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# RATIONALS WITH EXOTIC CONVERGENCES II

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ABSTRACT. We study the process of enlarging the metric convergence for rational numbers to a compatible group and ring convergences. We show how these enlargements are related to some basic algebraic characteristics of the field of rational numbers and their extensions.

We continue our investigations of rational numbers equipped with compatible group and ring convergences coarser than the usual metric one ([4]). We concentrate on the process of enlarging the metric convergence and show how it is related to some basic algebraic characteristics of the field of rational numbers and its extensions.

In notation and terminology we follow [4]. We start with some definitions and general remarks concerning the process of enlarging a convergence within a given class of convergences. Section 2 is devoted to group convergences of bounded sequences of rational numbers and the role of a Hamel basis (of the vector space of real numbers over the scalar field of rational numbers) in the process of enlarging the metric convergence. In Section 3 also unbounded group convergences for rational numbers are considered. In Section 4 we investigate ring convergences and in particular ring convergences on the field of algebraic numbers in connection with its transcendental extensions.

By a sequential convergence we usually understand a FLUSH-convergence (L stands for the compatibility of the group or ring structure, U for the Urysohn axiom and H for the uniqueness of limits) but we shall work with FUSH-, FLS-, or FLSH-convergences as well.

By R, A, Q, Z and N we denote the real numbers, the algebraic numbers, the rational numbers, the integers and the natural numbers (i.e. positive integers), respectively. By MON we denote the monotone mappings of N into N, by $S = \langle S(n) \rangle \in X^{\mathrm{N}}$ a sequence of points of $X$ and by $S \circ s$, $s \in \mathrm{MON}$, the corresponding subsequence $\langle S(s(n)) \rangle$. If $X$ is equipped with an algebraic operation, then the operation in $X^{\mathrm{N}}$ is defined pointwise.

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0. Remark. To make the paper more self-contained, recall that a compatible convergence in a group, resp. ring, is determined via its neutral sequences, i.e. sequences converging to its neutral element $e$, resp. 0. Given an FLS-convergence $\mathfrak{L}$ for $X$, $\mathfrak{L}^{\leftarrow}(e)$ ( $= \{S \in X^{\mathrm{N}}; (S, e) \in \mathfrak{L}\}$) is a distinguished subgroup, resp. $\mathfrak{L}^{\leftarrow}(0)$ is a distinguished subring of $X^{\mathrm{N}}$. It is, e.g., closed with respect to subsequences, i.e. $\delta$-closed, where for $\mathscr{A} \subset X^{\mathrm{N}}$ we define $\delta\mathscr{A}$ to be the set of all $S \circ s$ with $S \in \mathscr{A}$ and $s \in \mathrm{MON}$; $\mathfrak{L}$ satisfies axiom (H) iff $\mathfrak{L}^{\leftarrow}(e)$, resp. $\mathfrak{L}^{\leftarrow}(0)$, does not contain constant sequences except $\langle e \rangle$, resp. $\langle 0 \rangle$, axiom (U) iff $\mathfrak{L}^{\leftarrow}(e)$, resp. $\mathfrak{L}^{\leftarrow}(0)$, is $\zeta$-closed, where for $\mathscr{A} \subset X^{\mathrm{N}}$ we define $\zeta\mathscr{A}$ by $S \in X^{\mathrm{N}}$ belongs to $\zeta\mathscr{A}$ whenever for each $s \in \mathrm{MON}$ there is $t \in \mathrm{MON}$ such that $S \circ s \circ t \in \mathscr{A}$. On the other hand, given $\mathscr{A} \subset X^{\mathrm{N}}$, we can construct the minimal suitable subgroup, resp. subring, of $X^{\mathrm{N}}$ (as the intersection of all suitable subgroups, resp. subrings, containing $\mathscr{A}$) and this will be $\mathfrak{L}_{\mathscr{A}}^{\leftarrow}(e)$, resp. $\mathfrak{L}_{\mathscr{A}}^{\leftarrow}(0)$, of the smallest FLS-convergence $\mathfrak{L}_{\mathscr{A}}$ for $X$ such that $(S, e) \in \mathfrak{L}_{\mathscr{A}}$, resp. $(S, 0) \in \mathfrak{L}_{\mathscr{A}}$, for each $S \in \mathscr{A}$; axiom (H) is satisfied iff $\mathfrak{L}_{\mathscr{A}}^{\leftarrow}(e)$, resp. $\mathfrak{L}_{\mathscr{A}}^{\leftarrow}(0)$, does not contain constants except $\langle e \rangle$, resp. $\langle 0 \rangle$, and axiom (U) holds iff $\mathfrak{L}_{\mathscr{A}}^{\leftarrow}(e)$, resp. $\mathfrak{L}_{\mathscr{A}}^{\leftarrow}(0)$, is $\zeta$-closed. Note that $(S, x) \in \mathfrak{L}_{\mathscr{A}}$ iff $S\langle x^{-1} \rangle \in \mathfrak{L}_{\mathscr{A}}^{\leftarrow}(e)$ resp. $(S - \langle x \rangle) \in \mathfrak{L}_{\mathscr{A}}^{\leftarrow}(0)$. For details the reader is referred to [7] and [8], respectively.

