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REGULAR IDEALS IN AUTOMETRIZED ALGEBRAS

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K. L. N. Swamy and N. P. Rao introduced (in [8]) the notion of an ideal in autometrized algebras. (Autometrized algebras were introduced by SWAMY in [6]). Prime ideals in autometrized algebras were studied by the author in [4]. In this paper there is introduced the notion of a regular ideal in an autometrized algebra (it is a particular case of the notion of a prime ideal). The aim of the paper is to investigate the properties of regular ideals and their relations to prime ideals. The theory of autometrized algebras is a common generalization, e.g., of the theories of Brouwerian algebras and commutative lattice ordered groups. Hence we refer for the results of those theories to the books [1, 2, 3].

An autometrized algebra is any system $\mathscr{A} = (A, +, \leqq, *)$ such that
(1) $(A, +, \leqq)$ is an ordered commutative semigroup with zero element 0;
(2) $*: A \times A \to A$ is a mapping (a metric operation) such that

$$\forall a, b \in A;\ a * b \geqq 0 \text{ and } a * b = 0 \Leftrightarrow a = b,$$
$$\forall a, b \in A;\ a * b = b * a,$$
$$\forall a, b, c \in A;\ a * c \leqq (a * b) + (b * c).$$

If the ordered set $(A, \leqq)$ is a lattice and

$$\forall a, b, c \in A;\ a + (b \vee c) = (a + b) \vee (a + c),$$
$$a + (b \wedge c) = (a + b) \wedge (a + c),$$

then $\mathscr{A}$ is called an *autometrized l-algebra.*

We say that an autometrized algebra is

a) *normal* if

$$\forall a \in A;\ a \leqq a * 0,$$
$$\forall a, b, c, d \in A;\ (a + c) * (b + d) \leqq (a * b) + (c * d),$$
$$\forall a, b, c, d \in A;\ (a * c) * (b * d) \leqq (a * b) + (c * d),$$
$$\forall a, b \in A;\ (a \leqq b \Rightarrow \exists x \geqq 0;\ a + x = b);$$

b) *semiregular* if

$$\forall a \in A;\ a \geqq 0 \Rightarrow a * 0 = a.$$

Let $\mathscr{A}$ be an autometrized algebra, $\emptyset \neq I \subseteq A$. Then I is called an *ideal* in $\mathscr{A}$ if

$$\forall a, b \in I;\ a + b \in I;$$
$$\forall a \in I,\ x \in A;\ x * 0 \leqq a * 0 \Rightarrow x \in I.$$

Let us denote the set of all ideals in $\mathscr{A}$ by $\mathscr{I}(\mathscr{A})$. If $\mathscr{A}$ is a normal autometrized algebra, then $\mathscr{I}(\mathscr{A})$ ordered by set inclusion is (by [8, Theorem 1]) a complete algebraic lattice. Moreover, infima in $\mathscr{I}(\mathscr{A})$ are formed by intersections. Let $\mathscr{A}$ be an autometrized algebra, $I \in \mathscr{I}(\mathscr{A})$. Then we say that I is a *prime ideal* in $\mathscr{A}$ (see [4]) if

$$\forall J, K \in \mathscr{I}(\mathscr{A});\ J \cap K = I \Rightarrow J = I \text{ or } K = I.$$

Definition 1. Let $\mathscr{A}$ be an autometrized algebra, $I \in \mathscr{I}(\mathscr{A})$. Then I is called a *regular ideal* in $\mathscr{A}$ if $I = \bigcap_{\alpha \in \Gamma} J_\alpha$, where $J_\alpha \in \mathscr{I}(\mathscr{A})$ for each $\alpha \in \Gamma$ implies the existence of $\beta \in \Gamma$ such that $I = J_\beta$.

It is evident that any regular ideal is also a prime ideal. Now, let us consider a regular ideal I in a normal semiregular autometrized algebra $\mathscr{A}$, $I \neq A$. Denote I^* the intersection of all ideals in $\mathscr{A}$ strictly containing I. Evidently, $I \subset I^*$ and I^* is a unique cover of I in the lattice $\mathscr{I}(\mathscr{A})$.

