m be a measure in $M(X)$. Let $(m_{0\alpha})$ and m_0 be the restrictions of the m_α and m from $\mathscr{B}(X)$ to $\mathscr{B}_0(X)$, respectively. If the m_α are concentrated on a separable closed set $F \in \mathscr{F}_0(X)$, then (m_α) converges to m in $w(M(X), C(X))$ if and only if $(m_{0\alpha})$ converges to m_0 in $w(M_0(X), C_0(X))$.*

Proof: Using only some trivial modifications the reader can verify that the Tietze extension theorem [6, theorem 2.1.8.] can be modified by the following: Let f_F be a continuous function on $F \in \mathscr{F}_0(X)$ with the induced topology. If $f_F \in C_0(F)$, then f_F has an extension to a function $f \in C_0(X)$.

Now if the $m_{0\alpha}$ are concentrated on a separable set $F \in \mathscr{F}_0(X)$ and if they converge to m_0 in $w(M_0(X), C_0(X))$, then m_0 is concentrated on F (see the proof of 4.2.). By the Tietze theorem, to each $f \in C(X)$ there is $f_0 \in C_0(X)$ such that $f_0 = f$ on F. This implies that (m_α) converges to m in $w(M(X), C(X))$.

Since the reverse assertion is trivial, the proof is finished.

Theorem 4.5.: *If $(m_\alpha) \subset M_0^+(X)$ converges to $m \in M_0^+(X)$* in $w(M_0(X), C_0(X))$, *then*

a) $m \in M_0^+(X, \tau)(M_0^+(X, \sigma))$ *if and only if for each decreasing net (sequence) $(F_\beta) \subset \mathscr{F}_0(X)$ with $\bigcap_\beta F_\beta = \emptyset$*

i) $\inf_\beta \limsup m_\alpha F_\beta = 0$,

b) $m \in M_0^+(X, \mathrm{t})$ *if and only if*

ii) $\inf_{K \in \mathscr{K}(X)} \sup_{F \subset K^c} \limsup m_\alpha F = 0$

Proof: a) The necessity of i) is an immediate consequence of 3.1. iii). Conversely, let $(F_n) \subset \mathscr{F}_0(X)$ be a decreasing sequence with $\bigcap F_n = \emptyset$. By [13, part I, theorem 15] there are increasing $(F_n^*) \subset \mathscr{F}_0(X)$, $(G_n^*) \subset \mathscr{G}_0(X)$ such that $F_n^* \subset X - F_n$, $F_n^* \subset G_n^* \subset F_{n+1}^*$ for all n and $\bigcup_n F_n^* = X$. Using i) we obtain the relation $mX = \sup_n mX - F_n$, which proves the σ-smoothness.

If $(F_\beta) \subset \mathscr{F}_0(X)$ is a decreasing net with $\bigcap F_\beta = \emptyset$, then to each β there is at least one $F \in \mathscr{F}_0(X)$ and $G \in \mathscr{G}_0(X)$ for which $F \supseteq G \supseteq F_\beta$ and the class $\{F\colon F \supseteq G \supseteq F_\beta$ for some β and $G \in \mathscr{G}_0(X)\}$ forms a decreasing net filtering to $\emptyset$. Now, using i) and 3.1. iii), we can easily prove that $m \bigcap_\beta F_\beta = \inf_\beta mF_\beta$.

b) If $m \in M_0(X, \mathrm{t})$, then by 3.1. iii)

$$mX = \sup_{K \in \mathscr{K}(X)} mK = \sup_{K \in \mathscr{K}(X)} \inf_{G \supseteq K} mG \leqslant \sup_{K \in \mathscr{K}(X)} \inf_{G \supseteq K} \liminf m_\alpha G \leqslant \lim m_\alpha X,$$

which proves the necessity of ii).

