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GEODETIC LINE, MIDDLE AND TOTAL GRAPHS

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1. In the note presented by a graph we mean an undirected, finite graph without loops and multiple edges. By K_n we denote the complete graph on n vertices. A star, a cycle, and a path having n edges is denoted by $K_{1,n}$, C_n, and P_n, respectively. Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. The distance $d_G(v, u)$ between the vertices $v, u \in V(G)$ is the length of the shortest path between v and u in G; if v and u are not joined in G, we define $d_G(v, u) = \infty$. The diameter $d(G)$ of G is defined by $d(G) = \max\{d_G(v, u): : v, u \in V(G)\}$.

A graph G is said to be weakly geodetic if every two vertices v and u of G with distance $d_G(v, u) = 2$ are joined by exactly one shortest path. A graph G is geodetic if for any two vertices v and u of G there exists at most one shortest path between them. A graph G is called to be strongly geodetic if and only if every two vertices of G are joined by at most one path of length less than or equal to the diameter of G. Evidently, every strongly geodetic graph is geodetic, and every geodetic graph is weakly geodetic. Note that a graph G is weakly geodetic if and only if G contains no induced subgraph isomorphic to C_4 or $K_4 - e$ in Fig. 1. It is obvious that a graph G is geodetic if and only if each block of G is geodetic. A forest is a unique disconnected strongly geodetic graph. Moreover, it is easy to observe that if a strongly geodetic graph G contains a triangle, then G is a complete graph. The problem of characterizing geodetic graphs was first raised in [9] and is still open. We refer the reader to [5, 6, 10, 11] for surveys of results and open problems concerning geodetic graphs.

A question discussed in [3] is the following: For a graphical property A, what property must a graph G possess for the line graph $L(G)$ to have property A? This note answers that question in the case where A is the property that a graph is weakly geodetic, geodetic, and strongly geodetic, respectively. Additionally, we characterize all graphs G whose middle graphs $M(G)$ (total graphs $T(G)$, resp.) are weakly geodetic, geodetic, and strongly geodetic, respectively.

Let us recall that the line (total, resp.) graph $L(G)$ ($T(G)$, resp.) of a graph G is the graph whose set of vertices is in one-to-one correspondence with the set of edges (edges and vertices, resp.) of the graph G, with two vertices of $L(G)$ ($T(G)$, resp.) being adjacent if and only if the corresponding edges (elements, resp.) of G are adjacent (adjacent or incident, resp.). For and edge e of G, let $\bar{e}$

denote the vertex of $L(G)$ ($T(G)$, resp.) corresponding to the edge e. The middle graph of G, denoted $M(G)$, is the intersection graph $\Omega(F)$ on the set $V(G)$ of the family $F = \{\{x\} : x \in V(G)\} \cup E(G)$. $M(G)$ may also be defined as the line graph $L(G \circ K_1)$ (see [1, 8]), where $G \circ K_1$ is the graph obtained by taking G and $|V(G)|$ copies of K_1 and joining the i-th vertex of G to the i-th copy of K_1.

2. In this section we shall determine all graphs whose line graphs (middle graphs, total graphs, resp.) are weakly geodetic.

Theorem 1. *The line graph $L(G)$ of a graph G is weakly geodetic if and only if G does not contain any of the graphs C_4 and H_4 (see Figure 1) as a subgraph.*

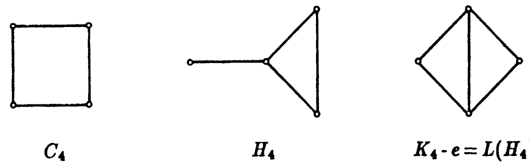

Fig. 1

Proof. Let us first observe that a graph G does not contain any of the graphs C_4 and H_4 as a subgraph if and only if $L(G)$ does not contain any of the graphs $C_4 = L(C_4)$ and $K_4 - e = L(H_4)$ as an induced subgraph. This fact coupled with the observation that a graph is weakly geodetic if and only if it contains neither C_4 nor $K_4 - e$ as an induced subgraph, implies the desired result. □

Corollary 1. *The middle graph $M(G)$ of a graph G is weakly geodetic if and only if neither C_3 nor C_4 is a subgraph of G.*

Proof. Since neither C_4 nor C_3 is a subgraph of G if and only if neither C_4 nor H_4 is a subgraph of $G \circ K_1$, the result follows from Theorem 1 and the fact that $M(G) = L(G \circ K_1)$. □

Theorem 2. *The total graph $T(G)$ of a graph G is weakly geodetic if and only if every connected component of G has at most one edge.*

Proof. One implication follows from the fact that the graphs $T(K_1) = K_1$ and $T(K_2) = K_3$ are weakly geodetic. On the other hand, if a graph G contains two adjacent edges, say uv and vw, then the subgraph of $T(G)$ induced by the vertices u, v, $\overline{uv}$, $\overline{vw}$ is isomorphic to $K_4 - e$ and $T(G)$ is not weakly geodetic. □

3. We now turn our attention to geodetic line graphs, geodetic middle graphs, and geodetic total graphs. We start with two auxiliary lemmas.

