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ON SEQUENCES OF REAL FUNCTIONS WITH THE DARBOUX PROPERTY

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ABSTRACT. It is proved that every function $f\colon \mathbb{R} \to \mathbb{R}$ with the Baire property (every cliquish function $f\colon \mathbb{R} \to \mathbb{R}$) is the limit of a sequence of Darboux cliquish functions (Darboux quasicontinuous functions).

It is know that every real function $f\colon \mathbb{R} \to \mathbb{R}$ with the Baire property is the limit of a sequence of pointwise discontinuous functions [1]. Moreover, in [1] it is shown that each pointwise discontinuous real function $f\colon \mathbb{R} \to \mathbb{R}$ is the limit of a sequence of quasi-continuous functions. It is our intention to demonstrate that in any of the above facts one can assume that the members of the mentioned sequence have the Darboux property. First let us recall some definitions.

A real function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be quasi-continuous at a point $x_0 \in \mathbb{R}$ iff for every nondegenerate interval U containing x_0 and every nondegenerate interval V containing $f(x_0)$ there is a nondegenerate interval $U_0 \subset U$ such that $f(U_0) \subset V$ [3].

A real function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be quasi-continuous if it is quasi-continuous at each point of its domain [3].

A real function $f\colon \mathbb{R} \to \mathbb{R}$ is called pointwise discontinuous if the set of continuity points is dense in the domain [4].

It is said that a real function $f\colon \mathbb{R} \to \mathbb{R}$ has the Baire property iff for every open set U the set $f^{-1}(U)$ is of the form $(G \setminus N) \cup M$, where G is open and N and M are of the first category [4].

For any real function f by O_f we denote the oscillation of the function f defined as follows:

$$O_f(x) = \inf_{\varepsilon > 0}(\sup_{y, z \in K(x, \varepsilon)} |f(z) - f(y)|).$$

It is easy to check that the function f is continuous at a point $x \in \mathbb{R}$ iff $O_f(x) = 0$.

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Theorem 1. *Each pointwise discontinuous real function $f\colon \mathbb{R} \to \mathbb{R}$ is the limit of a sequence of quasi-continuous functions with the Darboux property.*

Proof: (A modification of the proof of Theorem 1 in [1]).

Without loss of generality we may assume that the function f is bounded. Indeed, if f is not bounded, then it would be sufficient to consider a superposition $g \circ f$, where $g\colon \mathbb{R} \to (a, b)$ is any homeomorphism between $\mathbb{R}$ and some open interval (a, b) (it is easy to see that $g \circ f$ is pointwise discontinuous iff f is pointwise discontinuous).

There is a residual subset A of $\mathbb{R}$ of the type G_δ such that the function $f|_A = f_1$ is continuous. Putting

$$h(x) = \overline{\lim_{u \to x, u \in A}} f_1(u) \text{ we obtain the function } h\colon \mathbb{R} \to \mathbb{R}$$

which is bounded and of the first class of Baire. This function is such that

$$A \subset \{x \in \mathbb{R}: f(x) = h(x)\}.$$

Let $g = f - h$. Then the sets $A_n = \{x \in \mathbb{R}: O_g(x) \geq 1/n\}$ are closed for any $n \in \mathbb{N}$ and because they are of the first category they are nowhere dense. In addition $A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n \subseteq \ldots$. Let M be such nonnegative integer that $|g(x)| \leq M$ for every $x \in \mathbb{R}$. Let us choose a natural number n. Then for each $k = 1, 2, \ldots, n$ there is a sequence of closed intervals (nondegenerated) $\{I^n_{k,m}\}_{m=1}^{\infty}$ such that