Definition 2. Let $\mathscr{A}$ be a normal autometrized algebra, $0 \neq a \in A$. If $I \in \mathscr{I}(\mathscr{A})$ is a maximal ideal in $\mathscr{A}$ not containing a, then I is called a *value* of the element a in $\mathscr{A}$.

The set of all values of a will be denoted by $\operatorname{val}(a)$.

Theorem 1. *Let A be a normal autometrized algebra, $I \in \mathscr{I}(A)$. Then I is regular if and only if there exists $a \in A$ such that $I \in \operatorname{val}(a)$.*

Proof. Let I be a regular ideal in $\mathscr{A}$. Let us consider $a \in I^* \setminus I$. If $J \in \mathscr{I}(\mathscr{A})$ and $I \subset J$, then $a \in J$, hence I is a value of a.

Conversely, let $0 \neq a \in A$ and $I \in \operatorname{val}(a)$. If $J_\alpha \in \mathscr{I}(\mathscr{A})$, $\alpha \in \Gamma$, and $I = \bigcap_{\alpha \in \Gamma} J_\alpha$, then there exists $\beta \in \Gamma$ such that $a \notin J_\beta$. Moreover, $I \subseteq J_\beta$, and since $I \in \operatorname{val}(a)$, it must be $I = J_\beta$. Therefore I is a regular ideal.

Theorem 2. *If A is a normal autometrized algebra, $I \in \mathscr{I}(\mathscr{A})$, $a \in A$, $a \notin I$, then there exists $I \in \operatorname{val}(a)$ such that $I \subseteq J$.*

Proof. Denote $Z = \{K \in \mathscr{I}(\mathscr{A});\ I \subseteq K,\ a \notin K\}$. In [5, Proof of Theorem 3], it is shown that Z is an inductive set, and hence Z contains a maximal element J which is a value of A.

Consequently:

Theorem 3. *Any ideal of a normal autometrized algebra $\mathscr{A}$ is the intersection of regular ideals.*

Let us recall the notion of a dually residuated lattice ordered semigroup (*DRl*-semigroup) which has been introduced by Swamy in [7].

A system $\mathscr{A} = (A, +, \leqq, -)$ is called a *DRl-semigroup* if

(1) $(A, +, \leqq)$ is a commutative lattice ordered semigroup with zero element 0;

(2) for each $a, b \in A$ there exists the least element $x \in A$ such that $b + x \geqq a$ (x is denoted by $a - b$);

(3) $\forall a, b \in A$; $(a - b) \vee 0 + b \leqq a \vee b$;

(4) $\forall a \in A$; $a - a \geqq 0$.

If we denote $a * b = (a - b) \vee (b - a)$ for $a, b \in A$, then $(A, +, \leqq, *)$ is an autometrized l-algebra which is normal and semiregular. (See [7, 8].)

A *DRl*-semigroup $\mathscr{A}$ is called *representable* (see [9]) if

$$\forall a, b \in A; (a - b) \wedge (b - a) \leqq 0.$$

(Commutative l-groups and Boolean algebras are examples of representable *DRl*-semigroups.)

Theorem 4. *If I is a prime ideal in a representable DRl-semigroup $\mathscr{A}$, then the set of all ideals in $\mathscr{A}$ containing I is linearly ordered.*

Proof. Let I be a prime ideal in $\mathscr{A}$. Suppose that $J, K \in \mathscr{I}(\mathscr{A})$, $I \subset J$, $I \subset K$, and that $J \nsubseteq K$, $K \nsubseteq J$. Then there exist $0 < a \in J \backslash K$, $0 < b \in K \backslash J$. Let us consider the elements $a - (a \wedge b)$ and $b - (a \wedge b)$. By [7, Corollary of Lemma 4], $a - (a \wedge b) > 0$ and $b - (a \wedge b) > 0$. Moreover, from $a \wedge b \geqq 0$ we get $(a \wedge b) + a \geqq a$ and $(a \wedge b) + b \geqq b$, hence $a \geqq a - (a \wedge b) > 0$ and $b \geqq b - (a \wedge b) > 0$. Hence, the semiregularity of $\mathscr{A}$ implies $a - (a \wedge b) \in J$ and $b - (a \wedge b) \in K$.

