Lubomír Kubáček

Optimal elimination of nuisance parameters in mixed linear models

Mathematica Slovaca, Vol. 40 (1990), No. 3, 287--301

Persistent URL: http://dml.cz/dmlcz/136511

Math. Slovaca 40, 1990, No. 3, 287—301

OPTIMAL ELIMINATION OF NUISANCE PARAMETERS IN MIXED LINEAR MODELS

LUBOMÍR KUBÁČEK

ABSTRACT. The vector parameter of the mean value of an observation vector in a mixed linear model (MLM) is supposed to be divided into necessary and nuisance vector parameters. A class of linear transformations of the observation vector eliminating the nuisance vector parameter which do not cause a loss of information about the necessary parameter is considered. The problem which of these transformations has the property that the same locally best quadratic estimator of variance components from the original and the transformed MLM is obtained is solved. Several kinds of estimators are investigated.

Introduction

In a regression model

$$(\mathbf{Y}, \mathbf{X}\boldsymbol{\beta}, \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i), \tag{1}$$

where $\boldsymbol{Y}$ is an n-dimensional random vector with normal distribution, its mean value being $E(\boldsymbol{Y}|\boldsymbol{\beta}) = \mathbf{X}\boldsymbol{\beta}$, $\boldsymbol{\beta} \in \mathscr{R}^k$ (k-dimensional Euclidean space), $\mathbf{X}$ is a given $n \times k$ matrix of the structure $\mathbf{X} = (\mathbf{A}, \mathbf{S})$; the matrices $\mathbf{A}$, $\mathbf{S}$ are of the type $n \times k_1$, $n \times k_2$ and $k_1 + k_2 = k$. The rank $R(\mathbf{X})$ of the matrix $\mathbf{X}$ is $R(\mathbf{X}) = k$. The vector $\boldsymbol{\beta}$ is divided into two subvectors $\Theta \in \mathscr{R}^{k_1}$ and $\boldsymbol{\varkappa} \in \mathscr{R}^{k_2}$, $\boldsymbol{\beta} = (\boldsymbol{\Theta}', \boldsymbol{\varkappa}')'$; $\boldsymbol{\Theta}$ is a necessary vector parameter, $\boldsymbol{\varkappa}$ is a nuisance vector parameter. The covariance matrix $\boldsymbol{\Sigma}$ of the random vector $\boldsymbol{Y}$ is considered to have a form $\boldsymbol{\Sigma} = \mathrm{Var}(\boldsymbol{Y}|\boldsymbol{\vartheta}) = \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i$, where symmetric matrices $\mathbf{V}_1, \ldots, \mathbf{V}_p$ are known, the vector $\boldsymbol{\vartheta} = (\vartheta_1, \ldots, \vartheta_p)' \in \underline{\boldsymbol{\vartheta}}$ (open set) $\subset \mathscr{R}^p$, $p > 1$, is unknown. The set $\underline{\boldsymbol{\vartheta}}$ is assumed to have a property: $\boldsymbol{\vartheta} \in \underline{\boldsymbol{\vartheta}} \Rightarrow \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i$ is positive definite.

AMS Subject Classification (1980): Primary 62J05, Secondary 62F10
Key words: Model with variance components, Nuisance parameters, Optimal elimination transformation.

This model arises in various problems of research and practice. For example, let us consider a gravimetric network. This is created by a k_1-tuple of points on the surface of the earth. At each of them repeated measurements by a group of p gravimeters are carried out in order to obtain information on the actual value of the gravities Θ_i, $i = 1, 2, \ldots, k_1$, and on the rate of each individual gravimeters. The rate, which represents the disturbing component of the measurement, is usually modelled by a polynomial $\sum_{i=1}^{r} \varkappa_i^{(j,d)} t^i$ of a certain order. The coefficients $\varkappa_i^{(j,d)}$, $i = 1, \ldots, r$, $j = 1, \ldots, p$ (= number of gravimeters), $d = 1, \ldots, D$ (= number of days in which the measurement is carried out), of the polynomial represent nuisance parameters. The dispersion ϑ_j of the jth gravimeter is the jth component of the vector $\boldsymbol{\vartheta}$. The coefficients $\varkappa_i^{(j,d)}$ of the individual gravimeter can be considered to be constant for a given day. On another day they attain another values; $p \times r \times D = k_2$. The current size of k_1 is several hundreds or more, k_2 can be even greater. Thus the number of normal equations is $k_1 + k_2$ and this can be a rather huge number. It seems to be reasonable, in some cases, to eliminate the nuisance parameters from the input data performing a suitable transformation by a matrix $\mathbf{T}$, which leads to a solution of a linear system with k_2 unknowns, and then to solve a linear system with k_1 unknowns only.

A matrix $\mathbf{T}$ with properties $\mathbf{TA} = \mathbf{A}$, $\mathbf{TS} = \mathbf{O}$, transforms the original model (1) into the model $(\mathbf{T}\boldsymbol{Y}, \mathbf{A}\boldsymbol{\Theta}, \sum_{i=1}^{p} \vartheta_i \mathbf{T}\mathbf{V}_i\mathbf{T}')$. The matrix $\mathbf{T}$ or the transformation performed by it is called optimal if the last model enables us to determine the same best linear unbiased estimator of the vector parameter $\boldsymbol{\Theta}$ as the model (1), provided the matrix $\boldsymbol{\Sigma}$ is given. The necessary and sufficient conditions for a matrix $\mathbf{T}$ to be optimal in the mentioned sense are given in [5] (see also [1], [2], [4]). This optimality will be called the optimality with respect to $\boldsymbol{\Theta}$.

The aim is to find out if there exists such a matrix $\mathbf{T}$ optimal with respect to $\boldsymbol{\Theta}$, which in addition guarantees the possibility to construct the same estimators of parameters $\vartheta_1, \ldots, \vartheta_p$ in the model $(\mathbf{T}\boldsymbol{Y}, \mathbf{A}\boldsymbol{\Theta}, \sum_{i=1}^{p} \vartheta_i \mathbf{T}\mathbf{V}_i\mathbf{T}')$ as in the model (1). We restrict ourselves to estimators given in definitions 1.1 to 1.4.

1. Definitions and auxiliary statements

Definition 1.1. *An estimator* $\boldsymbol{Y}'\mathbf{U}\boldsymbol{Y}$ ($\mathbf{U}$ *is a symmetric* $n \times n$ *matrix*) *of a function* $g(\boldsymbol{\vartheta}) = \boldsymbol{f}'\boldsymbol{\vartheta}$, $\boldsymbol{\vartheta} \in \underline{\vartheta}$, *in the model* (1) *is* $\boldsymbol{\vartheta}^{(0)}$-LMVQUIE (*locally minimum variance quadratic unbiased invariant estimator*) *if*

(i) $\forall \{\boldsymbol{\vartheta} \in \underline{\vartheta}\}\ E(\boldsymbol{Y}'\mathbf{U}\boldsymbol{Y} \mid \boldsymbol{\vartheta}) = \boldsymbol{f}'\boldsymbol{\vartheta}$.

(ii) $\forall \{\boldsymbol{\beta} \in \mathscr{R}^k\}\ (\boldsymbol{Y} - \mathbf{X}\boldsymbol{\beta})'\mathbf{U}(\boldsymbol{Y} - \mathbf{X}\boldsymbol{\beta}) = \boldsymbol{Y}'\mathbf{U}\boldsymbol{Y}$.

(iii) $\mathrm{Var}(\boldsymbol{Y}'\mathbf{U}\boldsymbol{Y}|\vartheta^{(0)}) \le \mathrm{Var}(\boldsymbol{Y}'\tilde{\mathbf{U}}\boldsymbol{Y}|\vartheta^{(0)})$ *for every symmetric matrix* $\tilde{\mathbf{U}}$ *satisfying* (i) *and* (ii).

Lemma 1.1. *The* $\vartheta^{(0)}$-LMVQUIE *from Definition* 1.1 *exists iff the class* $\mathcal{U}_g^{(I)} = \{\boldsymbol{Y}'\mathbf{U}\boldsymbol{Y}\colon \mathbf{U}' = \mathbf{U}, \mathbf{UX} = \mathbf{O}, \mathrm{Tr}(\mathbf{UV}_i) = f_i, i = 1, \ldots, p\}$ *is not empty; this occurs iff* $\boldsymbol{f} \in \mathcal{M}(\mathbf{K}^{(I)})$, *where* $\mathcal{M}(\mathbf{K}^{(I)})$ *is the column space of the matrix* $\mathbf{K}^{(I)}$. *The* (i, j)*th element of the matrix* $\mathbf{K}^{(I)}$ *is* $\{\mathbf{K}^{(I)}\}_{i,j} = \mathrm{Tr}(\mathbf{M_x V}_i \mathbf{M_x V}_j)$, *where* $\mathrm{Tr}(.)$ *means the trace and* $\mathbf{M_x} = \mathbf{I} - \mathbf{P_x}$, $\mathbf{P_x} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$.

Proof *see in* [7].

Lemma 1.2 *The* $\vartheta^{(0)}$-LMVQUIE *from Definition 1.1 is*

$$\widehat{g(\vartheta)} = \sum_{i=1}^{p} \lambda_i \boldsymbol{Y}'(\mathbf{M_x}\Sigma_0\mathbf{M_x})^+ \mathbf{V}_i (\mathbf{M_x}\Sigma_0\mathbf{M_x})^+ \boldsymbol{Y}, \tag{1.1}$$

where $\Sigma_0 = \sum_{i=1}^{p} \vartheta_i^{(0)}\mathbf{V}_i$, $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_p)'$ *is a solution of the equation*

$$\mathbf{S}_{(\mathbf{M_x}\Sigma_0\mathbf{M_x})^+}\boldsymbol{\lambda} = \boldsymbol{f},$$

where

$$\{\mathbf{S}_{(\mathbf{M_x}\Sigma_0\mathbf{M_x})^+}\}_{i,j} = \mathrm{Tr}[(\mathbf{M_x}\Sigma_0\mathbf{M_x})^+\mathbf{V}_i(\mathbf{M_x}\Sigma_0\mathbf{M_x})^+\mathbf{V}_j],$$

$i, j = 1, \ldots, p$, *and* $(\mathbf{M_x}\Sigma_0\mathbf{M_x})^+$ *denotes the Moore—Penrose generalized inverse of the matrix* $\mathbf{M_x}\Sigma_0\mathbf{M_x}$ (*in detail see* [6]).

Proof see in [7].

Definition 1.2. *An estimator* $\boldsymbol{Y}'\mathbf{U}\boldsymbol{Y}$ *of a function* $g(\vartheta) = \boldsymbol{f}'\vartheta$, $\vartheta \in \underline{\vartheta}$, *in the model* (1) *is* $(\boldsymbol{\beta}^{(0)}, \vartheta^{(0)})$-LMVQUE (*locally minimum variance quadratic unbiased estimator) if*

(i) $\forall\{\vartheta \in \underline{\vartheta}\}\forall\{\boldsymbol{\beta} \in \mathcal{R}^k\}\, E(\boldsymbol{Y}'\mathbf{U}\boldsymbol{Y}|\boldsymbol{\beta}, \vartheta) = \boldsymbol{f}'\vartheta$,

(ii) $\mathrm{Var}(\boldsymbol{Y}'\mathbf{U}\boldsymbol{Y}|\boldsymbol{\beta}^{(0)}, \vartheta^{(0)}) \le \mathrm{Var}(\boldsymbol{Y}'\tilde{\mathbf{U}}\boldsymbol{Y}|\boldsymbol{\beta}^{(0)}, \vartheta^{(0)})$ *for every symmetric matrix* $\tilde{\mathbf{U}}$ *satisfying* (i).

Lemma 1.3. *The* $(\boldsymbol{\beta}^{(0)}, \vartheta^{(0)})$-LMVQUE *from Definition 1.2 exists iff the class* $\mathcal{U}_g = \{\boldsymbol{Y}'\mathbf{U}\boldsymbol{Y}\colon \mathbf{U}' = \mathbf{U}, \mathrm{Tr}(\mathbf{UV}_i) = f_i, i = 1, \ldots p, \mathbf{X}'\mathbf{UX} = \mathbf{O}\}$ *is not empty: this occurs iff* $\boldsymbol{f} \in M(\mathbf{K}^*)$, *where* $\{\mathbf{K}^*\}_{i,j} = \mathrm{Tr}(\mathbf{V}_i\mathbf{V}_j - \mathbf{P_x V}_i \mathbf{P_x V}_j)$, $i, j = 1, \ldots, p$.

Proof see in [7].