1. $I^n_{k_1,m_1} \cap I^n_{k_2,m_2} = \emptyset$ for $k_1 \neq k_2$ or $m_1 \neq m_2$;
2. $I^n_{k,m} \cap A_n = \emptyset$ for every $k = 1, 2, \ldots, n$, $m = 1, 2, \ldots$;
3. $I^n_{k,m} \subset \bigcup_{x \in A_k} B(x, 1/n)$ for every $m = 1, 2, \ldots$, $k = 1, 2, \ldots, n$,
   $(B(x, r) = (x - r, x + r))$;
4. any sequence $\{I^n_{k,m}\}_{m=1}^{\infty}$ is convergent on both sides to the set A_k $(k = 1, 2, \ldots, n)$; this means that for every open interval (a, b) which has one end belonging to the set A_k there are infinitely many natural numbers m such that $I^n_{k,m} \subset (a, b)$, moreover, if $x = \lim_m x_{t_m}$, where $x_{t_m} \in I^n_{k,t_m}$ then $x \in A_k$.

For any interval $I^n_{k,m}$ where n is fixed, $m \in \mathbb{N}$ and $1 < k \leq n (k = 1)$ let us define a surjective continuous function $g^n_{k,m}\colon I^n_{k,m} \to [-1/(k-1), 1/(k-1)]$ $(g^n_{1,m}\colon I^n_{1,m} \to [-M, M])$ such that if x is an end of the interval $I^n_{k,m}$, then $g^n_{k,m}(x) = 0$. Next, for every $n \in \mathbb{N}$ we define the function $g_n\colon \mathbb{R} \to \mathbb{R}$ as follows:

$$g_n(x) = \begin{cases} g^n_{k,m}(x) & \text{for } x \in I^n_{k,m} \ (k = 1, 2, \ldots, n,\ m = 1, 2, \ldots) \\ g(x) & \text{for } x \in A_n \\ 0 & \text{for } x \in \mathbb{R} \setminus \left(\bigcup_{k=1}^{n} \bigcup_{m=1}^{\infty} I^n_{k,m} \cup A_n \right). \end{cases}$$

We will prove that the functions g_n are quasi-continuous for $n \in \mathbb{N}$. Let n be a natural number. The function g_n is continuous at each point of the set $\mathbb{R} \setminus A_n$. If $x \in A_n$, then $x \in A_1$ or $x \in A_k \setminus A_{k-1}$ for some $k = 2, 3, \ldots, n$. Let $x \in A_k \setminus A_{k-1}$. Then the following inequality holds:

$$|g(x)| < 1/(k-1).$$

Since $A \subset \{x \in \mathbb{R}: g(x) = 0\}$ and A is residual in $\mathbb{R}$, the set $\{x \in \mathbb{R}: g(x) = 0\}$ is dense in $\mathbb{R}$ and therefore $O_g(x) < 1/(k-1)$ for each $x \in A_k \setminus A_{k-1}$. Let U be an open neighbourhood of the point x. Let us take $\varepsilon > 0$. By the condition 4. there is $i_0 \in \mathbb{N}$ such that $I^n_{k,i_0} \subseteq U$. Since $g_n(I^n_{k,i_0}) = [-1/(k-1), 1/(k-1)]$ there is an open interval $I \subset I^n_{k,i_0} \subset U$ such that for every $u \in I$ the following inequality holds

$$|g_n(x) - g_n(u)| < \varepsilon.$$

So, the function g_n is quasi-continuous at any point $x \in A_k \setminus A_{k-1}$.

In a similar way we prove that the function g_n is quasi-continuous at any point of the set A_1. Finally, we have shown that g_n is quasi-continuous on the set $\mathbb{R}$. Now, we will prove that the sequence of the functions $\{g_n\}_{n=1}^{\infty}$ is pointwise convergent to the function g. Let us take $x \in \mathbb{R}$ and a positive number ε.