Now let (m_α) converge weakly to m and let ii) hold. (m_α) contains a subnet (m_{α_β}) such that $\lim m_{\alpha_\beta} G$ exists for all $G \in \mathscr{G}_0(X)$. The relation

$$m'E = \sup_{K \subseteq E} \inf_{G \supseteq K} \liminf m_\alpha G \quad \text{for all} \quad E \in \mathscr{B}_0(X)$$

defines a measure $m' \in M_0(X, \mathrm{t})$ (see [14]). $m'G \leqslant \lim m_{\alpha_\beta} G$ for all $G \in \mathscr{G}_0(X)$ and since

$$\lim m_{\alpha_\beta} X = \lim m_{\alpha_\beta} X - \inf_{K \in \mathscr{K}(X)} \sup_{F \subseteq K^c} \limsup m_{\alpha_\beta} F =$$

$$= \sup_{K \in \mathscr{K}(X)} \inf_{G \supseteq K} \liminf m_{\alpha_\beta} G \leqslant \lim m_{\alpha_\beta} X$$

$\lim m_{\alpha_\beta} X = m'X$. However, this means that (m_α) converges weakly to m' and there must be $m' = m$.

Corollary 4.6.: Let $(m_\alpha) \subset M_0^+(X)$ be a net of measures with $\limsup m_\alpha X < < \infty$. Then (m_α) has a weakly convergent subnet with the limit point $m \in M_0^+(X)$.

a) If the condition i) from 4.5. holds for each decreasing net (sequence) $(F_\beta) \subset \mathscr{F}_0(X)$ with $\bigcap_\beta F_\beta = \emptyset$, then $m \in M_0^+(X, \tau)(m \in M_0^+(X, \sigma))$.

b) if ii) from 4.4. b) holds, then $m \in M_0^+(X, \mathrm{t})$.

If we consider the τ-smooth case in a) and the tight case in b), then we can say instead of net and subnet sequence and subsequence.

The first part of 4.6. follows from the Banach-Alaoglu theorem and from 4.5. The second part uses the fact that the limit measure must be concentrated on a separable subset in $\mathscr{F}_0(X)$. For more detail see [11, sec. 4.].

REFERENCES

[1] ALEXANDROFF, A. D.: Additive set functions in abstract spaces. Mat. Sb. 8, 1940, 307—348; 9, 1941, 563—628; 13, 1943, 169—238.

[2] BOURBAKI, N.: Éléments de mathématiques. Livre VI, Intégration, Hermann, Paris 1959—1969.

[3] LE-CAM, L.: Convergence in distribution of stochastic processes. Univ. California Publ. Statist. 2, no. 11, 1957, 207—236.

[4] DUDLEY, R. M.: Weak convergence of probabilities on nonseparable metric spaces and empirical measures on Euclidean spaces. Illinois J. Math. vol 10, 1966, 109—126.

[5] DUDLEY, R. M.: Measures on non-separable metric spaces, Illinois J. Math. vol 11, 1967, 449—453.

[6] ENGELKING, R.: General Topology. PWN, Warszawa 1977.

[7] FREMLIN, D. H.—GARLING, G. J. H.—HAYDON, R. G.: Bounded measures on topological spaces. Proc. London Math. Soc. 3, no. 25, 1972, 115—136.

[8] HALMOS, P. R.: Measure Theory. D. Van Nostrand, New York 1950.

[9] KELLEY, J. L.: General Topology. D. Van Nostrand, New York 1955.

[10] PACHL, J. K.: Measures as functionals on uniformly continuous functions. Pacific. J. Math. 82, no. 2, 1979, 515—521.

[11] POLLARD, D.: Weak convergence on non-separable metric spaces. J. Austral. Math. Soc. 28, 1979, 197—204.

[12] POLLARD, D.—TOPSØE, F.: A unified approach to the Riesz type representation theorems. Studia Math. 54, 1975, 173—190.

[13] TOPSØE, F.: Topology and Measure. Springer, Berlin 1970.

[14] TOPSØE, F.: Compactness in spaces of measures. Studia Math. 36, 1970, 195—212.

[15] TOPSØE, F.: Further results on integral representations. Studia Math. 55, 1976, 239—245.

[16] VARADARJAN, V. S.: Measures on topological spaces. Mat. Sb. 55, 1961, 35—100.

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