Lemma 1.4. *The* $(\boldsymbol{\beta}^{(0)}, \vartheta^{(0)})$-LMVQUE *from Definition* 1.2 *is*

$$\widehat{g(\vartheta)} = \sum_{i=1}^{p} \lambda_i \boldsymbol{Y}'(\Sigma_0 + \mathbf{X}\boldsymbol{\beta}^{(0)}\boldsymbol{\beta}^{(0)\prime}\mathbf{X}')^{-1}(\mathbf{V}_i - \mathbf{P}_{\mathbf{x}}^{\Sigma_0^{-1}}\mathbf{V}_i\mathbf{P}_{\mathbf{x}}^{\Sigma_0^{-1}\prime})(\Sigma_0 + \mathbf{X}\boldsymbol{\beta}^{(0)}\boldsymbol{\beta}^{(0)\prime}\mathbf{X}')^{-1}\boldsymbol{Y},$$

where $\mathbf{P}_{\mathbf{x}}^{\Sigma_0^{-1}} = \mathbf{X}(\mathbf{X}'\Sigma_0^{-1}\mathbf{X})^{-1}\mathbf{X}'\Sigma_0^{-1}$ *and the vector* $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_p)'$ *is a solution of the equation*

$$(\mathbf{S}_{(\Sigma_0 + \mathbf{X}\boldsymbol{\beta}^{(0)}\boldsymbol{\beta}^{(0)\prime}\mathbf{X}')^{-1}} - \mathbf{S}_{\mathbf{P}_{\mathbf{x}}^{\Sigma_0^{-1}\prime}(\Sigma_0 + \mathbf{X}\boldsymbol{\beta}^{(0)}\boldsymbol{\beta}^{(0)\prime}\mathbf{X}')^{-1}\mathbf{P}_{\mathbf{x}}^{\Sigma_0^{-1}}})\boldsymbol{\lambda} = \boldsymbol{f}$$

$(\{\mathbf{S}_R\}_{i,j} = \mathrm{Tr}(\mathbf{R}\mathbf{V}_i\mathbf{R}\mathbf{V}_j)$, $\mathbf{R} = (\Sigma_0 + \mathbf{X}\boldsymbol{\beta}^{(0)}\boldsymbol{\beta}^{(0)\prime}\mathbf{X}')^{-1}$ *and*

$\mathbf{P}_{\mathbf{X}}^{\Sigma_0^{-1}\prime}(\Sigma_0 + \mathbf{X}\boldsymbol{\beta}^{(0)}\boldsymbol{\beta}^{(0)\prime}\mathbf{X}')^{-1}\mathbf{P}_{\mathbf{X}}^{\Sigma_0^{-1}}$, *respectively*).

Proof. The dispersion of a random a variable $\boldsymbol{Y}'\mathbf{U}\boldsymbol{Y}$ for given $\boldsymbol{\beta}^{(0)}$ and Σ_0 is $\mathrm{Var}(\boldsymbol{Y}'\mathbf{U}\boldsymbol{Y}|\boldsymbol{\beta}^{(0)}, \boldsymbol{\vartheta}^{(0)}) = 2\,\mathrm{Tr}(\mathbf{U}\Sigma_0\mathbf{U}\Sigma_0) + 4\boldsymbol{\beta}^{(0)\prime}\mathbf{X}'\mathbf{U}\Sigma_0\mathbf{U}\mathbf{X}\boldsymbol{\beta}^{(0)}$. This quantity has to be minimized by the proper choice of a symmetric matrix $\mathbf{U}$ when simultaneously the conditions of unbiasedness $\mathbf{X}'\mathbf{U}\mathbf{X} = \mathbf{O}$, $\mathrm{Tr}(\mathbf{U}\mathbf{V}_i) = f_i$, $i = 1, \ldots, p$, have to be satisfied. This problem can be solved in a standard way using the Lagrange method of indefinite multipliers; the auxiliary Lagrange function is

$$\boldsymbol{\Phi}(\mathbf{U}) = \mathrm{Tr}(\Sigma_0\mathbf{U}\Sigma_0\mathbf{U}) + 2\boldsymbol{\beta}^{(0)\prime}\mathbf{X}'\mathbf{U}\Sigma_0\mathbf{U}\mathbf{X}\boldsymbol{\beta}^{(0)} - 2\sum_{i=1}^{p}\lambda_i[\mathrm{Tr}(\mathbf{U}\mathbf{V}_i) - f_i] -$$

$$- \mathrm{Tr}(\boldsymbol{\varkappa}'\mathbf{X}'\mathbf{U}\mathbf{X}),$$

where $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_p)'$ is a vector and $\boldsymbol{\varkappa}$ a matrix of the Lagrange multipliers.

$$\partial\boldsymbol{\Phi}(\mathbf{U})/\partial\mathbf{U} = \mathbf{O} \Leftrightarrow \Sigma_0\mathbf{U}\Sigma_0 + \Sigma_0\mathbf{U}\mathbf{X}\boldsymbol{\beta}^{(0)}\boldsymbol{\beta}^{(0)\prime}\mathbf{X}' + \mathbf{X}\boldsymbol{\beta}^{(0)}\boldsymbol{\beta}^{(0)\prime}\mathbf{X}'\mathbf{U}\Sigma_0 =$$

$$= \sum_{i=1}^{p}\lambda_i\mathbf{V}_i + \mathbf{X}[(\boldsymbol{\varkappa} + \boldsymbol{\varkappa}')/4]\mathbf{X}'.$$

By multiplying the last equation from l.h.s. by $\mathbf{X}'\Sigma_0^{-1}$ and from r.h.s. by $\Sigma_0^{-1}\mathbf{X}$ taking simultaneously into account that $\mathbf{X}'\mathbf{U}\mathbf{X} = \mathbf{O}$, we obtain

$$\mathbf{X}[(\boldsymbol{\varkappa} + \boldsymbol{\varkappa}')/4]\mathbf{X}' = -\sum_{i=1}^{p}\lambda_i\mathbf{P}_{\mathbf{X}}^{\Sigma_0^{-1}}\mathbf{V}_i\mathbf{P}_{\mathbf{X}}^{\Sigma_0^{-1}\prime}.$$

Thus

$$\Sigma_0\mathbf{U}\Sigma_0 + \Sigma_0\mathbf{U}\mathbf{X}\boldsymbol{\beta}^{(0)}\boldsymbol{\beta}^{(0)\prime}\mathbf{X}' + \mathbf{X}\boldsymbol{\beta}^{(0)}\boldsymbol{\beta}^{(0)\prime}\mathbf{X}'\mathbf{U}\Sigma_0 = \sum_{i=1}^{p}\lambda_i(\mathbf{V}_i - \mathbf{P}_{\mathbf{X}}^{\Sigma_0^{-1}}\mathbf{V}_i\mathbf{P}_{\mathbf{X}}^{\Sigma_0^{-1}\prime}).$$

If the zero term $\mathbf{X}\boldsymbol{\beta}^{(0)}\boldsymbol{\beta}^{(0)\prime}\mathbf{X}'\mathbf{U}\mathbf{X}\boldsymbol{\beta}^{(0)}\boldsymbol{\beta}^{(0)\prime}\mathbf{X}'$ is added to the l.h.s. of the last equation, then obviously

$$\mathbf{U} = \sum_{i=1}^{p}\lambda_i(\Sigma_0 + \mathbf{X}\boldsymbol{\beta}^{(0)}\boldsymbol{\beta}^{(0)\prime}\mathbf{X}')^{-1}(\mathbf{V}_i - \mathbf{P}_{\mathbf{X}}^{\Sigma_0^{-1}}\mathbf{V}_i\mathbf{P}_{\mathbf{X}}^{\Sigma_0^{-1}\prime})(\Sigma_0 + \mathbf{X}\boldsymbol{\beta}^{(0)}\boldsymbol{\beta}^{(0)\prime}\mathbf{X}')^{-1}.$$

Further it is to be proved that under the given conditions the matrix $\mathbf{U}$ minimizes $\mathrm{Tr}(\Sigma_0\mathbf{U}\Sigma_0\mathbf{U}) + 2\boldsymbol{\beta}^{(0)\prime}\mathbf{X}'\mathbf{U}\Sigma_0\mathbf{U}\mathbf{X}\boldsymbol{\beta}^{(0)}$. Let Δ be an $n \times n$ symmetric matrix possessing the properties $\mathbf{X}\Delta\mathbf{X}' = \mathbf{O}$ and $\mathrm{Tr}(\Delta\mathbf{V}_i) = 0$, $i = 1, \ldots, p$. Then

$$\mathrm{Tr}[\Sigma_0(\mathbf{U} + \Delta)\Sigma_0(\mathbf{U} + \Delta)] + 2\boldsymbol{\beta}^{(0)\prime}\mathbf{X}'(\mathbf{U} + \Delta)\Sigma_0(\mathbf{U} + \Delta)\mathbf{X}\boldsymbol{\beta}^{(0)} =$$

$$= \mathrm{Tr}(\mathbf{U}\Sigma_0\mathbf{U}\Sigma_0) + \mathrm{Tr}(\Delta\Sigma_0\Delta\Sigma_0) + 2\,\mathrm{Tr}(\Delta\Sigma_0\mathbf{U}\Sigma_0) + 2\boldsymbol{\beta}^{(0)\prime}\mathbf{X}'\mathbf{U}\Sigma_0\mathbf{U}\mathbf{X}\boldsymbol{\beta}^{(0)} +$$

$$+ 4\boldsymbol{\beta}^{(0)\prime}\mathbf{X}'\mathbf{U}\Sigma_0\Delta\mathbf{X}\boldsymbol{\beta}^{(0)} + 2\boldsymbol{\beta}^{(0)\prime}\mathbf{X}'\ \Delta\Sigma_0\Delta\mathbf{X}\boldsymbol{\beta}^{(0)}.$$

As

$$2\,\mathrm{Tr}\,(\Delta\Sigma_0\mathbf{U}\Sigma_0) + 4\boldsymbol{\beta}^{(0)\prime}\mathbf{X}'\mathbf{U}\Sigma_0\Delta\mathbf{X}\boldsymbol{\beta}^{(0)} =$$

$$= 2\,\mathrm{Tr}\left\{\sum_{i=1}^{p}\lambda_i\Delta\Sigma_0[\Sigma_0^{-1} - \Sigma_0^{-1}\mathbf{X}\boldsymbol{\beta}^{(0)}\boldsymbol{\beta}^{(0)\prime}\mathbf{X}'\Sigma_0^{-1}/(1+\boldsymbol{\beta}^{(0)\prime}\mathbf{X}'\Sigma_0^{-1}\mathbf{X}\boldsymbol{\beta}^{(0)})].\right.$$

$$\left.\cdot(\mathbf{V}_i - \mathbf{P}_{\mathbf{X}}^{\Sigma_0^{-1}}\mathbf{V}_i\mathbf{P}_{\mathbf{X}}^{\Sigma_0^{-1}\prime})[\Sigma_0^{-1} - \Sigma_0^{-1}\mathbf{X}\boldsymbol{\beta}^{(0)}\boldsymbol{\beta}^{(0)\prime}\mathbf{X}'\Sigma_0^{-1}/(1+\boldsymbol{\beta}^{(0)\prime}\mathbf{X}'\Sigma_0^{-1}\mathbf{X}\boldsymbol{\beta}^{(0)})]\Sigma_0\right\} +$$

$$+4\boldsymbol{\beta}^{(0)\prime}\mathbf{X}'\Delta\Sigma_0\sum_{i=1}^{p}\lambda_i[\Sigma_0^{-1} - \Sigma_0^{-1}\mathbf{X}\boldsymbol{\beta}^{(0)}\boldsymbol{\beta}^{(0)\prime}\mathbf{X}'\Sigma_0^{-1}/(1+\boldsymbol{\beta}^{(0)\prime}\mathbf{X}'\Sigma_0^{-1}\mathbf{X}\boldsymbol{\beta}^{(0)})].$$

$$.(\mathbf{V}_i - \mathbf{P}_{\mathbf{X}}^{\Sigma_0^{-1}}\mathbf{V}_i\mathbf{P}_{\mathbf{X}}^{\Sigma_0^{-1}\prime})[\Sigma_0^{-1} - \Sigma_0^{-1}\mathbf{X}\boldsymbol{\beta}^{(0)}\boldsymbol{\beta}^{(0)\prime}\mathbf{X}'\Sigma_0^{-1}/(1+\boldsymbol{\beta}^{(0)\prime}\mathbf{X}'\Sigma_0^{-1}\mathbf{X}\boldsymbol{\beta}^{(0)})]\mathbf{X}\boldsymbol{\beta}^{(0)} = 0$$

(here the equality $(\Sigma_0 + \mathbf{X}\boldsymbol{\beta}^{(0)}\boldsymbol{\beta}^{(0)\prime}\mathbf{X}')^{-1} = \Sigma_0^{-1} - \Sigma_0^{-1}\mathbf{X}\boldsymbol{\beta}^{(0)}\boldsymbol{\beta}^{(0)\prime}\mathbf{X}'\Sigma_0^{-1}/(1 + \boldsymbol{\beta}^{(0)\prime}\mathbf{X}'\Sigma_0^{-1}\mathbf{X}\boldsymbol{\beta}^{(0)})$ was used), then $\mathrm{Tr}\,[\Sigma_0(\mathbf{U}+\Delta)\Sigma_0(\mathbf{U}+\Delta)] + 2\boldsymbol{\beta}^{(0)\prime}\mathbf{X}'(\mathbf{U} + \Delta)\Sigma_0(\mathbf{U}+\Delta)\mathbf{X}\boldsymbol{\beta}^{(0)} \geqslant \mathrm{Tr}\,(\Sigma_0\mathbf{U}\Sigma_0\mathbf{U}) + 2\boldsymbol{\beta}^{(0)\prime}\mathbf{X}'\mathbf{U}\Sigma_0\mathbf{X}\boldsymbol{\beta}^{(0)}$, which proves the statement of the lemma.