Case 1: $x \notin \bigcup_{n=1}^{\infty} A_n$. Thus $g(x) = 0$. Let n_0 be such natural number that $1/n_0 < \varepsilon$. Since $x \notin \bigcup_{n=1}^{\infty} A_n$, so $x \notin A_{n_0}$. The set A_{n_0} is closed, therefore $\mathrm{d}(x, A_{n_0}) = \inf\{|x-y| : y \in A_{n_0}\} > 0$. There is an index $k > n_0$ such that $\mathrm{d}(x, A_{n_0}) > 1/k$, so $x \notin \bigcup_{t \in A_{n_0}} K(t, 1/k)$. From this and from the condition 3) it follows that $x \notin I^k_{n_0,l}$ for $l = 1, 2, \ldots$. Let us observe that if $m \leq n_0$, then, because $A_m \subseteq A_{n_0}$, we have $\bigcup_{t \in A_m} B(t, 1/k) \subseteq \bigcup_{t \in A_{n_0}} B(t, 1/k)$ and consequently $x \notin I^k_{m,l}$. Moreover, if $n > k$ and $m \leq n_0$, then $\mathrm{d}(x, A_m) > 1/n$. It means that for every $n > k$ we have $x \notin I^n_{m,l}$ for $m = 1, 2, \ldots, n_0$; $l = 1, 2, \ldots$, so $|g_n(x)| < 1/n_0$ for $n > k$. Finally for $x \notin \bigcup_{n=1}^{\infty} A_n$ the sequence $\{g_n(x)\}$ is convergent to $g(x) = 0$.

Case 2: $x \in \bigcup_{n=1}^{\infty} A_n$. Thus, there is a number $k \in \mathbb{N}$ such that $x \in A_k$ and in the result $g_n(x) = g(x)$ for every $n > k$. Therefore, $\lim_n g_n(x) = g(x)$.

Finally, for every $x \in \mathbb{R}$ the sequence $\{g_n(x)\}_{n=1}^{\infty}$ converges to $g(x)$.

The function h is of the first class of Baire so there is a sequence $\{h_n\}_{n=1}^{\infty}$ of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ such that $\lim_n h_n = h$. Let us put $f_n = g_n + h_n$, $n = 1, 2, \ldots$. For every $n \in \mathbb{N}$ the function f_n is the sum of a quasi-continuous function and a continuous function, hence it is quasi-continuous [see Theorem 1 in [2]].

It is easy to see that $\lim_n f_n(x) = f(x)$ for every $x \in \mathbb{R}$. It remains to show that the functions f_n have the Darboux property.

Let us suppose, conversely, that there is $n \in \mathbb{N}$ such that the function f_n does not have the Darboux property. It means that there are two elements x_1, x_2 of the set such that $f_n(x_1) < f_n(x_2)$ (without loss of generality we may assume that $x_1 < x_2$) and a number $c \in (f_n(x_1), f_n(x_2))$ such that the graph of the function $f_n|_{(x_1, x_2)}$ does not intersect the straight line $y = c$.

Let $G = \{x \in [x_1, x_2]; f_n(x) < c\}$ and $H = \{x \in [x_1, x_2]; f_n(x) > c\}$. Since $G \neq \emptyset$ and $H \neq \emptyset$ and f_n is quasicontinuous, both of the sets, G and H, have a momenpty interior. Obviously $G \cup H = [x_1, x_2]$. By connectivity of the interval $[x_1, x_2]$ we have that there is a point $t \in [x_1, x_2]$ such that for any open neighbourhood U of this point there hold $U \cap \operatorname{int} G \neq \emptyset$ and $U \cap \operatorname{int} H \neq \emptyset$. Because the set A_n is the set of discontinuity points of the function f_n we have $t \in A_n$. We may assume without decreasing the generality of the considerations, the following:

a) $h_n(t) = 0$,

b) $f_n(t) < c$,

c) $c > 0$.