Since $\mathscr{A}$ is representable, by [4, Lemma 6] we have $[a - (a \wedge b)] \wedge [b - (a \wedge b)] = 0$, but this is by [3, Theorem 4] a contradiction to the assumption that I is a prime ideal.

Therefore $J \subseteq K$ or $K \subseteq J$.

Let $\mathscr{A} = (A, +, \leqq)$ be an ordered semigroup with zero element 0. Then $\mathscr{A}$ is called an *interpolation semigroup* if

$$\forall a, b, c \in A; [(0 \leqq a, b, c \text{ and } a \leqq b + c) \Rightarrow$$
$$\Rightarrow (\exists 0 \leqq b_1 \leqq b, 0 \leqq c_1 \leqq c; a = b_1 + c_1)].$$

(For instance, commutative l-groups and Brouwerian algebras are interpolation semigroups.)

Theorem 5. *If I is an ideal in a semiregular normal interpolation autometrized l-algebra $\mathscr{A}$ such that the set of all ideals in $\mathscr{A}$ containing I is linearly ordered, then I is a prime ideal in $\mathscr{A}$.*

Proof. Let I be an ideal in $\mathscr{A}$ satisfying the condition of the assumption. Suppose that I is not prime. Then by [4, Theorem 4] there exist $a, b \in A$, $0 < a$, $0 < b$ such that $a \wedge b \in I$. Denote

$$J = \{x \in A;\ (x * 0) \wedge b \in I\},\ K = \{y \in A;\ (y * 0) \wedge a \in I\}.$$

If $x \in I$, then $x * 0 \in I$. Moreover, $0 \leqq (x * 0) \wedge b \leqq x * 0$, and since $[(x * 0) \wedge b] * 0 = (x * 0) \wedge b$, we get $(x * 0) \wedge b \in I$, hence $x \in J$. Therefore $I \subseteq J$. Similarly $I \subseteq K$.

Further, $(a * 0) \wedge b = a \wedge b \in I$, thus $a \in J$. In addition, $(a * 0) \wedge a \in I$, hence $a \notin K$, and so $a \in J \backslash K$. Analogously $b \in K \backslash J$.

Let us prove that $J, K \in \mathscr{I}(\mathscr{A})$. Let $x, y \in J$. Since $\mathscr{A}$ is a normal and interpolation algebra, we get

$$[(x + y) * 0] \wedge b \leqq [(x * 0) + (y * 0)] \wedge b \leqq [(x * 0) \wedge b] + [(y * 0) \wedge b] \in I,$$

hence $x + y \in J$.

Further, let $x \in J$, $z \in A$, $z * 0 \leqq x * 0$. Then from the semiregularity of $\mathscr{A}$ we get

$$[(z * 0) \wedge b] * 0 = (z * 0) \wedge b \leqq (x * 0) \wedge b = [(x * 0) \wedge b] * 0,$$

and since $(x * 0) \wedge b \in I$, we also have $(z * 0) \wedge b \in I$, thus $z \in J$.

Therefore $J \in \mathscr{I}(\mathscr{A})$ and similarly $K \in \mathscr{I}(\mathscr{A})$. But this means that $I \subseteq J$, $I \subseteq K$, $J \nsubseteq K$, $K \nsubseteq J$, a contradiction with the assumption. Hence I is a prime ideal in $\mathscr{A}$.