Definition 1.3. *An estimator* $\boldsymbol{Y}'\mathbf{U}\boldsymbol{Y} + 2\boldsymbol{u}'\boldsymbol{Y}$ *of a function* $g(\boldsymbol{\vartheta}) = \boldsymbol{f}'\boldsymbol{\vartheta}$, $\boldsymbol{\vartheta}\in\underline{\boldsymbol{\vartheta}}$, *in the model* (1) *is* $(\boldsymbol{\beta}^{(0)}, \boldsymbol{\vartheta}^{(0)})$LMVLQUE (*locally minimum variance linear-quadratic unbiased estimator*) *if*

(i) $\forall\{\boldsymbol{\beta}\in\mathscr{R}^k\}\forall\{\boldsymbol{\vartheta}\in\underline{\boldsymbol{\vartheta}}\}\,E(\boldsymbol{Y}'\mathbf{U}\boldsymbol{Y} + 2\boldsymbol{u}'\boldsymbol{Y}|\boldsymbol{\beta},\boldsymbol{\vartheta}) = \boldsymbol{f}'\boldsymbol{\vartheta}$,

(ii) $\mathrm{Var}(\boldsymbol{Y}'\mathbf{U}\boldsymbol{Y} + 2\boldsymbol{u}'\boldsymbol{Y}|\boldsymbol{\beta}^{(0)}, \boldsymbol{\vartheta}^{(0)}) \leq \mathrm{Var}(\boldsymbol{Y}'\tilde{\mathbf{U}}\boldsymbol{Y} + 2\tilde{\boldsymbol{u}}'\boldsymbol{Y}|\boldsymbol{\beta}^{(0)}, \boldsymbol{\vartheta}^{(0)})$

for every matrix $\tilde{\mathbf{U}}$ *and vector* $\tilde{\boldsymbol{u}}$ *satisfying* (i).

Lemma 1.5 *The* $(\boldsymbol{\beta}^{(0)}, \boldsymbol{\vartheta}^{(0)})$-LMVLQUE *from Definition 1.3 exists iff the class* $\mathscr{U}_g$ *from Lemma 1.3 is not empty; it has the form*

$$\widehat{g(\boldsymbol{\vartheta})} = \sum_{i=1}^{p}\lambda_i(\boldsymbol{Y} - \mathbf{X}\boldsymbol{\beta}^{(0)})'(\Sigma_0^{-1}\mathbf{V}_i\Sigma_0^{-1} - \Sigma_0^{-1}\mathbf{P}_{\mathbf{x}}^{\Sigma_0^{-1}}\mathbf{V}_i\mathbf{P}_{\mathbf{x}}^{\Sigma_0^{-1}\prime}\Sigma_0^{-1})(\boldsymbol{Y} - \mathbf{X}\boldsymbol{\beta}^{(0)}),$$

where the vector $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_p)'$ *is a solution of the equation*

$$(\mathbf{S}_{\Sigma_0^{-1}} - \mathbf{S}_{\Sigma_0^{-1}\mathbf{P}_{\mathbf{x}}^{\Sigma_0^{-1}}})\boldsymbol{\lambda} = \boldsymbol{f} \tag{2.1}$$

(*notations* $\mathbf{S}_{\Sigma_0^{-1}}$ *and* $\mathbf{S}_{\Sigma_0^{-1}\mathbf{P}_{\mathbf{x}}^{\Sigma_0^{-1}}}$ *have a meaning analogous to that in Lemma 1.4*).

Proof see in [7].

Definition 1.4. *If the vector* $\boldsymbol{\beta}^{(0)}$ *in the* $(\boldsymbol{\beta}^{(0)}, \boldsymbol{\vartheta}^{(0)})$-LMVLQUE *from Definition 1.3 is replaced by the vector* $\tilde{\boldsymbol{\beta}} = (\mathbf{X}'\Sigma_0^{-1}\mathbf{X})^{-1}\mathbf{X}'\Sigma_0^{-1}\boldsymbol{Y}$ ([3]), *then the new estimator is called* $\boldsymbol{\vartheta}^{(0)}$-mLMVQE (*modified locally minimum variance quadratic estimator*) *of the function* $g(.)$ *from Definition 1.3.*

Lemma 1.6. *The* $\boldsymbol{\vartheta}^{(0)}$-mLMVQE *from Definition 1.4 is*

$$\tau_g(\boldsymbol{Y}) = \sum_{i=1}^{p}\lambda_i\boldsymbol{Y}'(\mathbf{M}_{\mathbf{x}}\Sigma_0\mathbf{M}_{\mathbf{x}})^{+}\mathbf{V}_i(\mathbf{M}_{\mathbf{x}}\Sigma_0\mathbf{M}_{\mathbf{x}})^{+}\boldsymbol{Y},$$

where the vector $\lambda = (\lambda_1, \ldots, \lambda_p)'$ *is a solution of the equation (1.2). The bias of the estimator* $\tau_g(\boldsymbol{Y})$ *is*

$$b(\boldsymbol{\vartheta}) = E[\tau_g(\boldsymbol{Y})|\boldsymbol{\vartheta}] - \boldsymbol{f}'\boldsymbol{\vartheta} = -2\lambda'\{\mathrm{Tr}[(\mathbf{M}_x\Sigma_0\mathbf{M}_x)^+\mathbf{V}_1\Sigma_0^{-1}\mathbf{P}_X^{\Sigma_0^{-1}}\Sigma], \ldots$$

$$\ldots, \mathrm{Tr}[(\mathbf{M}_x\Sigma_0\mathbf{M}_x)^+\mathbf{V}_p\Sigma_0^{-1}\mathbf{P}_X^{\Sigma_0^{-1}}\Sigma]\}';$$

if $\boldsymbol{\vartheta} = \boldsymbol{\vartheta}^{(0)}$, *then* $b(\boldsymbol{\vartheta}^{(0)}) = 0$.

Proof. See [3].

Lemma 1.7. *If a transformation matrix* $\mathbf{T}$ *for the model* (1) *has a form* $\mathbf{T} = \mathbf{I} - \mathbf{SC}$, $\mathbf{TS} = \mathbf{O}$, $\mathbf{TA} = \mathbf{A}$, *then it is optimum with respect to* $\boldsymbol{\Theta}$.

Proof. Cf. Corollary 2.4 in [5].

Lemma 1.8. *Let* $\mathbf{V}$ *be a symmetric positive definite (p.d.)* $n \times n$ *matrix and the rank of the* $n \times k$ *matrix* $\mathbf{X} = (\mathbf{A}, \mathbf{S})$ *be* $R(\mathbf{X}) = k = k_1 + k_2$, *where* $R(\mathbf{A}) = k_1$, $R(\mathbf{S}) = k_2$. Let $\mathbf{L} = \mathbf{V} - \mathbf{VS}(\mathbf{S}'\mathbf{VS})^{-1}\mathbf{S}'\mathbf{V}$ and $\mathbf{K} = \mathbf{V} - \mathbf{VA}(\mathbf{A}'\mathbf{VA})^{-1}\mathbf{A}'\mathbf{V}$. If $\mathbf{P}_x^{\mathbf{V}} = \mathbf{X}(\mathbf{X}'\mathbf{VX})^{-1}\mathbf{X}'\mathbf{V}$, $\mathbf{P}_{\mathbf{A}}^{\mathbf{L}} = \mathbf{A}(\mathbf{A}'\mathbf{LA})^{-1}\mathbf{A}'\mathbf{L}$, $\mathbf{P}_{\mathbf{S}}^{\mathbf{K}} = \mathbf{S}(\mathbf{S}'\mathbf{KS})^{-1}\mathbf{S}'\mathbf{K}$, $\mathbf{M}_x^{\mathbf{V}} = \mathbf{I} - \mathbf{P}_{\mathbf{X}}^{\mathbf{V}}$, $\mathbf{M}_{\mathbf{A}}^{\mathbf{L}} = \mathbf{I} - \mathbf{P}_{\mathbf{A}}^{\mathbf{L}}$, $\mathbf{M}_{\mathbf{S}}^{\mathbf{K}} = \mathbf{I} - \mathbf{P}_{\mathbf{S}}^{\mathbf{K}}$, then $\mathbf{P}_x^{\mathbf{V}} = \mathbf{P}_{\mathbf{A}}^{\mathbf{L}} + \mathbf{P}_{\mathbf{S}}^{\mathbf{K}}$, $\mathbf{P}_{\mathbf{A}}^{\mathbf{L}}\mathbf{P}_{\mathbf{S}}^{\mathbf{K}} = \mathbf{P}_{\mathbf{S}}^{\mathbf{K}}\mathbf{P}_{\mathbf{A}}^{\mathbf{L}} = \mathbf{O}$, $\mathbf{M}_x^{\mathbf{V}} = \mathbf{M}_{\mathbf{A}}^{\mathbf{L}}\mathbf{M}_{\mathbf{S}}^{\mathbf{K}} = \mathbf{M}_{\mathbf{S}}^{\mathbf{K}}\mathbf{M}_{\mathbf{A}}^{\mathbf{L}}$.

Proof. The assertion is a consequence of Theorem 2.5 [5].

Lemma 1.9. *A matrix equation* $\mathbf{AXB} = \mathbf{C}$ *with an unknown matrix* $\mathbf{X}$ *has a solution iff* $\mathbf{AA}^-\mathbf{CB}^-\mathbf{B} = \mathbf{C}$. *The class of solutions is* $\{\mathbf{A}^-\mathbf{CB}^- + \mathbf{Z} - \mathbf{A}^-\mathbf{AZBB}^-: \mathbf{Z}$ *arbitrary*$\}$ ($\mathbf{A}^-$ *denotes a generalized inverse of the matrix* $\mathbf{A}$).

Proof. See Theorem 2.3.2 [6].

Remark 1.1. The class $\mathscr{U}_g^{(I)}$ from Lemma 1.1 is the class of all quadratic unbiased and invariant estimators of the function $g(.)$ from Definition 1.1 in the model (1) (in detail cf. [7]). If $\mathscr{M}_{n,n}$ is a space of all $n \times n$ matrices with an inner product $\langle \mathbf{A}, \mathbf{B} \rangle = \mathrm{Tr}(\mathbf{A}'\mathbf{B})$, $\mathbf{A}, \mathbf{B} \in \mathscr{M}_{n,n}$, then $\mathbf{K}^{(I)}$ from Lemma 1.1 is the Gram matrix of the p-tuple $\{\mathbf{M}_x\mathbf{V}_i\mathbf{M}_x\}_{i=1}^p$.

Remark 1.2. So far the matrix Σ has been assumed to be regular, which implies $\mathscr{M}(\mathbf{X}) \subset \mathscr{M}(\Sigma)$. If $\mathscr{M}(\mathbf{X}) \subset \mathscr{M}(\Sigma)$ and Σ is not regular, then instead of Σ^{-1} the matrix Σ^+ has to be used in the preceding lemmas. Analogously instead of $(\Sigma + \mathbf{X}\boldsymbol{\beta\beta}'\mathbf{X}')^{-1}$ the matrix $(\Sigma + \mathbf{X}\boldsymbol{\beta\beta}'\mathbf{X}')^+$ has to be used.

2. Optimality with respect to Θ and ϑ

Lemma 2.1. *Let* $\mathbf{V}$, $\mathbf{K}$, $\mathbf{L}$, $\mathbf{M}_{\mathbf{A}}^{\mathbf{L}}$, $\mathbf{M}_{\mathbf{S}}^{\mathbf{K}}$ *be matrices from Lemma* 1.8. *Let* $\mathbf{K}^{(I)}$ *be the matrix from Lemma* 1.1 *and* $\mathscr{M}_{n,n}$ *the space from Remark* 1.1. *Then* $\mathscr{M}(\mathbf{K}^{(I)}) = \mathscr{M}(\mathbf{G}_1)(= \mathscr{M}(\mathbf{G}_2))$, *where* $\mathbf{G}_1$ *is the Gram matrix of the p-tuple* $\{\mathbf{M}_x^{\mathbf{V}}\mathbf{V}_i\mathbf{M}_x^{\mathbf{V}\prime}\}_{i=1}^p$ ($\mathbf{V}_1, \ldots, \mathbf{V}_p$ *are matrices from* (1) *and* $\mathbf{G}_2$ *is the Gram matrix of the p-tuple* $\{\mathbf{M}_{\mathbf{A}}^{\mathbf{L}}\mathbf{M}_{\mathbf{S}}^{\mathbf{K}}\mathbf{V}_i\mathbf{M}_{\mathbf{S}}^{\mathbf{K}\prime}\mathbf{M}_{\mathbf{A}}^{\mathbf{L}\prime}\}_{i=1}^p$).