There is $\beta > 0$ such that $h_n(x) < c/2$ for every $x \in B(t, \beta)$. Because of $B(t, \beta) \cap \operatorname{int} H \neq \emptyset$ there is an open interval L such that for any $x \in L$ we have $f_n(x) > c$. From this $g_n(L) \subseteq (c/2, \infty)$. By the definition of the function g_n we can see that $L \subseteq I^n_{k,m} \subset (x_1, x_2)$ for some $k \in \{1, 2, \ldots, n\}$ and $m \in \mathbb{N}$. At the ends of the interval $I^n_{k,m}$ the function g_n takes 0 and the function $g_n|_{I^n_{k,m}}$ is continuous. So the function f_n takes on the ends of $I^n_{k,m}$ values smaller than $c/2$ and it is also continuous on this interval. However because there is $x_0 \in L \cap I^n_{k,m}$ such that $f_n(x_0) > c$, there must be $x \in I^n_{k,m}$ such that $f_n(x) = c$. We obtain the contradiction to the assumption that the graph of $f_n|_{[x_1, x_2]}$ and the line $y = c$ do not intersect. This proves that for every $n \in \mathbb{N}$ the function f_n has the Darboux property and it completes the proof.

Theorem 2. *Any real function f: $\to$ having the Baire property is the limit of a sequence of pointwise discontinuous functions with the Darboux property.*

Proof. Similarly as in the proof of the Theorem 1 we can restrict ourselves to consider bounded functions. Because the function f has the Baire property we may find a residual set $A \subset \mathbb{R}$ of the type G_δ such that there is a function g: $\mathbb{R} \to \mathbb{R}$ of the first class of Baire such that the function $f|_A = g|_A$ is continuous. We have $\mathbb{R} \setminus A = \bigcup_{n=1}^{\infty} B_n$, where B_n are closed and nowhere dense. Moreover, we may assume that $B_n \subset B_{n+1}$ for any $n \in \mathbb{N}$. For every $n \in \mathbb{N}$ we can define a sequence of perfect nowhere dense sets $\{C_k^n\}_{k=1}^{\infty}$ such that the following conditions hold:

1. $C_k^n \cap C_l^m = \emptyset$ for $n \neq m$ or $k \neq l$ and $C_k^n \subset A$ for all k, n;
2. for any number $\varepsilon > 0$ there is a natural number k_n such that for every $k > k_n$ we have $C_k^n \subset \bigcup_{x \in B_n} B(x, \varepsilon)$;
3. for any $n \in \mathbb{N}$ the set $\bigcup_{l=1}^{n} \bigcup_{k=1}^{\infty} C_k^l \cup B_l$ is a nowhere dense closed set;
4. the sequence $\{C_k^n\}_{k=1}^{\infty}$ is on both sides convergent to B_n.

For every pair n, k of natural numbers there is a sequence $\{D_{k,l}^n\}_{l=1}^{\infty}$ of bilateral c-dense-in-itself sets contained in the set C_k^n such that $D_{k,l}^n \cap D_{k,m}^n = \emptyset$ if only $m \neq l$. In addition let the following condition hold for every open set $U \subset \mathbb{R}$:

If $U \cap C_k^n \neq \emptyset$, then there is an infinite number of indices l such that $D_{k,l}^n \subset U$.

Let $g_{k,l}^n : D_{k,l}^n \to \mathbb{R}$ be any bijective mapping for every $n, k, l \in \mathbb{N}$. Let us put:

$$f_n(x) = \begin{cases} f(x) & \text{for } x \in B_n \\ g_{k,l}^n(x) & \text{for } x \in D_{k,l}^n \\ g_n(x) & \text{in other cases}, \end{cases}$$

where $g_n : \mathbb{R} \to \mathbb{R}$, $n = 1, 2, \ldots$ is a sequence of continuous functions such that $\lim_n g_n(x) = g(x)$ for every $x \in \mathbb{R}$. It is obvious that for every $n \in \mathbb{N}$ the function f_n is pointwise discontinuous. Also it is easy to check that these functions have the Darboux property. Since for every $x \in \mathbb{R}$ we have $\lim f_n(x) = f(x)$ the proof is completed.

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