## 1. Enlargements of a convergence

**1.1. Definition.** *Let $X$ be a set and let $\Lambda$ be a class of convergences for $X$. If $\mathfrak{L}$, $\mathfrak{L}' \in \Lambda$ and $\mathfrak{L} \subset \mathfrak{L}'$, then $\mathfrak{L}'$ is said to be an* enlargement *of $\mathfrak{L}$ in $\Lambda$. If $\mathfrak{A}$ is a subset of $X^{\mathrm{N}} \times X$ and $\mathfrak{A} \subset \mathfrak{L}' \setminus \mathfrak{L}$, then $\mathfrak{L}'$ is said to be an* $\mathfrak{A}$-enlargement *of $\mathfrak{L}$ in $\Lambda$. We say that $\mathfrak{A} \subset X^{\mathrm{N}} \times X$ is* $\mathfrak{L}$-free *in $\Lambda$ if there is an $\mathfrak{A}$-enlargement of $\mathfrak{L}$. If $\mathscr{E}$ is a subset of $X^{\mathrm{N}}$ and $x$ a point of $X$, then $\mathscr{E}$ is said to be* $\mathfrak{L}$-free at $x$ *in $\Lambda$ if $\mathscr{E} \times \{x\}$ is $\mathfrak{L}$-free. We say that $\mathscr{E} \subset X^{\mathrm{N}}$ is* totally $\mathfrak{L}$-free *in $\Lambda$ if for each mapping $f$ of $\mathscr{E}$ into $X$ the set $\{(S, f(S)); S \in \mathscr{E}\}$ is $\mathfrak{L}$-free in $\Lambda$. If $\mathfrak{L}$ has no proper enlargement in $\Lambda$, then $\mathfrak{L}$ is said to be* coarse *in $\Lambda$. If $\mathfrak{A}$ is $\mathfrak{L}$-free and each $\mathfrak{A}$-extension of $\mathfrak{L}$ in $\Lambda$ is coarse in $\Lambda$, then $\mathfrak{A}$ is said to be* $\mathfrak{L}$-saturated *in $\Lambda$. If $\mathfrak{A}$ is $\mathfrak{L}$-free in $\Lambda$ and for each $\mathfrak{B} \subset \mathfrak{A}$ there exists a $\mathfrak{B}$-extension $\mathfrak{L}'$ of $\mathfrak{L}$ in $\Lambda$ such that $\mathfrak{A} \setminus \mathfrak{B}$ is $\mathfrak{L}'$-free in $\Lambda$, then $\mathfrak{A}$ is said to be* $\mathfrak{L}$-independent *in $\Lambda$. If the class $\Lambda$ is fixed, then the phrase "in $\Lambda$" will be omitted.*

1.2. Remark. If $\Lambda$ is the class of all FSH-convergences for $X$, then $\mathfrak{L}$ is coarse in $\Lambda$ iff it is sequentially compact. If $\mathfrak{L}$ is an FSH-convergence for $X$ and $\Lambda$ is the class of all FSH-convergences for $X$ inducing the same sequential closure operator for $X$ as $\mathfrak{L}$, then $\mathfrak{L}$ is coarse in $\Lambda$ iff it satisfies the Urysohn axiom. If $\mathfrak{L}$ does not satisfy the Urysohn axiom, then the Urysohn modification $\mathfrak{L}^*$ of $\mathfrak{L}$ is a proper enlargement of $\mathfrak{L}$ in $\Lambda$. As shown in [6] (Example 5), if $\mathfrak{M}$

is the metric convergence for Q and $\Lambda$ is the class of all group FLUSH-convergences for Q and $S \in Q^N$ is a sequence converging in the real line to any irrational number, then $\{S\}$ is $\mathfrak{M}$-free at 0 (in fact at any $q \in Q$, and hence totally $\mathfrak{M}$-free) in $\Lambda$. On the other hand (cf. [4]), if $\Lambda$ is the class of all ring FLUSH-convergences for Q and $S \in Q^N$ is a sequence converging in the real line to an algebraic number, then $\{S\}$ fails to be $\mathfrak{M}$-free at any $q \in Q$ in $\Lambda$. In the present paper we continue our investigations of the metric convergence $\mathfrak{M}$ for Q in terms of the notions introduced in Definition 1.

There is a natural way how to enlarge a compatible convergence in a group. The idea has been communicated to the author by D. Dikranjan and also appeared in [1].

Let $H$ be a group equipped with a FLUSH-convergence $\mathfrak{H}$. Let $G$ and $F$ be subgroups of $H$ and let $gFg^{-1} \subset F$ for each $g \in G$. Put $\mathscr{A} = \{S \in G^N; (S, x) \in \mathfrak{H}, x \in F\}$ and $\mathfrak{G} = \{(S, x) \in G^N \times G; S\langle x^{-1} \rangle \in \mathscr{A}\}$. The proof of the next lemma is a straightforward consequence of Theorem 3.3 in [7] (see Remark 0) and is omitted.

**1.3. Lemma.** (i) *$\mathscr{A}$ is a subgroup of $G^N$.*
(ii) *If $S \in \mathscr{A}$, then $S \circ s \in \mathscr{A}$ for each $s \in$* MON.
(iii) *$\langle g \rangle \mathscr{A} \langle g^{-1} \rangle \subset \mathscr{A}$ for each $g \in G$.*
(iv) *$\mathfrak{G}$ is an* FLS-*convergence for $G$ and $\mathfrak{H} \restriction G \subset \mathfrak{G}$.*
(v) *$G \cap F = \{e\}$ iff $\mathscr{A}$ contains no constant sequence except $\langle e \rangle$.*
(vi) *$\mathfrak{H} \restriction G = \mathfrak{G}$ iff ($\mathfrak{H}$-cl $G) \cap F = \{e\}$.*

**1.4. Proposition and definition.** *Let $H$ be a group equipped with a* FLUSH-*convergence $\mathfrak{H}$. Let $G$ and $F$ be subgroups of $H$, let $gFg^{-1} \subset F$ for each $g \in G$, let $G \cap F = \{e\}$, let ($\mathfrak{H}$-cl $G) \cap F \neq \{e\}$, let $\mathscr{A}$ and $\mathfrak{G}$ be as defined above. Then $\mathfrak{G}$ is an* FLSH-*convergence for $G$ strictly coarser than $\mathfrak{H} \restriction G$. Let $\mathfrak{G}^*$ be the Urysohn modification of $\mathfrak{G}$. Then $\mathfrak{G}^*$ is said to be the ($\mathfrak{H}$, $F$)-enlargement of $\mathfrak{H} \restriction G$.*

A compatible convergence in a ring can be enlarged in a similar way. Indeed, let $K$ be a commutative ring equipped with a FLUSH-convergence $\mathfrak{K}$. Let $L$ and $M$ be subrings of $K$ and let $LM \subset M$. Put $\mathscr{A} = \{S \in L^N; (S, x) \in \mathfrak{K}, x \in M\}$ and $\mathfrak{L} = \{(S, x) \in L^N \times L; S - \langle x \rangle \in \mathscr{A}\}$. A straightforward proof of the next lemma is omitted.