Proof. Considering Lemma 1.9 an equation $\mathbf{UX} = \mathbf{O}$ with an unknown matrix $\mathbf{U}$ has a solution of the form $\mathbf{U} = \mathbf{Z} - \mathbf{ZXX}^-$, where $\mathbf{X}^-$ is an arbitrary fixed g-inverse of the matrix $\mathbf{X}$ and $\mathbf{Z}$ is an arbitrary $n \times n$ matrix. As $\mathbf{U} = \mathbf{U}'$, $\mathbf{Z}(\mathbf{I} - \mathbf{XX}^-) = (\mathbf{I} - \mathbf{XX}^-)'\mathbf{Z}' = (\mathbf{I} - \mathbf{XX}^-)'\mathbf{Z}'(\mathbf{I} - \mathbf{XX}^-)$ (the matrix $\mathbf{I} - \mathbf{XX}^-$ is idempotent) $= (\mathbf{I} - \mathbf{XX}^-)'\mathbf{Z}(\mathbf{I} - \mathbf{XX}^-) = (\mathbf{I} - \mathbf{XX}^-)'(1/2)(\mathbf{Z} + \mathbf{Z}')(\mathbf{I} - \mathbf{XX}^-)$. Thus the solution is $\mathbf{U} = (\mathbf{I} - \mathbf{XX}^-)'\mathbf{S}(\mathbf{I} - \mathbf{XX}^-)$, where $\mathbf{S}$ is an arbitrary symmetric matrix. If $\mathbf{X}^+$ is chosen for $\mathbf{X}^-$, then $\mathbf{U} = \mathbf{M}_x\mathbf{S}\mathbf{M}_x$; if $\mathbf{X}^- = [(\mathbf{X}')^-_{m(\mathbf{V}^{-1})}]' =$ $= (\mathbf{X}'\mathbf{VX})^{-1}\mathbf{X}'\mathbf{V}$, then $\mathbf{U} = \mathbf{M}_x^{\mathbf{V}\prime}\mathbf{S}\mathbf{M}_x^{\mathbf{V}}$. Thus the class of matrices $\mathbf{U}$ with properties $\mathbf{U} = \mathbf{U}'$, $\mathbf{UX} = \mathbf{O}$, $\mathrm{Tr}(\mathbf{UV}_i) = f_i$, $i = 1, \ldots, p$, is $\{\mathbf{M}_x^{\mathbf{V}\prime}\mathbf{S}\mathbf{M}_x^{\mathbf{V}}: \mathbf{S} = \mathbf{S}', \mathrm{Tr}(\mathbf{S}\mathbf{M}_x^{\mathbf{V}}\mathbf{V}_i\mathbf{M}_x^{\mathbf{V}\prime}) = f_i, i = 1, \ldots, p\}$ and this class can be rewritten in the form $\{\mathbf{M}_x\mathbf{S}\mathbf{M}_x: \mathbf{S} = \mathbf{S}', \mathrm{Tr}(\mathbf{S}\mathbf{M}_x\mathbf{V}_i\mathbf{M}_x) = f_i, i = 1, \ldots, p\}$. A matrix $\mathbf{S}_1 = \mathbf{S}_1'$ with a property $\mathrm{Tr}(\mathbf{S}_1\mathbf{M}_x^{\mathbf{V}}\mathbf{V}_i\mathbf{M}_x^{\mathbf{V}\prime}) = f_i$, $i = 1, \ldots, p$, exists iff there exists a matrix $\mathbf{S}_2 = \mathbf{S}_2'$ with a property $\mathrm{Tr}(\mathbf{S}_2\mathbf{M}_x\mathbf{V}_i\mathbf{M}_x) = f_i$, $i = 1, \ldots, p$; thus $\boldsymbol{f} \in \mathcal{M}(\mathbf{K}^{(I)})$ iff $\boldsymbol{f} \in \mathcal{M}(\mathbf{G}_1)$. The equality $\mathbf{G}_1 = \mathbf{G}_2$ is an obvious consequence of Lemma 1.8.

Theorem 2.1. *Let* $\mathbf{V}$ *be an arbitrary symmetric p.d. matrix of the type* $n \times n$ *and* $\mathbf{K} = \mathbf{V} - \mathbf{VA}(\mathbf{A}'\mathbf{VA})^{-1}\mathbf{A}'\mathbf{V}$, *where* $\mathbf{A}$ *is a matrix from the model* (1). *Then* $\mathbf{T} = \mathbf{M}_S^{\mathbf{K}} = \mathbf{I} - \mathbf{S}(\mathbf{S}'\mathbf{KS})^{-1}\mathbf{S}'\mathbf{K}$ *is optimum with respect to* Θ *and the model* $(\mathbf{M}_S^{\mathbf{K}}\boldsymbol{Y}; \mathbf{A}\Theta, \sum_{i=1}^{p}\vartheta_i\mathbf{M}_S^{\mathbf{K}}\mathbf{V}_i\mathbf{M}_S^{\mathbf{K}\prime})$ *enable us to construct the quadratic unbiased and invariant estimator of each function* $g(\boldsymbol{\vartheta}) = \boldsymbol{f}'\boldsymbol{\vartheta}$, $\boldsymbol{\vartheta} \in \underline{\boldsymbol{\vartheta}}$, *such that a quadratic unbiased and invariant estimator of it can be obtined in the model* (1).

Proof. According to Lemma 1.7 $\mathbf{M}_S^{\mathbf{K}} = \mathbf{I} - \mathbf{S}(\mathbf{S}'\mathbf{KS})^{-1}\mathbf{S}'\mathbf{K}$ is optimum with respect to Θ. Regarding Lemma 1.1 and Remark 1.1 a function $f(\boldsymbol{\vartheta}) = \boldsymbol{f}'\boldsymbol{\vartheta}$, $\boldsymbol{\vartheta} \in \underline{\boldsymbol{\vartheta}}$, is unbiasedly and invariantly estimable iff $\boldsymbol{f} \in \mathcal{M}(\mathbf{G}_1)$, where $\mathbf{G}_1$ is the Gram matrix of the p-tuple $\{\mathbf{M}_X^{\mathbf{V}}\mathbf{V}_i\mathbf{M}_X^{\mathbf{V}\prime}\}_{i=1}^{p}$. In the model after transformation the analogous matrix is given by the p-tuple $\{\mathbf{M}_A^{\mathbf{L}}(\mathbf{M}_S^{\mathbf{K}}\mathbf{V}_i\mathbf{M}_S^{\mathbf{K}\prime})\mathbf{M}_A^{\mathbf{L}\prime}\}_{i=1}^{p}$, which is the matrix $\mathbf{G}_2$ from Lemma 2.1. By the equality $\mathcal{M}(\mathbf{G}_1) = \mathcal{M}(\mathbf{G}_2)$ (c.f. Lemma 2.1) the proof is completed.

The problem is if the LMVQUIEs from the model $(\mathbf{M}_S^{\mathbf{K}}\boldsymbol{Y}, \mathbf{A}\Theta, \sum_{i=1}^{p}\vartheta_i\mathbf{M}_S^{\mathbf{K}}\mathbf{V}_i\mathbf{M}_S^{\mathbf{K}\prime})$ are the same as the LMVQUIEs from the model (1).

Theorem 2.2. *The* $\boldsymbol{\vartheta}^{(0)}$-LMVQUIE *of the function* $g(\boldsymbol{\vartheta}) = \boldsymbol{f}'\boldsymbol{\vartheta}$, $\boldsymbol{\vartheta} \in \underline{\boldsymbol{\vartheta}}$, *where* $\boldsymbol{f} \in \mathcal{M}(\mathbf{K}^{(I)})$, *in the model* (1) *is identical with the* $\boldsymbol{\vartheta}^{(0)}$-LMVQUIE *of the same function in the model* $\left(\mathbf{M}_S^{\mathbf{K}}\boldsymbol{Y}, \mathbf{A}\Theta, \sum_{i=1}^{p}\vartheta_i\mathbf{M}_S^{\mathbf{K}}\mathbf{V}_i\mathbf{M}_S^{\mathbf{K}\prime}\right)$.

Proof. With respect to Lemma 1.2 the $\boldsymbol{\vartheta}^{(0)}$-LMVQUIE in the model (1) is given by (1.1). This expression can be rewritten in the form

$$\sum_{i=1}^{p}\lambda_i\boldsymbol{Y}'\mathbf{M}_X^{\mathbf{V}\prime}(\mathbf{M}_X^{\mathbf{V}}\Sigma_0\mathbf{M}_X^{\mathbf{V}\prime})^+\mathbf{M}_X^{\mathbf{V}}\mathbf{V}_i\mathbf{M}_X^{\mathbf{V}\prime}(\mathbf{M}_X^{\mathbf{V}}\Sigma_0\mathbf{M}_X^{\mathbf{V}\prime})^+\mathbf{M}_X^{\mathbf{V}}\boldsymbol{Y}. \qquad (2.1)$$

It results from the following. The matrix $\mathbf{U}$ in the estimator $\boldsymbol{Y}'\mathbf{U}\boldsymbol{Y}$, where $\mathbf{UX} = \mathbf{O}$ and $\mathrm{Tr}(\mathbf{UV}_j) = f_j, j = 1, \ldots, p$, can be expressed as $\mathbf{U} = \mathbf{M}_{\mathbf{X}}^{\mathbf{V}\prime}\mathbf{S}\mathbf{M}_{\mathbf{X}}^{\mathbf{V}}$ (cf. Lemma 2.1 and its proof). Let us minimize the quantity $\mathrm{Tr}(\mathbf{U}\,\Sigma_0\mathbf{U}\,\Sigma_0)$ under the side conditions $\mathrm{Tr}(\mathbf{UV}_j) = f_j, j = 1, \ldots, p$, where $\mathbf{U} = \mathbf{M}_{\mathbf{X}}^{\mathbf{V}\prime}\mathbf{S}\mathbf{M}_{\mathbf{X}}^{\mathbf{V}}$. The auxiliary Lagrange function is

$$\boldsymbol{\Phi}(\mathbf{S}) = \mathrm{Tr}(\mathbf{S}\mathbf{M}_{\mathbf{X}}^{\mathbf{V}}\Sigma_0\mathbf{M}_{\mathbf{X}}^{\mathbf{V}\prime}\mathbf{S}\mathbf{M}_{\mathbf{X}}^{\mathbf{V}}\Sigma_0\mathbf{M}_{\mathbf{X}}^{\mathbf{V}\prime}) - 2\sum_{j=1}^{p}\lambda_j[\mathrm{Tr}(\mathbf{S}\mathbf{M}_{\mathbf{X}}^{\mathbf{V}}\mathbf{V}_j\mathbf{M}_{\mathbf{X}}^{\mathbf{V}\prime}) - f_j];$$

$$\partial\boldsymbol{\Phi}(\mathbf{S})/\partial\mathbf{S} = \mathbf{O} \Leftrightarrow \mathbf{M}_{\mathbf{X}}^{\mathbf{V}}\Sigma_0\mathbf{M}_{\mathbf{X}}^{\mathbf{V}\prime}\mathbf{S}\mathbf{M}_{\mathbf{X}}^{\mathbf{V}}\Sigma_0\mathbf{M}_{\mathbf{X}}^{\mathbf{V}\prime} = \sum_{j=1}^{p}\lambda_j\mathbf{M}_{\mathbf{X}}^{\mathbf{V}}\mathbf{V}_j\mathbf{M}_{\mathbf{X}}^{\mathbf{V}\prime} \Rightarrow$$

$$\mathbf{S} = \sum_{j=1}^{p}\lambda_j(\mathbf{M}_{\mathbf{X}}^{\mathbf{V}}\Sigma_0\mathbf{M}_{\mathbf{X}}^{\mathbf{V}\prime})^{+}\mathbf{M}_{\mathbf{X}}^{\mathbf{V}}\mathbf{V}_j\mathbf{M}_{\mathbf{X}}^{\mathbf{V}\prime}(\mathbf{M}_{\mathbf{X}}^{\mathbf{V}}\Sigma_0\mathbf{M}_{\mathbf{X}}^{\mathbf{V}\prime})^{+} \Rightarrow$$

$$\mathbf{U} = \mathbf{M}_{\mathbf{X}}^{\mathbf{V}\prime}\mathbf{S}\mathbf{M}_{\mathbf{X}}^{\mathbf{V}} = \sum_{j=1}^{p}\lambda_j\mathbf{M}_{\mathbf{X}}^{\mathbf{V}\prime}(\mathbf{M}_{\mathbf{X}}^{\mathbf{V}}\Sigma_0\mathbf{M}_{\mathbf{X}}^{\mathbf{V}\prime})^{+}\mathbf{M}_{\mathbf{X}}^{\mathbf{V}}\mathbf{V}_j\mathbf{M}_{\mathbf{X}}^{\mathbf{V}\prime}(\mathbf{M}_{\mathbf{X}}^{\mathbf{V}}\Sigma_0\mathbf{M}_{\mathbf{X}}^{\mathbf{V}\prime})^{+}\mathbf{M}_{\mathbf{X}}^{\mathbf{V}}.$$