Theorems 4 and 5 and [4, Theorem 4] now imply:

Theorem 6. *If $\mathscr{A}$ is a representable interpolation DRl-semigroup, $I \in \mathscr{I}(\mathscr{A})$, then the following conditions are equivalent:*

(1) *I is a prime ideal in $\mathscr{A}$.*

(2) *$\forall J, K \in \mathscr{I}(\mathscr{A});\ J \cap K \subseteq I \Rightarrow J \subseteq I$ or $K \subseteq I$.*

(3) *$\forall a, b \in A;\ 0 \leqq a \wedge b \in I \Rightarrow a \in I$ or $b \in I$.*

(4) *$\{J \in \mathscr{I}(\mathscr{A});\ I \subseteq J\}$ is linearly ordered.*

Let us recall that a subset S of a lattice $\mathscr{L}$ is called a *root system* (see [1, p. 27], [3, p. 51]) if for each $x \in S$ the set of all $y \in L$ such that $x \leqq y$ is linearly ordered and contained in S.

Corollary. *The set of all prime ideals in a representable interpolation DRl-semigroup $\mathscr{A}$ forms a root system in the lattice $\mathscr{I}(\mathscr{A})$.*

We know that any regular ideal is prime. Now let us show a more complete connection between these notions.

Theorem 7. *If $\mathscr{A}$ is a representable DRl-semigroup, $I \in \mathscr{I}(\mathscr{A})$, then I is a prime ideal if and only if it is the intersection of a linearly ordered system of regular ideals.*

Proof. Let I be a prime ideal. Then by Theorem 3, I is the intersection of regular ideals. Moreover, since $\mathscr{A}$ is a representable DRl-semigroup, the ideals containing I form, by Theorem 4, a chain.

The converse implication follows from the fact that by [4, Theorem 8] the intersesction of any linearly ordered system of prime ideals in a semiregular normal autometrized l-algebra $\mathscr{A}$ is a prime ideal in $\mathscr{A}$, too.

Theorem 8. *Let $\mathscr{A}$ be a semiregular interpolation normal autometrized l-algebra, $I \in \mathscr{I}(\mathscr{A})$, $0 \neq a \in I$. Then the mapping φ: $J \mapsto J \cap I$, for any $J \in \mathrm{val}_A(a)$, is a bijection of the set $\mathrm{val}_A(a)$ onto the set $\mathrm{val}_I(a)$.*

Proof. Let $I \in \mathscr{I}(\mathscr{A})$, $a \in I$. According to [4, Theorem 10], the mapping ψ: $P \mapsto P \cap I$ is a bijection of the set of all prime ideals in $\mathscr{A}$ not containing I onto the set of all proper prime ideals in I which is an isomorphism between those sets ordered by set inclusion. Evidently, φ is a restriction of ψ on the set $\mathrm{val}_A(a)$.

Let $J \in \mathrm{val}_A(a)$. Since $J \cap I \in \mathscr{I}(I)$ and $a \notin J \cap I$, there exists $K \in \mathrm{val}_I(a)$ such that $J \cap I \subseteq K$. And since $J = \varphi^{-1}(J \cap I)$, we have $J \subseteq \varphi^{-1}(K)$. Moreover, $a \in \varphi^{-1}(K)$, but that implies $J = \varphi^{-1}(K)$. Therefore $J \cap I = \varphi^{-1}(K) \cap I = K$, i.e. $J \cap I \in \mathrm{val}_I(a)$.

Conversely, let $M \in \mathrm{val}_I(a)$. Then $\varphi^{-1}(M)$ is contained in some $N \in \mathrm{val}_A(a)$. We have $M = \varphi^{-1}(M) \cap I \subseteq N \cap I$ and $a \in N \cap I$, hence $M = N \cap I$, which means $\varphi^{-1}(M) = N$. Therefore $\varphi^{-1}(M) \in \mathrm{val}_A(a)$.

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РЕГУЛЯРНЫЕ ИДЕАЛЫ В АВТОМЕТРИЗОВАННЫХ АЛГЕБРАХ

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Резюме

В статье введены регулярные идеалы в автометризованных алгебрах и показаны их свойства в некоторых классах этих алгебр.