**1.5. Lemma.** (i) *$\mathscr{A}$ is a subring of $L^N$.*
(ii) *If $S \in \mathscr{A}$, then $S \circ s \in \mathscr{A}$ for each $s \in$* MON.
(iii) *$\langle x \rangle \mathscr{A} \subset \mathscr{A}$ for each $x \in L$.*
(iv) *$\mathfrak{L}$ is an* FLS-*convergence for $L$ and $\mathfrak{K} \restriction L \subset \mathfrak{L}$.*
(v) *$L \cap M = \{0\}$ iff $\mathscr{A}$ contains no constant sequence except $\langle 0 \rangle$.*
(vi) *$\mathfrak{K} \restriction L = \mathfrak{L}$ iff ($\mathfrak{K}$-cl $L) \cap M = \{0\}$.*

**1.6. Proposition and definition.** *Let $K$ be a commutative ring equipped with a* FLUSH-*convergence $\mathfrak{K}$. Let $L$ and $M$ be subrings of $K$, let $LM \subset M$. let*

*$L \cap M = \{0\}$, let $(\mathfrak{K}\text{-cl}\, L) \cap M \neq \{0\}$, let $\mathscr{A}$ and $\mathfrak{L}$ be as defined above. Then $\mathfrak{L}$ is an* FLSH-*convergence for $L$ strictly coarser than $\mathfrak{K} \restriction L$. Let $\mathfrak{L}^*$ be the Urysohn modification of $\mathfrak{L}$. Then $\mathfrak{L}^*$ is said to be the* $(\mathfrak{K}, M)$-enlargement *of $\mathfrak{K} \restriction L$.*

1.7. Remark. Let $L$ be a group and let $\mathfrak{L}$ be a FLUSH-convergence for $L$. If $(S, x) \in L^{\mathrm{N}} \times L$ and $\mathfrak{L}'$ is an $\{(S, x)\}$-enlargement of $\mathfrak{L}$ in the class of all FLUSH-convergences for $L$, then $(T, x) \in \mathfrak{L}'$ whenever $(ST^{-1}, e) \in \mathfrak{L}$. Since $(ST^{-1}, e) \in \mathfrak{L}$ defines an equivalence relation $\sim$ on $L$, every enlargement $\mathfrak{L}'$ of $\mathfrak{L}$ matches nicely with the equivalence classes induced by $\mathfrak{L}$.

## 2. Bounded group enlargements

Denote by $\mathfrak{M}_{\mathrm{R}}$ the usual metric convergence for R and let $\mathfrak{M} = \mathfrak{M}_{\mathrm{R}} \restriction \mathrm{Q}$ be its restriction to Q. Let $\mathfrak{G}$ be a coarse group FLUSH-convergence for Q such that $\mathfrak{M} \subset \mathfrak{G}$. Then (cf. [4]), the sequence $\langle 2^n \rangle$ is $\mathfrak{M}$-free at 0 and there are unbounded sequences in Q which $\mathfrak{G}$-converge to 0. Let $\mathfrak{G}_b$ be the bounded part of $\mathfrak{G}$, i.e., $(S, x) \in \mathfrak{G}$ belongs to $\mathfrak{G}_b$ iff $S$ is a bounded sequence of rational numbers.

2.1. Remark. It follows immediately that Q equipped with $\mathfrak{G}_b$ inherits many properties of Q equipped with $\mathfrak{G}$. In particular, it is complete and no two points of Q can be separated by disjoint neighborhoods (cf. [3]), its sequential order is $\omega_1$ (cf. [5]) and if $(S, 0) \in \mathfrak{G}_b$, then $(\langle q \rangle S, 0) \in \mathfrak{G}_b$ for each $q \in \mathrm{Q}$ (cf. [4]).

**2.2. Theorem.** *There is a set $B$ of irrational numbers such that $\{1\} \cup B$ is a Hamel basis of* R *over* Q *and if $F$ is the subspace of* R *generated by $B$, then $\mathfrak{G}_b$ is the $(\mathfrak{M}_{\mathrm{R}}, F)$-enlargement of $\mathfrak{M}$.*

Proof. Define $Y \subset \mathrm{R} \setminus \{0\}$ as follows: $y \in Y$ iff there exists $T \in \mathrm{Q}^{\mathrm{N}}$ such that $(T, y) \in \mathfrak{M}_{\mathrm{R}}$ and $(T, 0) \in \mathfrak{G}_b$. Let $B$ be a maximal linearly independent subset of $Y$.

First, we shall prove that the set $\{1\} \cup B$ is linearly independent. Suppose that $a \in \mathrm{Q}$, $k \in \mathrm{N}$, $a(i) \in \mathrm{Q}$, $y(i) \in B$, $y(i) \neq y(j)$ for $i \neq j$, $i, j = 1, \ldots, k$, and $a + a(1)y(1) + \ldots + a(k)y(k) = 0$. Choose $T(i) \in \mathrm{Q}^{\mathrm{N}}$ such that $(T(i), y(i)) \in \mathfrak{M}_{\mathrm{R}}$ and $(T(i), 0) \in \mathfrak{G}_b$, $i = 1, \ldots, k$. Consider the sequence $\langle a \rangle + \langle a(1) \rangle R(1) + \ldots$ $\ldots + \langle a(k) \rangle T(k)$. By Remark 2.1 we have $(\langle a(i) \rangle T(i), 0) \in \mathfrak{G}_b$, $i = 1, \ldots, k$. Thus $a = 0$ and hence, by the assumption on $B$, $a(i) = 0$ for all $i$, $i = 1, \ldots, k$.

Second, let $y \in \mathrm{R} \setminus \mathrm{Q}$ and let $T \in \mathrm{Q}^{\mathrm{N}}$, $(T, y) \in \mathfrak{M}_{\mathrm{R}}$. Then $T$ is $\mathfrak{G}_b$-Cauchy and hence, by Remark 2.1, $(T, x) \in \mathfrak{G}_b$ for some $x \in \mathrm{Q}$. Since $(T - \langle x \rangle, y - x) \in \mathfrak{M}_{\mathrm{R}}$ and $(T - \langle x \rangle, 0) \in \mathfrak{G}_b$, we have $x - y \in Y$. If $y - x \in B$, then $y = x + (y - x)$. If $y - x \notin B$, then there are $a \in \mathrm{Q} \setminus \{0\}$, $k \in \mathrm{N}$, $a(i) \in \mathrm{Q} \setminus \{0\}$, $y(i) \in B$, $y(i) \notin y(j)$ for $i \neq j$, $i, j = 1, \ldots, k$, such that $a(y - x) = a(1)y(1) + \ldots + a(k)y(k)$. Thus $y = x + b(1)y(1) + \ldots + b(k)y(k)$, where $b(i) = a(i)/a$, $i = 1, \ldots, k$.