Now it is easy to see that (1.1) and (2.1) are identical. After taking into acount the relationship $\mathbf{M}_{\mathbf{X}}^{\mathbf{V}} = \mathbf{M}_{\mathbf{A}}^{\mathbf{L}}\mathbf{M}_{\mathbf{S}}^{\mathbf{K}}$ (cf. Lemma 1.8) (2.1) can be rewritten as

$$\sum_{i=1}^{p}\lambda_i(\mathbf{M}_{\mathbf{S}}^{\mathbf{K}}\boldsymbol{Y})'\mathbf{M}_{\mathbf{A}}^{\mathbf{L}\prime}[\mathbf{M}_{\mathbf{A}}^{\mathbf{L}}(\mathbf{M}_{\mathbf{S}}^{\mathbf{K}}\Sigma_0\mathbf{M}_{\mathbf{S}}^{\mathbf{K}\prime})\mathbf{M}_{\mathbf{A}}^{\mathbf{L}\prime}]^{+}\,.$$

$$.\,\mathbf{M}_{\mathbf{A}}^{\mathbf{L}}(\mathbf{M}_{\mathbf{S}}^{\mathbf{K}}\mathbf{V}_i\mathbf{M}_{\mathbf{S}}^{\mathbf{K}\prime})\mathbf{M}_{\mathbf{A}}^{\mathbf{L}\prime}[\mathbf{M}_{\mathbf{A}}^{\mathbf{L}}(\mathbf{M}_{\mathbf{S}}^{\mathbf{K}}\Sigma_0\mathbf{M}_{\mathbf{S}}^{\mathbf{K}\prime})\mathbf{M}_{\mathbf{A}}^{\mathbf{L}\prime}]^{+}\mathbf{M}_{\mathbf{A}}^{\mathbf{L}}(\mathbf{M}_{\mathbf{S}}^{\mathbf{K}}\boldsymbol{Y}),$$

which is the $\boldsymbol{\vartheta}^{(0)}$-LMVQUIE in the model $(\mathbf{M}_{\mathbf{S}}^{\mathbf{K}}\boldsymbol{Y}, \mathbf{A}\boldsymbol{\Theta}, \sum_{i=1}^{p}\vartheta_i\mathbf{M}_{\mathbf{S}}^{\mathbf{K}}\mathbf{V}_i\mathbf{M}_{\mathbf{S}}^{\mathbf{K}\prime})$ after elimination transformation by the matrix $\mathbf{M}_{\mathbf{S}}^{\mathbf{K}}$.

Remark 2.1. Let $\mathbf{T} = \mathbf{I} - \mathbf{SC}$ and the matrices $\mathbf{X}$ and Σ_0 are at our disposal except the vector $\mathbf{T}\boldsymbol{Y}$. With respect to the invariance (Definition 1.1 (ii)) of the estimator (1.1) it is obvious that the estimator

$$\sum_{i=1}^{p}\lambda_i(\mathbf{T}\boldsymbol{Y})'(\mathbf{M}_{\mathbf{x}}\Sigma_0\mathbf{M}_{\mathbf{x}})^{+}\mathbf{V}_i(\mathbf{M}_{\mathbf{x}}\Sigma_0\mathbf{M}_{\mathbf{x}})^{+}(\mathbf{T}\boldsymbol{Y})$$

is identical with (1.1).

Remark 2.2. It is easy to find an optimal with respect to the $\boldsymbol{\Theta}$ transformation $\mathbf{T}$ which is not optimal with respect to $\boldsymbol{\Theta}$. If $\mathbf{T} = \mathbf{P}_{\mathbf{A}}^{\mathbf{L}}$ (Lemma 1.8), then for $\mathbf{V} = \Sigma_0^{-1}$ the tranfsormation $\mathbf{T}$ is optimal with respect to $\boldsymbol{\Theta}$ (cf. Theorem 2.5 in [5]). However, the expression

$$(\mathbf{P}_{\mathbf{A}}^{\mathbf{L}}\boldsymbol{Y})'\mathbf{M}_{\mathbf{A}}^{\mathbf{L}\prime}[\mathbf{M}_{\mathbf{A}}^{\mathbf{L}}(\mathbf{P}_{\mathbf{A}}^{\mathbf{L}}\Sigma_0\mathbf{P}_{\mathbf{A}}^{\mathbf{L}\prime})\mathbf{M}_{\mathbf{A}}^{\mathbf{L}\prime}]^{+}\mathbf{M}_{\mathbf{A}}^{\mathbf{L}}(\mathbf{P}_{\mathbf{A}}^{\mathbf{L}}\mathbf{V}_i\mathbf{P}_{\mathbf{A}}^{\mathbf{L}\prime})\mathbf{M}_{\mathbf{A}}^{\mathbf{L}\prime}\,.$$

$$.\,[\mathbf{M}_{\mathbf{A}}^{\mathbf{L}}(\mathbf{P}_{\mathbf{A}}^{\mathbf{L}}\Sigma_0\mathbf{P}_{\mathbf{A}}^{\mathbf{L}\prime})\mathbf{M}_{\mathbf{A}}^{\mathbf{L}\prime}]^{+}\mathbf{M}_{\mathbf{A}}^{\mathbf{L}}(\mathbf{P}_{\mathbf{A}}^{\mathbf{L}}\boldsymbol{Y})$$

(analogous with (2.1) in the model $(\mathbf{P}_A^{\mathbf{L}}\boldsymbol{Y}, \mathbf{A}\boldsymbol{\Theta}, \sum_{i=1}^{p} \mathbf{P}_A^{\mathbf{L}}\mathbf{V}_i\mathbf{P}_A^{\mathbf{L}\prime}))$ is obviously zero. The matrix $\mathbf{G}_2$ (Lemma 2.1) in this transformed model is zero as well; thus no nonzero function $g(.)$ of the parameters $\vartheta_1, \ldots, \vartheta_p$ can be estimed in the transformed model.

Remark 2.3 If the identity matrix $\mathbf{I}$ is chosen for $\mathbf{V}$ in Theorem 2.2, then $\mathbf{L} = \mathbf{M_S}$, $\mathbf{K} = \mathbf{M_A}$ and the $\boldsymbol{\vartheta}^{(0)}$-LMVQUIE in the transformed model $(\mathbf{M}_S^{\mathbf{M}_A}\boldsymbol{Y}, \mathbf{A}\boldsymbol{\Theta}, \sum_{i=1}^{p} \vartheta_i \mathbf{M}_S^{\mathbf{M}_A}\mathbf{V}_i\mathbf{M}_S^{\mathbf{M}_A\prime})$ is

$$\sum_{i=1}^{p} \lambda_i (\mathbf{M}_S^{\mathbf{M}_A}\boldsymbol{Y})'\mathbf{M}_A^{\mathbf{M}_S\prime}[\mathbf{M}_A^{\mathbf{M}_S}(\mathbf{M}_S^{\mathbf{M}_A}\Sigma_0\mathbf{M}_S^{\mathbf{M}_A\prime})\mathbf{M}_A^{\mathbf{M}_S\prime}]^+ \mathbf{M}_A^{\mathbf{M}_S}(\mathbf{M}_S^{\mathbf{M}_A}\mathbf{V}_i\mathbf{M}_S^{\mathbf{M}_A\prime})\mathbf{M}_A^{\mathbf{M}_S\prime}.$$

$$.[\mathbf{M}_A^{\mathbf{M}_S}(\mathbf{M}_S^{\mathbf{M}_A}\Sigma_0\mathbf{M}_S^{\mathbf{M}_A\prime})\mathbf{M}_A^{\mathbf{M}_S\prime}]^+ \mathbf{M}_A^{\mathbf{M}_S}(\mathbf{M}_S^{\mathbf{M}_A}\boldsymbol{Y}).$$

If $\mathbf{V} = \Sigma_0^{-1}$, then $\mathbf{M}_A^{\mathbf{L}}\mathbf{M}_S^{\mathbf{K}}\boldsymbol{Y} = [\mathbf{I} - \mathbf{X}(\mathbf{X}'\Sigma_0^{-1}\mathbf{X})^{-1}\mathbf{X}'\Sigma_0^{-1}]\boldsymbol{Y} = \boldsymbol{v}$, where $\boldsymbol{v} = \boldsymbol{Y} - \mathbf{X}\hat{\boldsymbol{\beta}}$, $\hat{\boldsymbol{\beta}}$ being the $\boldsymbol{\vartheta}^{(0)}$-locally best linear estimator of the vector $\boldsymbol{\beta}$. Thus $\boldsymbol{\vartheta}^{(0)}$-LMVQUIE in the transformed model is

$$\sum_{i=1}^{p} \eta_i \boldsymbol{v}'[\mathbf{M}_A^{\mathbf{L}}(\mathbf{M}_S^{\mathbf{K}}\Sigma_0\mathbf{M}_S^{\mathbf{K}\prime})\mathbf{M}_A^{\mathbf{L}\prime}]^+ \mathbf{M}_A^{\mathbf{L}}(\mathbf{M}_S^{\mathbf{K}}\mathbf{V}_i\mathbf{M}_S^{\mathbf{K}\prime})\mathbf{M}_A^{\mathbf{L}\prime}[\mathbf{M}_A^{\mathbf{L}}(\mathbf{M}_S^{\mathbf{K}}\Sigma_0\mathbf{M}_S^{\mathbf{K}\prime})\mathbf{M}_A^{\mathbf{L}\prime}]^+ \boldsymbol{v},$$

where $\mathbf{L} = (\mathbf{M_S}\Sigma_0\mathbf{M_S})^+$, $\mathbf{K} = (\mathbf{M_A}\Sigma_0\mathbf{M_A})^+$. In this case $\mathbf{T}\boldsymbol{Y} = \boldsymbol{Y} - \mathbf{S}\hat{\boldsymbol{\varkappa}}$, where $\hat{\boldsymbol{\varkappa}}$ is the $\boldsymbol{\vartheta}^{(0)}$-locally best linear unbiased estimator of the nuisance parameters.

So far the case of the function $g(\boldsymbol{\vartheta}) = \boldsymbol{f}'\boldsymbol{\vartheta}$, $\boldsymbol{\vartheta} \in \underline{\boldsymbol{\vartheta}}$, where $\boldsymbol{f} \in \mathcal{M}(\mathbf{K}^{(I)})$, has been considered. If $\boldsymbol{f} \notin \mathcal{M}(\mathbf{K}^{(I)})$, no unbiased and simultaneously invariant estimator exists; however, an unbiased estimator can exist. This case occurs when $\boldsymbol{f} \in \mathcal{M}(\mathbf{K}^*)$ ($\mathbf{K}^*$ is the matrix from Lemma 1.3). The problem is if $\mathbf{T} = \mathbf{M}_S^{\mathbf{K}}$ will keep its optimal properties also in this case. Before answering this question let us give two lemmas and an example.

Lemma 2.2. *In the model* (1) *the dispersions of the* $\boldsymbol{\vartheta}^{(0)}$-LMVQUIE, $(\boldsymbol{\beta}^{(0)}, \boldsymbol{\vartheta}^{(0)})$-LMVQUIE *and* $(\boldsymbol{\beta}^{(0)}, \boldsymbol{\vartheta}^{(0)})$-LMVLQUE, *respectively, are*

$$\mathrm{Var}\left[\sum_{i=1}^{p} \lambda_i \boldsymbol{Y}'(\mathbf{M_X}\Sigma_0\mathbf{M_X})^+ \mathbf{V}_i(\mathbf{M_X}\Sigma_0\mathbf{M_X})^+ \boldsymbol{Y}|\boldsymbol{\vartheta}^{(0)}\right] = 2\boldsymbol{\lambda}'\mathbf{S}_{(\mathbf{M_X}\Sigma_0\mathbf{M_X})^+}\boldsymbol{\lambda},$$

$$(= 2\boldsymbol{f}'\mathbf{S}^{-1}_{(\mathbf{M_X}\Sigma_0\mathbf{M_X})^+}\boldsymbol{f}),$$

where $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_p)'$ *is the vector from Lemma* 1.2;

$$\mathrm{Var}\left[\sum_{i=1}^{p} \lambda_i \boldsymbol{Y}'(\Sigma_0 + \mathbf{X}\boldsymbol{\beta}^{(0)}\boldsymbol{\beta}^{(0)\prime}\mathbf{X}')^{-1}(\mathbf{V}_i - \mathbf{P}_X^{\Sigma_0^{-1}}\mathbf{V}_i\mathbf{P}_X^{\Sigma_0^{-1}\prime})(\Sigma_0 + \right.$$

$$\left. + \mathbf{X}\boldsymbol{\beta}^{(0)}\boldsymbol{\beta}^{(0)\prime}\mathbf{X}')^{-1}\boldsymbol{Y}|\boldsymbol{\beta}^{(0)}, \boldsymbol{\vartheta}^{(0)}\right] = 2\boldsymbol{\lambda}'(\mathbf{S}_{(\Sigma_0 + \mathbf{X}\boldsymbol{\beta}^{(0)}\boldsymbol{\beta}^{(0)\prime}\mathbf{X}')^{-1}} -$$

$$- \mathbf{S}_{\mathbf{P}_X^{\Sigma_0^{-1}\prime}(\Sigma_0 + \mathbf{X}\boldsymbol{\beta}^{(0)}\boldsymbol{\beta}^{(0)\prime}\mathbf{X}')^{-1}\mathbf{P}_X^{\Sigma_0^{-1}}})\boldsymbol{\lambda}$$

$$(= 2f'(\mathbf{S}_{(\Sigma_0 + \mathbf{X}\beta^{(0)}\beta^{(0)\prime}\mathbf{X}')^{-1}} - \mathbf{S}_{\mathbf{P}_x^{\Sigma_0^{-1}}{}'(\Sigma_0 + \mathbf{X}\beta^{(0)}\beta^{(0)\prime}\mathbf{X}')^{-1}\mathbf{P}_x^{\Sigma_0^{-1}}})^{-1} f),$$

where $\lambda = (\lambda'_1, \ldots, \lambda'_p)'$ *is the vector from Lemma* 1.4 *and*

$$\operatorname{Var}\left[\sum_{i=1}^{p} \lambda_i (Y - \mathbf{X}\beta^{(0)})'(\Sigma_0^{-1}\mathbf{V}_i\Sigma_0^{-1} - \Sigma_0^{-1}\mathbf{P}_x^{\Sigma_0^{-1}}\mathbf{V}_i\mathbf{P}_x^{\Sigma_0^{-1}\prime}\Sigma_0^{-1})(Y - \mathbf{X}\beta^{(0)})|\beta^{(0)}, \vartheta^{(0)}\right] =$$

$$= 2\lambda'(\mathbf{S}_{\Sigma_0^{-1}} - \mathbf{S}_{\Sigma_0^{-1}\mathbf{P}_x^{\Sigma_0^{-1}}})\lambda (= 2f'(\mathbf{S}_{\Sigma_0^{-1}} - \mathbf{S}_{\Sigma_0^{-1}\mathbf{P}_x^{\Sigma_0^{-1}}})^{+} f),$$

where $\lambda = (\lambda_1, \ldots, \lambda_p)'$ *is the vector from Lemma 1.5.*

Proof. The relationship $\operatorname{cov}[(Y'\mathbf{U}Y, b'Y)|\beta, \Sigma] = 2b'\Sigma\mathbf{U}\mathbf{X}\beta$, $\operatorname{cov}[(Y'\mathbf{U}_1Y, Y'\mathbf{U}_2Y)|\beta, \Sigma] = 2\operatorname{Tr}(\mathbf{U}_1\Sigma\mathbf{U}_2\Sigma) + 4\beta'\mathbf{X}'\mathbf{U}_1\Sigma\mathbf{U}_2\mathbf{X}\beta$ and the definition of the matrix $\mathbf{S}_{(\,)}$ have to be used. Thus the first statement is obvious. If

$$\mathbf{U}_i = (\Sigma_0 + \mathbf{X}\beta^{(0)}\beta^{(0)\prime}\mathbf{X}')^{-1}\mathbf{V}_i(\Sigma_0 + \mathbf{X}\beta^{(0)}\beta^{(0)\prime}\mathbf{X}')^{-1} - (\Sigma_0 +$$

$$+ \mathbf{X}\beta^{(0)}\beta^{(0)\prime}\mathbf{X}')^{-1}\mathbf{P}_x^{\Sigma_0^{-1}}\mathbf{V}_i\mathbf{P}_x^{\Sigma_0^{-1}\prime}(\Sigma_0 + \mathbf{X}\beta^{(0)}\beta^{(0)\prime}\mathbf{X}')^{-1},$$

then it can be easily proved that $\mathbf{X}'\mathbf{U}_i\Sigma_0 = \mathbf{X}'\mathbf{U}_i(\Sigma_0 + \mathbf{X}\beta^{(0)}\beta^{(0)\prime}\mathbf{X}')$, which implies $\operatorname{cov}[(Y'\mathbf{U}_iY, Y'\mathbf{U}_jY)|\beta^{(0)}, \Sigma_0] = 2\operatorname{Tr}[\mathbf{U}_i(\Sigma_0 + \mathbf{X}\beta^{(0)}\beta^{(0)\prime}\mathbf{X}')\mathbf{U}_j(\Sigma_0 + \mathbf{X}\beta^{(0)}\beta^{(0)\prime}\mathbf{X}')]$. From this the expression for the dispersion of the LMVQUE follows immediately. We proceed analogously in the case of the LMVLQUE.

Lemma 2.3. *The generator* $\{\mathbf{V}_i - \mathbf{P}_x\mathbf{V}_i\mathbf{P}_x\}_{i=1}^{p}$ *of the Gram matrix* $\mathbf{K}^*$ *in the model* (1) (*Lemma 1.3*) *can be rewritten in the form*

$$\{\mathbf{M}_S^K\mathbf{V}_i\mathbf{M}_S^{K\prime} - \mathbf{P}_A^L\mathbf{V}_i\mathbf{P}_A^{L\prime} + \mathbf{M}_x^{\Sigma_0^{-1}}\mathbf{V}_i\mathbf{P}_S^{K\prime} + \mathbf{P}_S^K\mathbf{V}_i\mathbf{M}_x^{\Sigma_0^{-1}\prime}\}_{i=1}^{p};$$

the analogous generator of the matrix $\mathbf{K}^*$ *in the transformed model* $(\mathbf{M}_S^K Y, \mathbf{A}\Theta, \sum_{i=1}^{p}\vartheta_i\mathbf{M}_S^K\mathbf{V}_i\mathbf{M}_S^{K\prime})$ *is given by the p-tuple* $\{\mathbf{M}_S^K\mathbf{V}_i\mathbf{M}_S^{K\prime} - \mathbf{P}_A^L\mathbf{V}_i\mathbf{P}_A^{L\prime}\}_{i=1}^{p}$. (*The matrices* $\mathbf{K}$, $\mathbf{L}$ *correspond with* Σ_0.)

Proof. It is implied by Lemmas 1.8 and 1.3.

Corollary 2.1. *If* $\mathbf{V}_i = \mathbf{M}_x^{\Sigma_0^{-1}}\mathbf{P}_S^{K\prime} + \mathbf{P}_S^K\mathbf{M}_x^{\Sigma_0^{-1}\prime}$, *then the transformation of the model* (1) *by the matrix* $\mathbf{M}_S^K$ *causes the vanishing of* ϑ_i.

Example. Let

$$\mathbf{A} = \begin{pmatrix}1\\1\\0\end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix}0\\1\\1\end{pmatrix}, \quad \beta^{(0)} = \begin{pmatrix}100\\50\end{pmatrix}, \quad \vartheta^{(0)} = \begin{pmatrix}10\\1\end{pmatrix},$$

$$\mathbf{V}_1 = \begin{pmatrix} 0, & -0.013\,314, & -0.013\,314 \\ -0.013\,314, & 0.023\,669, & -0.001\,479 \\ -0.013\,314, & -0.001\,479, & -0.026\,627 \end{pmatrix},$$

$$\mathbf{V}_2 = \begin{pmatrix} 10 & 1.133\,136, & 0.133\,136 \\ 1.133\,136, & 9.763\,314, & 1.014\,793 \\ 0.133\,136, & 1.014\,793, & 10.266\,272 \end{pmatrix},$$

thus

$$\boldsymbol{\Sigma}_0 = \begin{pmatrix} 10, & 1, & 0 \\ 1, & 10, & 1 \\ 0, & 1, & 10 \end{pmatrix},$$

It can easily be verified that $\mathbf{V}_1 = \mathbf{M}_{\mathbf{x}}^{\Sigma_0^{-1}\prime} \mathbf{P}_{\mathbf{S}}^{\mathbf{K}\prime} + \mathbf{P}_{\mathbf{S}}^{\mathbf{K}} \mathbf{M}_{\mathbf{x}}^{\Sigma_0^{-1}\prime}$,

$$\mathbf{K} = \begin{pmatrix} 0.055\,866, & -0.055\,866, & 0.005\,587 \\ -0.055\,866, & 0.055\,866, & -0.005\,587 \\ 0.005\,587, & -0.005\,587, & 0.100\,559 \end{pmatrix},$$

$$\mathbf{P}_{\mathbf{S}}^{\mathbf{K}} = \begin{pmatrix} 0, & 0, & 0 \\ -0.346\,154, & 0.346\,154, & 0.653\,846 \\ -0.346\,154, & 0.346\,154, & 0.653\,846 \end{pmatrix},$$

$$\mathbf{M}_{\mathbf{x}}^{\Sigma_0^{-1}\prime} = \begin{pmatrix} 0.346\,154, & -0.346\,154, & 0.346\,154 \\ -0.307\,692, & 0.307\,692, & -0.307\,692 \\ 0.346\,154, & -0.346\,154, & 0.346\,154 \end{pmatrix}.$$

For the model (1) we have

$$\mathbf{S}_{(\mathbf{M}_{\mathbf{x}}\Sigma_0\mathbf{M}_{\mathbf{x}})^+} = \begin{pmatrix} 0, & 0 \\ 0, & 1 \end{pmatrix},$$

$$2\mathbf{S}^{+}_{(\mathbf{M}_{\mathbf{x}}\Sigma_0\mathbf{M}_{\mathbf{x}})^+} = \begin{pmatrix} 0, & 0 \\ 0, & 2 \end{pmatrix} \Rightarrow \mathrm{Var}(\hat{\vartheta}_2|\boldsymbol{\vartheta}^{(0)}) = 2),$$

$$\mathbf{K}^* = \begin{pmatrix} 0.001\,983, & -0.045\,543 \\ -0.045\,543, & 76.268\,811 \end{pmatrix},$$

$$(\{\mathbf{K}^*\}_{i,j} = \mathrm{Tr}\,[(\mathbf{V}_i - \mathbf{P}_{\mathbf{x}}^{\Sigma_0^{-1}}\mathbf{V}_i\mathbf{P}_{\mathbf{x}}^{\Sigma_0^{-1}\prime})(\mathbf{V}_j - \mathbf{P}_{\mathbf{x}}^{\Sigma_0^{-1}}\mathbf{V}_j\mathbf{P}_{\mathbf{x}}^{\Sigma_0^{-1}\prime})],\ i, j = 1, \dots, p),$$

$$\mathbf{W}_1 = \mathbf{S}_{(\Sigma_0 + \mathbf{X}\boldsymbol{\beta}^{(0)}\boldsymbol{\beta}^{(0)\prime}\mathbf{X}')^{-1}} - \mathbf{S}_{\mathbf{P}_{\mathbf{x}}^{\Sigma_0^{-1}\prime}(\Sigma_0 + \mathbf{X}\boldsymbol{\beta}^{(0)}\boldsymbol{\beta}^{(0)\prime}\mathbf{X}')^{-1}\mathbf{P}_{\mathbf{x}}^{\Sigma_0^{-1}}} =$$

$$= \begin{pmatrix} 0.009\,712, & -0.097\,118 \\ -0.097\,118, & 1\,000.971\,18 \end{pmatrix} 10^{-3},$$

$$2\mathbf{W}_1^{-1} = \begin{pmatrix} 2.061\,355\,10^5, & 2 \\ 2, & 2 \end{pmatrix} \left(= \mathrm{Var}\left[\begin{pmatrix} \hat{\vartheta}_1 \\ \hat{\vartheta}_2 \end{pmatrix} \middle| \boldsymbol{\beta}^{(0)}, \boldsymbol{\vartheta}^{(0)} \right] \right),$$

$$\mathbf{W}_2 = \mathbf{S}_{\Sigma_0^{-1}} - \mathbf{S}_{\Sigma_0^{-1}\mathbf{P}_{\mathbf{x}}^{\Sigma_0^{-1}}} = \begin{pmatrix} 0.020\,784, & -0.207\,844 \\ -0.207\,844, & 1\,002.078\,437 \end{pmatrix} 10^{-3},$$

$$2\mathbf{W}_2^{-1} = \begin{pmatrix} 0.964\,261\,10^5, & 20 \\ 20, & 2 \end{pmatrix} \left(= \operatorname{Var}\left[\begin{pmatrix} \hat{\vartheta}_1 \\ \hat{\vartheta}_2 \end{pmatrix} \middle| \boldsymbol{\beta}^{(0)}, \boldsymbol{\vartheta}^{(0)} \right] \right).$$