Third, let $\mathfrak{L}$ be the $(\mathfrak{M}_{\mathrm{R}}, F)$-enlargement of $M$. Let $S \in \mathrm{Q}^{\mathrm{N}}$, $x \in F \setminus \{0\}$ and $(S, x) \in \mathfrak{M}_{\mathrm{R}}$. Hence $(S, 0) \in \mathfrak{L}$. Also, there are $k \in \mathrm{N}$, $a(i) \in \mathrm{Q} \setminus \{0\}$, $S(i) \in \mathrm{Q}^{\mathrm{N}}$, $y(i) \in B$,

$y(i) \neq y(j)$ for $i \neq j$, $i,j = 1, \ldots, k$, such that $x = a(1)y(1) + \ldots + a(k)y(k)$, $S = \langle a(1)\rangle S(1) + \ldots + \langle a(k)\rangle S(k)$ and $(S(i), y(i)) \in \mathfrak{M}_R$ for all $i$, $i = 1, \ldots, k$. Then, by the definition of $B$, we have $(S(i), 0) \in \mathfrak{G}_b$ and, by Remark 2.1, we have also $(\langle a(i)\rangle S(i), 0) \in \mathfrak{G}_b$, $i = 1, \ldots, k$. Thus $(S, 0) \in \mathfrak{G}_b$. Since $\mathfrak{G}_b$ satisfies the Urysohn axiom (U), we have $\mathfrak{L} \subset \mathfrak{G}_b$. Now, let $(S, 0) \in \mathfrak{G}_b$. To prove that $\mathfrak{G}_b \subset \mathfrak{L}$, it suffices to find $s \in$ MON such that $(S \circ s, 0) \in \mathfrak{L}$. Since $S$ is a bounded sequence of rational numbers, there are $x \in$ R and $s \in$ MON such that $(S \circ s, x) \in \mathfrak{M}_R$. If $x = 0$, then $(S \circ s, 0) \in \mathfrak{G}_b$. So, let $x \neq 0$. Clearly $x \in$ R $\setminus$ Q. Since $\{1\} \cup B$ is a Hamel basis of R over Q, there are $k \in$ N, $a \in$ Q, $a(i) \in$ Q $\setminus \{0\}$, $T(i) \in$ Q$^N$, $y(i) \in B$, $y(i) \neq y(j)$ for $i \neq j$, $i,j = 1, \ldots, k$, such that $x = a + a(1)y(1) + \ldots + a(k)y(k)$, $S \circ s = \langle a\rangle + \langle a(1)\rangle T(1) + \ldots + \langle a(k)\rangle T(k)$ and $(T(i), y(i)) \in \mathfrak{M}_R$ for each $i$, $i = 1, \ldots, k$. By the definition of $B$, we have $(T(i), 0) \in \mathfrak{G}_b$ and hence, by Remark 2.1, also $(\langle a(i)\rangle T(i), 0) \in \mathfrak{G}_b$. Thus $a = 0$ and $(S \circ s, 0) \in \mathfrak{L}$. This completes the proof of Theorem 2.2.

**2.3. Definition.** *Let $\mathfrak{L}$ be an enlargement of $\mathfrak{M}$ in the class of all group* FLUSH-*convergences for* Q. *We say that $\mathfrak{L}$ is* bounded *if $(S, x) \in \mathfrak{L}$ for no unbounded sequence $S$ of rational numbers. If $(\langle q\rangle S, 0) \in \mathfrak{L}$ for each $q \in$ Q whenever $(S, 0) \in \mathfrak{L}$, then $\mathfrak{L}$ is said to be* Q-productive.

**2.4. Theorem.** *Let $\mathfrak{L}$ be a proper enlargement of $\mathfrak{M}$ in the class of all group* FLUSH-*convergences for* Q. *Then the following are equivalent*:

(i) *$\mathfrak{L}$ is bounded and* Q-*productive*;

(ii) *There is $B \subset$ R $\setminus$ Q, such that $\{1\} \cup B$ is* Q-*linearly independent in* R, *$F$ is the linear subspace of* R *over* Q *generated by $B$ and $\mathfrak{L}$ is the $(\mathfrak{M}_R, F)$-enlargement of $\mathfrak{M}$.*

Proof. "(i) implies (ii)" can be proved in a similar way as Theorem 2.2. We omit details. The converse implication is obvious.

2.5. Example. Choose $S \in$ Q$^N$ such that $(S, \sqrt{2}) \in \mathfrak{M}_R$. Then (cf. Example 5 in [6]) $(S, 0)$ is $\mathfrak{M}$-free. Put $\mathscr{A} = \{S\} \cup \mathfrak{M}^{\leftarrow}(0)$. Let $\mathfrak{L}_{\mathscr{A}}$ be the generated convergence in the class of all group FLUSH-convergenes for Q (see Remark 0). It can be easily verified that for each $s \in$ MON we have $(\langle 1/2\rangle S, 0) \notin \mathfrak{L}_{\mathscr{A}}$. Hence $\mathfrak{L}_{\mathscr{A}}$ fails to be Q-productive.

**2.6. Theorem.** *Let $\{1\} \cup B$ be a Hamel basis for* R *over* Q. *Let $F$ be the linear subspace of* R *generated by $B$ and let $\mathfrak{L}$ be the $(\mathfrak{M}_R, F)$-enlargement of $\mathfrak{M}$. Then $\mathfrak{L}$ is coarse in the class of all bounded group* FLUSH-*convergences for* Q.

Proof. Let $\mathfrak{G}$ be a coarse FLUSH-convergence for Q coarser than $\mathfrak{L}$ and let $\mathfrak{G}_b$ be its bounded part. Virtually in the same way as in the last part of the proof of Theorem 2.2 it can be proved that $\mathfrak{G}_b \subset \mathfrak{L}$. Thus $\mathfrak{L}$ is bounded coarse.