Using $\mathbf{T} = \mathbf{M}_{\mathbf{S}}^{\mathbf{K}}$ the transformed model obtains the form

$$\mathbf{S}_{[\mathbf{M}_{\mathbf{A}}(\mathbf{M}_{\mathbf{S}}^{\mathbf{K}}\Sigma_0\mathbf{M}_{\mathbf{S}}^{\mathbf{K}\prime})\mathbf{M}_{\mathbf{A}}]^+} = \begin{pmatrix} 0, & 0 \\ 0, & 1 \end{pmatrix},$$

$$2\mathbf{S}^+_{[\mathbf{M}_{\mathbf{A}}(\mathbf{M}_{\mathbf{S}}^{\mathbf{K}}\Sigma_0\mathbf{M}_{\mathbf{S}}^{\mathbf{K}\prime})\mathbf{M}_{\mathbf{A}}]^+} = \begin{pmatrix} 0, & 0 \\ 0, & 2 \end{pmatrix} (\Rightarrow \operatorname{Var}(\hat{\vartheta}_2 | \hat{\vartheta}_2^{(0)}) = 2),$$

$$\mathbf{K}^* = \begin{pmatrix} 0, & 0 \\ 0, & 75.556\,21 \end{pmatrix},$$

$$(\{\mathbf{K}^*\}_{i,j} = \operatorname{Tr}[(\mathbf{M}_{\mathbf{S}}^{\mathbf{K}}\mathbf{V}_i\mathbf{M}_{\mathbf{S}}^{\mathbf{K}\prime} - \mathbf{P}_{\mathbf{A}}^{\mathbf{L}}\mathbf{V}_i\mathbf{P}_{\mathbf{A}}^{\mathbf{L}\prime})(\mathbf{M}_{\mathbf{S}}^{\mathbf{K}}\mathbf{V}_j\mathbf{M}_{\mathbf{S}}^{\mathbf{K}\prime} - \mathbf{P}_{\mathbf{A}}^{\mathbf{L}}\mathbf{V}_j\mathbf{P}_{\mathbf{A}}^{\mathbf{L}\prime})], \qquad i, j = 1, 2),$$

$$\mathbf{W}_1 = \mathbf{S}_{(\mathbf{M}_{\mathbf{S}}^{\mathbf{K}}\Sigma_0\mathbf{M}_{\mathbf{S}}^{\mathbf{K}\prime} + \mathbf{A}\Theta^{(0)}\Theta^{(0)\prime}\mathbf{A}')^+} -$$

$$- \mathbf{S}_{\mathbf{P}_{\mathbf{A}}^{(\mathbf{M}_{\mathbf{S}}^{\mathbf{K}}\Sigma_0\mathbf{M}_{\mathbf{S}}^{\mathbf{K}\prime})^+\prime}(\mathbf{M}_{\mathbf{S}}^{\mathbf{K}}\Sigma_0\mathbf{M}_{\mathbf{S}}^{\mathbf{K}\prime} + \mathbf{A}\Theta^{(0)}\Theta^{(0)\prime}\mathbf{A}')^+\mathbf{P}_{\mathbf{A}}^{(\mathbf{M}_{\mathbf{S}}^{\mathbf{K}}\Sigma_0\mathbf{M}_{\mathbf{S}}^{\mathbf{K}\prime})^+}} = \begin{pmatrix} 0, & 0 \\ 0, & 1 \end{pmatrix},$$

$$2\mathbf{W}_1^+ = \begin{pmatrix} 0, & 0 \\ 0, & 2 \end{pmatrix} (\Rightarrow \operatorname{Var}(\hat{\vartheta}_2 | \boldsymbol{\beta}^{(0)}, \vartheta_2^{(0)}) = 2),$$

$$\mathbf{W}_2 = \mathbf{S}_{(\mathbf{M}_{\mathbf{S}}^{\mathbf{K}}\Sigma_0\mathbf{M}_{\mathbf{S}}^{\mathbf{K}\prime})^+} - \mathbf{S}_{(\mathbf{M}_{\mathbf{S}}^{\mathbf{K}}\Sigma_0\mathbf{M}_{\mathbf{S}}^{\mathbf{K}\prime})^+\mathbf{P}_{\mathbf{A}}^{(\mathbf{M}_{\mathbf{S}}^{\mathbf{K}}\Sigma_0\mathbf{M}_{\mathbf{S}}^{\mathbf{K}\prime})^+}} = \begin{pmatrix} 0, & 0 \\ 0, & 1 \end{pmatrix},$$

$$2\mathbf{W}_2^+ = \begin{pmatrix} 0, & 0 \\ 0, & 2 \end{pmatrix} (\Rightarrow \operatorname{Var}(\hat{\vartheta}_2 | \boldsymbol{\beta}^{(0)}, \vartheta_2^{(0)}) = 2).$$

Comparing the matrices $\mathbf{K}^*$, $\mathbf{W}_1$ and $\mathbf{W}_2$ before and after the transformation we see that the optimality of the matrix $\mathbf{M}_{\mathbf{S}}^{\mathbf{K}}$ is not preserved if $\boldsymbol{f} \notin \mathcal{M}(\mathbf{K}^{(I)})$, $\boldsymbol{f} \in \mathcal{M}(\mathbf{K}^*)$.

Lemma 2.4 *The dispersion of the* $(\boldsymbol{\beta}^{(0)}, \boldsymbol{\vartheta}^{(0)})$-LMVLQUE *at the point* $(\boldsymbol{\beta}, \boldsymbol{\vartheta})$ *is*

$$\operatorname{Var}[(\boldsymbol{Y} - \mathbf{X}\boldsymbol{\beta}^{(0)})'\mathbf{T}(\boldsymbol{Y} - \mathbf{X}\boldsymbol{\beta}^{(0)}) | \boldsymbol{\beta}, \boldsymbol{\vartheta}] =$$

$$= 2\operatorname{Tr}(\mathbf{T}\Sigma\mathbf{T}\Sigma) + 4(\boldsymbol{\beta} - \boldsymbol{\beta}^{(0)})'\mathbf{X}'\mathbf{T}\Sigma\mathbf{T}\mathbf{X}(\boldsymbol{\beta} - \boldsymbol{\beta}^{(0)}),$$

where $\mathbf{T} = \sum_{i=1}^{p} \lambda_i (\Sigma_0^{-1}\mathbf{V}_i\Sigma_0^{-1} - \Sigma_0^{-1}\mathbf{P}_{\mathbf{x}}^{\Sigma_0^{-1}}\mathbf{V}_i\mathbf{P}_{\mathbf{x}}^{\Sigma_0^{-1}\prime}\Sigma_0^{-1})$ (Lemma 1.5); *the dispersion of the* $(\boldsymbol{\beta}^{(0)}, \boldsymbol{\vartheta}^{(0)})$-LMVQUE *at the point* $(\boldsymbol{\beta}, \boldsymbol{\vartheta})$ *is*

$$\operatorname{Var}(\boldsymbol{Y}'\mathbf{U}\boldsymbol{Y} | \boldsymbol{\beta}, \boldsymbol{\vartheta}) = 2\operatorname{Tr}(\mathbf{U}\Sigma\mathbf{U}\Sigma) + 4\boldsymbol{\beta}'\mathbf{X}'\mathbf{U}\Sigma\mathbf{U}\mathbf{X}\boldsymbol{\beta},$$

where $\mathbf{U} = \sum_{i=1}^{p} \lambda_i (\Sigma_0 + \mathbf{X}\boldsymbol{\beta}^{(0)}\boldsymbol{\beta}^{(0)\prime}\mathbf{X}')^{-1}(\mathbf{V}_i - \mathbf{P}_{\mathbf{x}}^{\Sigma_0^{-1}}\mathbf{V}_i\mathbf{P}_{\mathbf{x}}^{\Sigma_0^{-1}\prime})(\Sigma_0 + \mathbf{X}\boldsymbol{\beta}^{(0)}\boldsymbol{\beta}^{(0)\prime}\mathbf{X}')^{-1}$ (*Lemma 1.4*).

Proof. It is an obvious consequence of the implication $\boldsymbol{Y} \sim N_n(\boldsymbol{\mu}, \Sigma) \Rightarrow$

$\Rightarrow \mathrm{Var}(\boldsymbol{a}'\boldsymbol{Y} + \boldsymbol{Y}'\mathbf{A}\boldsymbol{Y}|\boldsymbol{\mu}, \quad \Sigma) = \boldsymbol{a}'\Sigma\boldsymbol{a} + 2\boldsymbol{a}'\Sigma\mathbf{A}\boldsymbol{\mu} + 4\boldsymbol{\mu}'\mathbf{A}\Sigma\mathbf{A}\boldsymbol{\mu} + 2\,\mathrm{Tr}(\mathbf{A}\Sigma\mathbf{A}\Sigma)$, which can be easily proved. Here $\mathbf{A} = \mathbf{A}'$ is an $n \times n$ matrix and $\boldsymbol{a} \in \mathscr{R}^n$.

The value of the second term in the expression for the variance of the LMVLQUE decreases when $\boldsymbol{\beta}$ tends to $\boldsymbol{\beta}^{(0)}$ (the matrix $\mathbf{X}'\mathbf{T}\Sigma\mathbf{T}\mathbf{X}$ is obviously positive semidefinite); that is why it is natural to use a $\boldsymbol{\vartheta}^{(0)}$-mLMVQE (Definition 1.4) instead of the $(\boldsymbol{\beta}^{(0)}, \boldsymbol{\vartheta}^{(0)})$-LMVLQUE. The structure of the $\boldsymbol{\vartheta}^{(0)}$-mLMVQE (Lemma 1.6) and Theorem 2.2 show the possibility that a function $g(.)$ unbiasedly estimable in the model (1) can be estimated by the help of the $\boldsymbol{\vartheta}^{(0)}$-mLMVQE in the model after transformation.

Theorem 2.3. *Let a function* $g(\boldsymbol{\vartheta}) = \boldsymbol{f}'\boldsymbol{\vartheta}$, $\boldsymbol{\vartheta} \in \underline{\boldsymbol{\vartheta}}$, *be unbiasedly estimable in the model* (1). *Let* $\mathbf{T} = \mathbf{M}_{\mathbf{S}}^{\mathbf{K}}$, *where* $\mathbf{K} = \Sigma_0^{-1} - \Sigma_0^{-1}\mathbf{A}(\mathbf{A}'\Sigma_0^{-1}\mathbf{A})^{-1}\mathbf{A}'\Sigma_0^{-1}$. *Then it is possible to obtain* $\boldsymbol{\vartheta}^{(0)}$-mLMVQE *from the model* (1) *by the help of the vector* $\mathbf{M}_{\mathbf{S}}^{\mathbf{K}}\boldsymbol{Y}$ *provided the matrices* Σ_0 *and* $\mathbf{X}$ *in the transformed model are known.*

Proof. The $\boldsymbol{\vartheta}^{(0)}$-mLMVQE from the model (1) is

$$\sum_{i=1}^{p} \lambda_i(\boldsymbol{Y} - \mathbf{X}\hat{\boldsymbol{\beta}})'[\Sigma_0^{-1}\mathbf{V}_i\Sigma_0^{-1} - \Sigma_0^{-1}\mathbf{P}_{\mathbf{x}}^{\Sigma_0^{-1}}\mathbf{V}_i\mathbf{P}_{\mathbf{x}}^{\Sigma_0^{-1}\prime}\Sigma_0^{-1}](\boldsymbol{Y} - \mathbf{X}\hat{\boldsymbol{\beta}}),$$

where $\hat{\boldsymbol{\beta}} = (\mathbf{X}'\Sigma_0^{-1}\mathbf{X})^{-1}\mathbf{X}'\Sigma_0^{-1}\boldsymbol{Y}$ and the vector $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_p)'$ is a solution of the equation $(\mathbf{S}_{\Sigma_0^{-1}} - \mathbf{S}_{\Sigma_0^{-1}\mathbf{P}_{\mathbf{x}}^{\Sigma_0^{-1}}})\boldsymbol{\lambda} = \boldsymbol{f}$. As $\boldsymbol{Y} - \mathbf{X}\hat{\boldsymbol{\beta}} = \mathbf{M}_{\mathbf{x}}^{\Sigma_0^{-1}}\boldsymbol{Y}$, $\Sigma_0^{-1}\mathbf{M}_{\mathbf{x}}^{\Sigma_0^{-1}} =$ $= (\mathbf{M}_{\mathbf{x}}\Sigma_0\mathbf{M}_{\mathbf{x}})^+$ and $\mathbf{P}_{\mathbf{x}}^{\Sigma_0^{-1}\prime}\Sigma_0^{-1}\mathbf{M}_{\mathbf{x}}^{\Sigma_0^{-1}} = \mathbf{O}$, the $\boldsymbol{\vartheta}^{(0)}$-mLMVQE can be rewritten in the form

$$\sum_{i=1}^{p} \lambda_i\boldsymbol{Y}'(\mathbf{M}_{\mathbf{x}}\Sigma_0\mathbf{M}_{\mathbf{x}})^+\mathbf{V}_i(\mathbf{M}_{\mathbf{x}}\Sigma_0\mathbf{M}_{\mathbf{x}})\boldsymbol{Y}$$