**2.7. Theorem.** *Let $X$ be a group, resp. ring, equipped with a group, resp. ring,* FUSH-*convergence $\mathfrak{L}$. Let $\mathscr{E} \subset X^N$ be totally $\mathfrak{L}$-free in the class $\Lambda$ of all group, resp. ring,* FLUSH-*convergences for $X$. Let $f$ be a mapping of $\mathscr{E}$ into $X$. Then $\mathfrak{A} = \{(S, f(S, f(S)); S \in \mathscr{E}\}$ is $\mathfrak{L}$-independent in $\Lambda$.*

Proof. We shall prove the assertion for groups. The proof for rings is analogous and is omitted. Let $\mathscr{D} \subset \mathscr{E}$ and let $\mathfrak{B} = \{(S, f(S));\ S \in \mathscr{D}\}$. Let $\mathfrak{L}'$ be the smallest group FLUSH-convergence for $X$ such that $\mathfrak{L} \subset \mathfrak{L}'$ and $\mathfrak{B} \subset \mathfrak{L}' \setminus \mathfrak{L}$. By Remark 0 (see Theorem 3.3 in [7]), $\mathfrak{L}'$ is generated by the group of $\mathfrak{L}'$-neutral sequences (the subgroup of $X^{\mathrm{N}}$ of all sequences $\mathfrak{L}'$-converging to the neutral element of $X$). Let $S \in \mathscr{E} \setminus \mathscr{D}$ and $s \in \mathrm{MON}$. then $S \circ s$ does not $\mathfrak{L}'$-converge in $X$. This follows from the fact that $\mathscr{E}$ is totally $\mathfrak{L}$-free. Indeed, changing $f$ at $S$ we can construct a suitable subgroup of $X^{\mathrm{N}}$ not containing $S \circ s$ and being the group of all $\mathfrak{L}'$-neutral sequences. Clearly, $\mathfrak{L}'$ can be enlarged in $\Lambda$ to $\mathfrak{L}''$ such that $\mathfrak{A} \setminus \mathrm{B} \subset (\mathfrak{L}'' \setminus \mathfrak{L}')$.

**2.8. Theorem.** *Let* $\{1\} \cup B$ *be a Hamel basis of* R *over* Q. *For each* $y \in B$ *choose a sequence* $S(y) \in \mathrm{Q}^{\mathrm{N}}$ *such that* $(S(y), y) \in \mathfrak{M}_{\mathrm{R}}$. *Then* $\{S(y);\ y \in B\}$ *is totally* $\mathfrak{M}$*-free in the class of all group* FLUSH*-convergences for* Q *and it is maximal totally* $\mathfrak{M}$*-free in the class of all bounded group* FLUSH*-convergences for* Q.

Proof. The first assertion follows from the fact that for each mapping $f$ of $B$ into Q the set $\{1\} \cup \{y + f(y);\ y \in B\}$ is a Hamel basis as well. The second assertion follows directly from Theorem 2.7.

2.9. Example. Let $\mathfrak{L}$ be a bounded enlargement of $\mathfrak{M}$. Consider the sequence $S = \langle 2^n \rangle$. Then $\{S\}$ is $\mathfrak{L}$-free at 0, but fails to be $\mathfrak{L}$-free at any point $x \neq 0$. The latter follows easily from $\langle 2S(n+1) \rangle = \langle S(n) \rangle$. Similarly for each $\langle q^n \rangle, q \in \mathrm{Q}, q > 1$. Consequently if $\{S(y); y \in B\}$ is as in Theorem 2.8, then it fails to be saturated. Observe that it is $\mathfrak{M}$-independent but there are $\mathfrak{M}$-independent sets which are not totally $\mathfrak{M}$-free.

## 3. Unbounded group enlargements

It follows from example 2.9 that no coarse enlargement of $\mathfrak{M}$ in the class of all group convergences for Q is bounded. In this section we present some simple observations about the unbounded enlargement of $\mathfrak{M}$.

Let $G$ be a group, let $\Lambda$ be the class of all group FLUSH-convergences for $G$ and let $\mathfrak{G} \in \Lambda$.

**3.1. Lemma.** *Let* $\{\mathscr{A}_a;\ a \in I\}$ *be a chain of subsets of* $\mathrm{Q}^{\mathrm{N}}$ $\mathfrak{G}$*-free at e in* $\Lambda$. *Then* $\bigcup_{a \in} \mathscr{A}_a$ *is* $\mathfrak{G}$*-free at e in* $\Lambda$.

Proof. The assertion follows from Theorem 3.3 in [7] (cf. Remark 0).

**3.2. Theorem.** (i) *Each* $\mathscr{A} \subset G^{\mathrm{N}}$ $\mathfrak{G}$*-free at e in* $\Lambda$ *is contained in a maximal set* $\mathfrak{G}$*-free at e in* $\Lambda$.

(ii) *Each* $\mathscr{A} \subset G^{\mathrm{N}}$ *totally* $\mathfrak{G}$*-free in* $\Lambda$ *is contained in a maximal set totally* $\mathfrak{G}$*-free in* $\Lambda$.

Proof. The proof is a straightforward application of the Kuratowski-Zorn lemma. the details are omitted.

**3.3. Lemma.** *Let $\mathfrak{G}$ be a coarse enlargement of $\mathfrak{M}$ in the class of all group* FLUSH-*convergences for* Q *and let $\mathfrak{B}_b$ be the bounded part of $\mathfrak{G}$. Let $T \in \mathrm{Q}^{\mathrm{N}}$ be a sequence with no bounded subsequence. Then there exists $t \in$* MON *such that $\{T \circ t\}$ is totally $\mathfrak{G}_b$-free.*

Proof. For each $q \in \mathrm{Q}$ and each $s \in$ MON the sequence $T \circ s - \langle q \rangle$ is unbounded and hence does not $\mathfrak{G}_b$-converge. Hence, according to the group coarseness criterion $(C)$ (cf. [4]), it suffices to find $t \in$ MON such that for each $q \in \mathrm{Q}$ no sequence $S(q) = T \circ s - \langle q \rangle$ satisfies condition

$(C2)$ *There are* $k \in \mathrm{N}$, $z(i) \in \mathrm{Z}$, $s(i) \in$ MON,

$i = 1, \ldots, k, \quad x \in \mathrm{Q}, \ x \neq 0,$ *such that*

$$\left( \sum_{i=1}^{k} z(i)\, S(q) \circ s(i),\ x \right) \in \mathfrak{G}_b.$$

Thus, it suffices to guarantee that each such sequence $U(q) = \sum_{i=1}^{k} z(i)\, S(y) \circ s(i)$ contains a subsequence which is either unbounded or constantly 0. Choose $t \in$ MON in such a way that $|T(t(n+1))| > n^n(1 + |T(t(n))|)$. Indeed, then for each fixed $U(q)$ and for all sufficiently large $n \in \mathrm{N}$ we have either $U(q, n) = 0$ or for some $i$, $i = 1, \ldots, k$, $z(i)\, S(q, n) \circ s(i, n)$ is much greater than the rest of the summands.