(cf. Lemma 1.6). Let $\mathbf{V}$ be an arbitrary symmetric and p.d. $n \times n$ matrix. It can be easily verified that $(\mathbf{M}_{\mathbf{x}}^{\mathbf{V}}\Sigma_0\mathbf{M}_{\mathbf{x}}^{\mathbf{V}\prime})^+ = \mathbf{P}_{\mathbf{M}_{\mathbf{x}}^{\mathbf{V}}}^{(\mathbf{I})}(\mathbf{M}_{\mathbf{x}}\Sigma_0\mathbf{M}_{\mathbf{x}})^+\mathbf{P}_{\mathbf{M}_{\mathbf{x}}^{\mathbf{V}}}^{(\mathbf{I})}$. If $\mathbf{V} = \Sigma_0^{-1}$, then $(\mathbf{M}_{\mathbf{x}}^{\Sigma_0^{-1}}\Sigma_0\mathbf{M}_{\mathbf{x}}^{\Sigma_0^{-1}\prime})^+ = \mathbf{P}_{\mathbf{M}_{\mathbf{x}}^{\Sigma_0^{-1}}}^{(\mathbf{I})}(\mathbf{M}_{\mathbf{x}}\Sigma_0\mathbf{M}_{\mathbf{x}})^+\mathbf{P}_{\mathbf{M}_{\mathbf{x}}^{\Sigma_0^{-1}}}^{(\mathbf{I})}$ and simultaneously $\mathbf{M}_{\mathbf{x}}^{\Sigma_0^{-1}\prime}(\mathbf{M}_{\mathbf{x}}^{\Sigma_0^{-1}}\Sigma_0\mathbf{M}_{\mathbf{x}}^{\Sigma_0^{-1}\prime})^+\mathbf{M}_{\mathbf{x}}^{\Sigma_0^{-1}} = \mathbf{M}_{\mathbf{x}}^{\Sigma_0^{-1}\prime}(\mathbf{M}_{\mathbf{x}}\Sigma_0\mathbf{M}_{\mathbf{x}})^+\mathbf{M}_{\mathbf{x}}^{\Sigma_0^{-1}} = \mathbf{M}_{\mathbf{x}}^{\Sigma_0^{-1}\prime}\mathbf{M}_{\mathbf{x}}\,.$ $.(\mathbf{M}_{\mathbf{x}}\Sigma_0\mathbf{M}_{\mathbf{x}})^+\mathbf{M}_{\mathbf{x}}\mathbf{M}_{\mathbf{x}}^{\Sigma_0^{-1}} = \mathbf{M}_{\mathbf{x}}(\mathbf{M}_{\mathbf{x}}\Sigma_0\mathbf{M}_{\mathbf{x}})^+\mathbf{M}_{\mathbf{x}} = (\mathbf{M}_{\mathbf{x}}\Sigma_0\mathbf{M}_{\mathbf{x}})^+$. Thus

$$① = \sum_{i=1}^{p} \lambda_i\boldsymbol{Y}'(\mathbf{M}_{\mathbf{x}}\Sigma_0\mathbf{M}_{\mathbf{x}})^+\mathbf{V}_i(\mathbf{M}_{\mathbf{x}}\Sigma_0\mathbf{M}_{\mathbf{x}})^+\boldsymbol{Y} =$$

$$= \sum_{i=1}^{p} \lambda_i\boldsymbol{Y}'\mathbf{M}_{\mathbf{x}}^{\Sigma_0^{-1}\prime}(\mathbf{M}_{\mathbf{x}}^{\Sigma_0^{-1}}\mathbf{V}_i\mathbf{M}_{\mathbf{x}}^{\Sigma_0^{-1}\prime})^+\mathbf{M}_{\mathbf{x}}^{\Sigma_0^{-1}}\mathbf{V}_i\mathbf{M}_{\mathbf{x}}^{\Sigma_0^{-1}\prime}(\mathbf{M}_{\mathbf{x}}^{\Sigma_0^{-1}}\mathbf{V}_i\mathbf{M}_{\mathbf{x}}^{\Sigma_0^{-1}\prime})^+\mathbf{M}_{\mathbf{x}}^{\Sigma_0^{-1}}\boldsymbol{Y}.$$

Using the relationship $\mathbf{M}_{\mathbf{x}}^{\Sigma_0^{-1}} = \mathbf{M}_{\mathbf{A}}^{\mathbf{L}}\mathbf{M}_{\mathbf{S}}^{\mathbf{K}}$ from Lemma 1.8 we obtain

$$① = \sum_{i=1}^{p} \lambda_i Y' \mathbf{M}_S^{K'} \mathbf{M}_A^{L'} [\mathbf{M}_A^L (\mathbf{M}_S^K \Sigma_0 \mathbf{M}_S^{K'}) \mathbf{M}_A^{L'}]^+ \mathbf{M}_A^L (\mathbf{M}_A^K \mathbf{V}_i \mathbf{M}_S^{K'}) \mathbf{M}_A^{L'} .$$

$$. [\mathbf{M}_A^L (\mathbf{M}_S^K \Sigma_0 \mathbf{M}_S^{K'}) \mathbf{M}_A^{L'}]^+ \mathbf{M}_A^L \mathbf{M}_S^K Y .$$

As $\mathbf{M}_A^L \mathbf{P}_A^L = \mathbf{O}$, the term $\mathbf{M}_S^K Y$ can be replaced by $\mathbf{M}_S^K Y - \mathbf{P}_A^L Y = \mathbf{M}_S^K Y - - \mathbf{A}\hat{\Theta}$, where $\hat{\Theta} = (\mathbf{A}'\mathbf{L}\mathbf{A})^{-1} \mathbf{A}'\mathbf{L}\mathbf{Y}$, $\mathbf{L} = \Sigma_0^{-1} - \Sigma_0^{-1}\mathbf{S}(\mathbf{S}'\Sigma_0^{-1}\mathbf{S})^{-1}\mathbf{S}'\Sigma_0^{-1} = = (\mathbf{M}_S \Sigma_0 \mathbf{M}_S)^+ = \mathbf{M}_S^{K'} (\mathbf{M}_S^K \Sigma_0 \mathbf{M}_S^{K'})^+ \mathbf{M}_S^K$. Thus $\hat{\Theta} = [\mathbf{A}'(\mathbf{M}_S^K \Sigma_0 \mathbf{M}_S^{K'})^+ \mathbf{A}]^{-1} \mathbf{A}' . (\mathbf{M}_S^K \Sigma_0 \mathbf{M}_S^{K'})^+ \mathbf{M}_S^K Y$, because of $\mathbf{M}_S^K \mathbf{A} = \mathbf{A}$; therefore $\hat{\Theta}$ is the $(\mathbf{M}_S^K \Sigma_0 \mathbf{M}_S^{K'})$-locally best linear unbiased estimator of the necessary parameter Θ in the model after transformation. The expression ① can be rewritten as follows

$$① = \sum_{i=1}^{p} \lambda_i (\mathbf{M}_S^K Y - \mathbf{A}\hat{\Theta})' \{[\mathbf{M}_A (\mathbf{M}_S^K \Sigma_0 \mathbf{M}_S^{K'}) \mathbf{M}_A]^+ .$$

$$. (\mathbf{M}_S^K \mathbf{V}_i \mathbf{M}_S^{K'}) [\mathbf{M}_A (\mathbf{M}_S^K \Sigma_0 \mathbf{M}_S^{K'}) \mathbf{M}_A]^+\} (\mathbf{M}_S^K Y - \mathbf{A}\hat{\Theta}) .$$

Let the term $-(\mathbf{M}_S^K \Sigma_0 \mathbf{M}_S^{K'})^+ \mathbf{P}_A^{(\mathbf{M}_S^K \Sigma_0 \mathbf{M}_S^{K'})^+} \mathbf{V}_i \mathbf{P}_A^{(\mathbf{M}_S^K \Sigma_0 \mathbf{M}_S^{K'})^+ \prime} (\mathbf{M}_S^K \Sigma_0 \mathbf{M}_S^{K'})^+$ be added into the brackets $\{\}$ in ①. The expression is unchanged because of

$$\mathbf{P}_A^{(\mathbf{M}_S^K \Sigma_0 \mathbf{M}_S^{K'})^+ \prime} (\mathbf{M}_S^K \Sigma_0 \mathbf{M}_S^{K'})^+ (\mathbf{M}_S^K Y - \mathbf{P}_A^L Y) =$$

$$= (\mathbf{M}_S^K \Sigma_0 \mathbf{M}_S^{K'})^+ \{\mathbf{A} [\mathbf{A}'(\mathbf{M}_S^K \Sigma_0 \mathbf{M}_S^{K'})^+ \mathbf{A}]^{-1} .$$

$$. \mathbf{A}'(\mathbf{M}_S^K \Sigma_0 \mathbf{M}_S^{K'})^+ Y - \mathbf{P}_A^{(\mathbf{M}_S^K \Sigma_0 \mathbf{M}_S^{M'})^+} \mathbf{P}_A^L Y\} = \mathbf{0} .$$

Thus ① is the $(\mathbf{M}_S^K \Sigma_0 \mathbf{M}_S^{K'})$-mLMVQE of the function $g(.)$ in the transformed model $(\mathbf{M}_S^K Y, \mathbf{A}\Theta, \sum_{i=1}^{p} \vartheta_i \mathbf{M}_S^K \mathbf{V}_i \mathbf{M}_S^{K'})$. However, the vector $\lambda = (\lambda_1, \ldots, \lambda_p)'$ has to be a solution of the equation $(\mathbf{S}_{\Sigma_0^{-1}} - \mathbf{S}_{\Sigma_0^{-1}\mathbf{P}_x^{\Sigma_0^{-1}}})\lambda = \boldsymbol{f}$. Thus we have to know the matrices Σ_0 and $\mathbf{X}$; we are not able to establish this equation using the matrices $\mathbf{M}_S^K \Sigma_0 \mathbf{M}_S^{K'}$ and $\mathbf{A}$ known in the transformed model.

Remark 2.4. The vector $\boldsymbol{\beta}^{(0)}$ can be replaced by $\hat{\boldsymbol{\beta}}$ in the $(\boldsymbol{\beta}^{(0)}, \boldsymbol{\vartheta}^{(0)})$-LMVQUE as well. An investigation of such an estimator is difficult, that is why it was not dealt with.

Acknowledgement. I should like to express my cordial thanks to RNDr. M. Bognárová for her help with numerical calculations.

REFERENCES

[1] FELLMAN, J.: Linear models with nuisance parameters — some theoretical results applied to balanced designs. Meddelanden working papers. Swedish School of Economics and Business Administrations. Working papers 147, 1986.

[2] FELLMAN, J.: Estimation in linear models with nuisance parameters. Statistics & Decisions, Supplement issue No 2, 1985, 161—164.
[3] KLEFFE, J.: Simultaneous estimation of expectation and covariance matrix in linear models. Zentralinstitut für Mathematik und Mechanik der Akademie der Wissenschaften der DDR, Berlin 1977.
[4] KUBÁČEK, L.—KUBÁČKOVÁ, L.: Statistical aspect of eliminating systematic effects. Studia geoph. et geod. 30, 1986, 1—12.
[5] KUBÁČEK, L.: Elimination of nuisance parameters in a regression model. Math. Slovaca 36, 1986, 137—144.
[6] RAO, C. R.—MITRA, S. K.: Generalized Inverse of Matrices and Its Application. N. York, J. Wiley 1971.
[7] RAO, C. R.—KLEFFE, J.: Estimation of variance components. In: P. R. Krishnaiah (Ed.), Handbook of Statistics. Vol. I. North-Holland, Amsterdam 1980.

Received May, 10, 1988

Matematický ústav SAV
Štefánikova ul. 49
814 73 Bratislava