**3.4. Corollary.** *Let $\{1\} \cup B$ be a Hamel basis for* R *over* Q. For each $y \in B$ *choose a sequence $S(y) \in \mathrm{Q}^{\mathrm{N}}$ such that $(S(y), y) \in \mathfrak{M}_{\mathrm{R}}$. Then the set $\{S(y);\ y \in B\}$ is totally $\mathfrak{M}$-free and is properly contained in a maximal totally $\mathfrak{M}$-free subset of $\mathrm{Q}^{\mathrm{N}}$ in the class of all group* FLUSH-*convergences for* Q.

3.5. Remark. Consider the vector space $\mathrm{Q}^{\mathrm{N}}$ over the scalar field Q and its quotient space $P$ of all equivalence classes $[S]$, $S \in \mathrm{Q}^{\mathrm{N}}$, where $S, T \in \mathrm{Q}^{\mathrm{N}}$ are defined to be equivalent whenever $(S - T, 0) \in \mathfrak{M}$. Then R is a subspace of $P$. For $X \subset P$ define $\delta X = \{[S \circ s] \in P;\ [S] \in X,\ s \in \mathrm{MON}\}$, define $\zeta X$ by: $[S] \in \zeta X$ if for each $s \in$ MON there exists $t \in$ MON such that $[S \circ s \circ t] \in X$ and define $\varepsilon X = \{[S] \in P;\ [S \circ s] \in X;\ s \in \mathrm{MON}\}$. Let $\mathfrak{G}$ be a coarse group enlargement of $\mathfrak{M}$ and let $\mathfrak{G}_b$ be the bounded part of $\mathfrak{G}$. Let $\{1\} \cup B$ be a Hamel basis of R over Q such that $B$ defines $\mathfrak{G}_b$ via $\mathfrak{G}_b^{\leftarrow}(0)$. Notice that $B$ is not a Hamel basis of $X = \{[S] \in P;\ S \in \mathfrak{G}_b^{\leftarrow}(0)\}$. In fact, if $Y$ is the subspace generated by $B$, then $X = \zeta Y$. To enlarge $\mathfrak{M}$ to a group FLSH-convergence $\mathfrak{L}$ for Q it suffices to find a subspace $P'$ of $P$ not containing 1 and such that $\delta P' = P'$. Then $\mathfrak{L}^{\leftarrow}(0) = \{S \in \mathrm{Q}^{\mathrm{N}};\ [S] \in P'\}$ and $\{S \in \mathrm{Q}^{\mathrm{N}};\ [S] \in \zeta P'\}$ is the set of all neutral sequences of the Urysohn modification $\mathfrak{L}^*$ of $\mathfrak{L}$.

**3.6. Problem.** *Is there a nice relationship between Hamel bases of P and coarse group enlargements of $\mathfrak{M}$?*

3.7. Remark. Answering Question 2 from [6] in [9] a coarse commutative group has been constructed (under CH) which cannot be embedded into a sequentially compact group; as shown in [2], the group in question is complete. Recall here the following problem concerning the relationship between the coarseness and the sequential compactness.

**3.8. Problem.** *Let* $\mathfrak{G}$ *be a coarse group enlargement of* $\mathfrak{M}$. *Is* Q *equipped with* $\mathfrak{G}$ *sequentially compact* (*i.e.* $P = \varepsilon\{[S]\in P;\ (S, x)\in\mathfrak{G},\ x\in \mathrm{Q}\}$)?

3.9. Remark. It follows from Theorem 2. and Theorem 2.6 that if $S\in \mathrm{Q}^{\mathrm{N}}$ converges in the real line to a real number, then $S$ converges in Q in each coarse group enlargement of $\mathfrak{M}$. In this sense R is a common $\delta$-closed subspace of $P$ representing equivalence classes $[S]$ of sequences $S$ converging in all such coarse group enlargements. Are there larger $\delta$-closed nice subspaces of $P$ representing sequences converging in all coarse group enlargements of $\mathfrak{M}$? In particular, we have the following problem.

**3.10. Problem.** *Characterize the set of all* $[S]\in P$ *such that* $S$ (*and hence each* $T\in[S]$) *converges in* Q *in each coarse group enlargement of* $\mathfrak{M}$. *Is it* R?

3.11. Remark. As observed in Example 2.9, sequences $\langle q^n\rangle$, $q\in \mathrm{Q}$, $q > 1$, are $\mathfrak{L}$-free at 0 in each bounded group enlargement $\mathfrak{L}$ of $\mathfrak{M}$. It might be interesting to find out more about subspaces of $P$ related to coarse group enlargements of $\mathfrak{M}$ and containing $[\langle q^n\rangle]$ (and similar equivalence classes of "strange" unbounded sequences).

## 4. Ring enlargements

Basic properties of coarse ring enlargements of the metric convergences $\mathfrak{M}$ for Q have been established in [4]. Recall that $\mathfrak{M}$ is not ring coarse and, using the Kuratowski—Zorn lemma, $\mathfrak{M}$ can be enlarged to a coarse ring FLUSH-convergence. Further, if $\mathfrak{R}$ is a coarse ring enlargement of $\mathfrak{M}$, then no unbounded sequence $S\in \mathrm{Q}^{\mathrm{N}}$ is $\mathfrak{R}$-Cauchy and if $S\in \mathrm{Q}^{\mathrm{N}}$ $\mathfrak{M}_{\mathrm{R}}$-converges (in the real line) to an algebraic number which is not rational, then no subsequence $S\circ s$ of $S$ can $\mathfrak{R}$-converge in Q (note that no $S\circ s$ converges in any ring enlargement of $\mathfrak{M}$). This in turn implies that Q equipped with $\mathfrak{R}$ fails to be complete. Since a coarse ring convergence is a field convergence whenever the underlying ring is a field (cf. [8]), convergence properties established for Q equipped with $\mathfrak{R}$ are interesting from the viewpoint of the field convergence theory. E.g., since Q equipped with $\mathfrak{R}$ has no ring completion (cf. Corollary 2 in [4]), there are also convergence fields having no completion.

4.1. Remark. Let $X$ be a commutative ring, let $\Lambda$ be the class of all ring FLUSH-convergences for $X$ and let $\mathfrak{L}\in\Lambda$. Then $\mathfrak{L}$ is coarse in $\Lambda$ iff $\mathfrak{L}$ satisfies the following ring coarseness criterion (cf. [8], [4]):

(*CR*) *For each* $S\in X^{\mathrm{N}}$ *either*

($CR\,1$) *For some* $s \in \mathrm{MON}$ *we have* $(S \circ s, 0) \in \mathfrak{L}$;

*or*

($CR\,2$) *There are* $p \in X$, $p \neq 0$, $m$, $k(j) \in \mathrm{N}$,

$$T(i, j) \in \mathfrak{L}^{\leftarrow}(0) \cup \{\langle x \rangle S \circ s;\ x \in X,\ s \in \mathrm{MON}\},$$

$j = 1, \ldots, m,\ i = 1, \ldots, k(j)$, *such that*

$$\langle p \rangle = \sum_{j=1}^{m} T(j, 1) \ldots T(j, k(j));$$

*holds true.*

Further, for $S \in X^{\mathrm{N}}$, $\{S\}$ is $\mathfrak{L}$-free at 0 iff $S$ satisfies neither ($CR\,1$) nor ($CR\,2$), and $\{S\}$ is totally $\mathfrak{L}$-free iff for each $x \in X$ the sequence $S(x) = S - \langle x \rangle$ satisfies neither ($CR\,1$) nor ($CR\,2$).

The next theorem generalizes Proposition 5 in [4].

**4.2. Theorem.** *Let* $y \in \mathrm{R} \setminus \mathrm{Q}$ *be a transcendental number and let* $S \in \mathrm{Q}^{\mathrm{N}}$ *be a sequence such that* $(S, y) \in \mathfrak{M}_{\mathrm{R}}$. *Then* $\{S\}$ *is totally* $\mathfrak{M}$*-free in the class of all ring* FLUSH-*convergences for* Q.

Proof. Fix $q \in \mathrm{Q}$. We have to prove that for $\mathfrak{L} = \mathfrak{M}$ the sequence $S(q) = S - \langle q \rangle$ satisfies neither ($CR\,1$) nor ($CR\,2$). Clearly, $S(q)$ does not satisfy ($CR\,1$). But $S(q)$ cannot satisfy ($CR\,2$) either. For, otherwise, passing in ($CR\,2$) to $\mathfrak{M}_{\mathrm{R}}$-limits (in the real line), we would get a polynomial $P(x) = a(n)\,x^n + \ldots$ $\ldots + a(1)\,x + a(0)$ over Q, $a(n) \neq 0$, $n > 0$, such that $P(y) = p \in \mathrm{Q}$, contradicting the assumption that $y$ is transcendental.

In a similar way as for groups we can prove that if a ring convergence $\mathfrak{L}$ admits an $\mathfrak{L}$-free set or a totally $\mathfrak{L}$-free set, then the set is contained in a maximal $\mathfrak{L}$-free set or in a maximal totally $\mathfrak{L}$-free set, respectively. The proof is omitted.

**4.3. Theorem.** *Let* $X$ *be a commutative ring, let* $\Lambda$ *be the class of all ring* FLUSH-*convergences for* $X$ *and let* $\mathfrak{L} \in \Lambda$. *Then*

(i) *Each* $\mathscr{A} \subset X^{\mathrm{N}}$ $\mathfrak{L}$*-free at* 0 *in* $\Lambda$ *is contained in a maximal set* $\mathfrak{L}$*-free at* 0 *in* $\Lambda$;

(ii) *Each* $\mathscr{A} \subset X^{\mathrm{N}}$ *totally* $\mathfrak{L}$*-free in* $\Lambda$ *is contained in a maximal set totally* $\mathfrak{L}$*-free in* $\Lambda$.

Taking into account that ring enlargements of $\mathfrak{M}$ discriminate algebraic numbers, instead of Q we shall investigate the field A of all algebraic numbers and ring enlargements of the metric convergence $\mathfrak{M}_{\mathrm{A}} = \mathfrak{M}_{\mathrm{R}} \upharpoonright \mathrm{A}$. Each assertion of the next theorem can be proved virtually in the same way as the corresponding assertion for Q (viz. Theorem 4.2 of the present paper, Proposition 2, Proposition 7 and Corollary 2 in [4]) and we omit the proofs. Throughout the rest of the paper $\Lambda$ denotes the class of all ring FLUSH convergences for A.

**4.4. Theorem.** *Let* $\mathfrak{N}$ *be a coarse enlargement of* $\mathfrak{M}_{\mathrm{A}}$ *in* $\Lambda$.

(i) *Let* $y \in \mathrm{R} \setminus \mathrm{A}$ *and let* $S \in \mathrm{A}^{\mathrm{N}}$ *be a sequence* $\mathfrak{M}_{\mathrm{R}}$*-converging to* $y$. *Then* $\{S\}$ *is totally* $\mathfrak{M}_{\mathrm{A}}$*-free in* $\Lambda$.

(ii) *If* $S \in \mathrm{A}^{\mathrm{N}}$ *is* $\mathfrak{R}$*-Cauchy, then it is bounded.*

(iii) *Let* $(T, 0) \in \mathfrak{R} \setminus \mathfrak{M}_{\mathrm{A}}$. *Then there is* $t \in \mathrm{MON}$ *such that* $(T \circ t)^{-1}$ *is a totally divergent* $\mathfrak{R}$*-Cauchy sequence.*

(iv) A *equipped with* $\mathfrak{R}$ *cannot be embedded into a complete* FLUSH*-convergence ring.*

According to Theorem 4.4, there is a totally $\mathfrak{M}_{\mathrm{A}}$-free in $\Lambda$ sequence $S \in \mathrm{A}^{\mathrm{N}}$ and, by Theorem 4.3, there is a maximal totally $\mathfrak{M}_{\mathrm{A}}$-free in $\Lambda$ set $\mathscr{S}$ containing $S$. Moreover, by Theorem 2.7, for every mapping $f\colon \mathscr{S} \to \mathrm{A}$, the set $\{(\mathrm{S}, \mathrm{f(S)});\ \mathrm{S} \in \mathscr{S}\}$ is independent in $\Lambda$.

Recall ([10]) that $x \in \mathrm{R}$ is a l g e b r a i c a l l y d e p e n d e n t on $B \subset \mathrm{R}$ over A whenever there are a finite subset $\{b(1), \ldots, b(n)\} \subset B$ and polynomials $P(0)$, $P(1), \ldots, P(n)$ in variables $b(1), \ldots, b(n)$ with coefficients from A, at least one $p(i)$ being nonzero, such that $P(n)x^n + \ldots + P(1)x + P(0) = 0$. Further, a subset $B$ of R is a l g e b r a i c a l l y i n d e p e n d e n t over A if no element $b \in B$ is algebraically dependent on $B \setminus \{b\}$ and, if $B = \{b\}$, $b \in \mathrm{R}$, then $B$ is algebraically independent over A provided $b$ is transcendental over A.

Our final goal is to characterize certain maximal totally $\mathfrak{M}_{\mathrm{A}}$-free (in $\Lambda$) sets. We show that there is a nice correspondence between such sets and maximal sets of algebraically independent transcendental elements of R considered as a field extension of A. In particular, all such maximal totally $\mathfrak{M}_{\mathrm{A}}$-free sets have the same cardinality, viz. the degree of transcendence of R over A.

Let $B \subset \mathrm{R}$, $B \neq \emptyset$, be algebraically independent over A. Let $M(B)$ be the smallest subset of R containing all elements of the form $ab$, $a \in \mathrm{A}$, $b \in B$, and closed with respect to sums and products. It is easy to verify that $\mathrm{A}M \subset M$, $\mathrm{A} \cap M = \{0\}$ and $(\mathfrak{M}\text{-cl}\,\mathrm{A}) \cap M \neq \{0\}$. Hence (cf. Proposition and definition 1.4), $M(B)$ defines the (A, $M(B)$)-enlargement of $\mathfrak{M}_{\mathrm{A}}$; denote it by $\mathfrak{L}_B$. Clearly $\mathfrak{L}_B \setminus \mathfrak{M}_{\mathrm{A}} \neq \emptyset$. For each $b \in B$, let $S_b$ be a sequence in A converging in the real line to $b$. Put $\mathscr{S}_B = \{S_b;\ b \in B\}$. Let $x \in \mathrm{R} \setminus \mathrm{A}$ be algebraically dependent on $B$ and let $S_x$ be a sequence in A converging in the real line to $x$. Put $\mathscr{S} = \{S_x\} \cup \mathscr{S}_B$.

**4.5. Theorem.** (i) $\mathscr{S}_B$ *is totally* $\mathfrak{M}_{\mathrm{A}}$*-free.*

(ii) *Let* $f$ *be a mapping of* $\mathscr{S}_B$ *into* A. *Then* $\{(S_b, f(S_b));\ b \in B\}$ *is* $\mathfrak{M}_{\mathrm{A}}$*-independent.*

P r o o f. (i) Let $h$ be a mapping of $B$ into A. Observe that for $b, b' \in B$ we have $b - h(b) \neq b' - h(b')$ whenever $b \neq b'$; further $B(h) = \{b - h(b);\ b \in B\}$ is algebraically independent over A. Let $f$ be a mapping of $\mathscr{S}_B$ into A. We have to prove that $\{(S_b, f(S_b));\ b \in B\}$ is $\mathfrak{M}_{\mathrm{A}}$-free or, equivalently, that $\{(S_b - \langle f(S_b)\rangle, 0);\ b \in B\}$ is $\mathfrak{M}_{\mathrm{A}}$-free. For $b \in B$ put $h(b) = f(S_b)$. Now, the (A, $h(B)$)-enlargement $\mathfrak{L}_{h(B)}$ of $\mathfrak{M}_{\mathrm{A}}$ has the property that $(S_b, f(S_b)) \in \mathfrak{L}_{h(B)}$.

(ii) follows from (i) and Theorem 2.7.

**4.6. Theorem.** *$\mathcal{S}$ fails to be totally $\mathfrak{M}_A$-free.*

Proof. Suppose, on the contrary, that $\mathcal{S}$ is totally $\mathfrak{M}_A$-free. According to Theorem 2.7, for each mapping $g$ of $\mathcal{S}$ into A, the set $\{(S_x, g(S_x))\} \cup \{(S_b, g(S_b)); b \in B\}$ is $\mathfrak{M}_A$-independent. Hence we can first construct an $\{(S_b, g(S_b)); b \in B\}$-enlargement of $\mathfrak{M}_A$ and then continue with its $\{(S_x, g(S_x))\}$-enlargement.

Denote by $f$ the restriction of $g$ to $\mathcal{S}_B$. For each $b \in B$, put $f(b) = h(S_b) = 1$. As shown in the proof of (i) of Theorem 4.5, the (A, $h(B)$)-enlargement $\mathfrak{L}_{h(B)}$ of $\mathfrak{M}_A$ has the property that $(S_b, 1) \in \mathfrak{L}_{h(B)}$. Since $x \in R \setminus A$ is algebraically dependent on $B$, there are a nonempty finite set $\{b(1), \ldots, b(n)\} \subset B$ and polynomials $P(0)$, $P(1), \ldots, P(n)$ in variables $b(1), \ldots, b(n)$ with coefficients from A such that at least one $P(i)$ is nonzero and $\mathcal{P}(x) = P(n)x^n + \ldots + P(1)x + P(0) = 0$. If in $\mathcal{P}(x)$ we replace $x$ by $S_x$, $b(i)$ by $S_{b(i)}$, $i = 1, \ldots, n$, and each constant $a \in A$ by $\langle a \rangle$, the resulting sequence $\mathcal{P}(S_x)$ converges in the real line and hence in $\mathfrak{L}_{h(B)}$ to 0. Since for each $a \in A$ and $k \in N$ the sequence $\langle a \rangle S_b^k$ $\mathfrak{L}_{h(B)}$-converges to $a$, the sequence corresponding to the polynomial $P(i)$ (it is a sum of sequences of the form $\langle a \rangle S_{b(1)}^{k(1)} \ldots S_{b(n)}^{k(n)}$), $\mathfrak{L}_{h(B)}$-converges to some $a(i) \in A$, $i = 0, 1, \ldots, n$. Clearly, at least one $a(i)$ is nonzero. Hence the sequence $S_x$ can converge in any enlargement of $\mathfrak{L}_{h(B)}$ only to such a $y \in A$, for which we have $a(n)y^n + \ldots + a(1)y + a(1) = 0$. Thus $S_x$ fails to be totally $\mathfrak{L}_{h(B)}$-free. Consequently, $\mathcal{S}$ fails to be totally $\mathfrak{M}_A$-free.

**4.7. Corollary.** *Let $B \subset R \setminus A$, $B \neq \emptyset$. For each $b \in B$, let $S_b$ be a sequence in* A *converging in the real line to $b$. Then*

(i) *The set $\{S_b; b \in B\}$ is totally $\mathfrak{M}_A$-free (in $\Lambda$) iff $B$ is algebraically independent over* A;

(ii) *The set $\{S_b; b \in B\}$ is a maximal totally $\mathfrak{M}_A$-free (in $\Lambda$) iff $B$ is a maximal algebraically independent set over* A *of transcendental elements of* R.

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