Lemma 1. *If $\overline{vu}$ and $\overline{wt}$ are different vertices of the line graph $L(G)$ of G, then*

$$d_{L(G)}(\overline{uv}, \overline{wt}) = \min\{d_G(v, w), d_G(v, t), d_G(u, w), d_G(u, t)\} + 1.$$

Proof. Let $\overline{vu}$ and $\overline{wt}$ be different vertices of $L(G)$ and suppose that

$$\min\{d_G(v, w), d_G(v, t), d_G(u, w), d_G(u, t)\} + 1 = d_G(v, w) + 1 = m + 1.$$

The result is obvious if $m = 0$. Thus assume that $m > 0$ and let $(v = v_0, v_1, \dots, w_m = w)$ be any shortest $v - w$ path in G. Then $(\overline{uv}, \overline{v_0v_1}, \dots, \overline{v_{m-1}v_m}, \overline{wt})$ is a $\overline{uv} - \overline{wt}$ path in $L(G)$ and therefore $d_{L(G)}(\overline{uv}, \overline{wt}) \leq m + 1$. We now claim that $d_{L(G)}(\overline{uv}, \overline{wt}) = m + 1$. Suppose to the contrary that $d_{L(G)}(\overline{uv}, \overline{wt}) = k < m + 1$. Let $(\overline{x_0y_0} = \overline{uv}, \overline{x_1y_1}, \dots, \overline{x_ky_k} = \overline{wt})$ be any shortest $\overline{uv} - \overline{wt}$ path in $L(G)$ and let z_i be a unique common vertex of the edges x_iy_i and $x_{i+1}y_{i+1}$ in G $(i = 0, 1, \dots, k - 1)$. Then $(z_0, z_1, \dots, z_{k-1})$ is a path joining $z_0 \in \{v, u\}$ to $z_{k-1} \in \{w, t\}$ in G and therefore

$$\min\{d_G(v, w), d_G(v, t), d_G(u, w), d_G(u, t)\} + 1 = d_G(v, w) + 1 \leq k < m + 1,$$

a contradiction to $m = \min\{d_G(v, w), d_G(v, t), d_G(u, w), d_G(u, t)\}$. □

Lemma 2. *If a graph G is nongeodetic, then its line graph L(G) is nongeodetic.*

Proof. Assume that G is a nongeodetic graph. Then G contains two vertices v and u joined by two different shortest paths, say, by the paths $P = (x_0 = v, x_1, \dots, x_n = u)$ and $Q = (y_0 = v, y_1, \dots, y_n = u)$. Without loss of generality, we may assume that the paths P and Q are internally disjoint. (For if not, then the vertices v, u and the paths P, Q may be replaced by the vertices $x_i = y_i$, $x_{i+k} = y_{i+k}$, and the paths $P' = (x_i, x_{i+1}, \dots, x_{i+k})$, $Q' = (y_i, y_{i+1}, \dots, y_{i+k})$, respectively, where i is the smallest integer belonging to $\{0, 1, \dots, n - 2\}$ and such that $x_{i+1} \neq y_{i+1}$, while k is the greatest positive integer such that $i + k \leq n$ and $x_j \neq y_j$ for every $j \in \{i + 1, \dots, i + k - 1\}$.) Then we have $d_G(v, u) = n$, $d_G(v, y_{n-1}) = d_G(x_1, u) = n - 1$, $d_G(x_1, y_{n-1}) \geq n - 1$ and therefore $d_{L(G)}(\overline{vx_1}, \overline{uy_{n-1}}) = n$ by Lemma 1. Thus $(\overline{vx_1}, \overline{x_1x_2}, \dots, \overline{x_{n-1}u}, \overline{uy_{n-1}})$ and $(\overline{vx_1}, \overline{vy_1}, \overline{y_1y_2}, \dots, \overline{y_{n-1}u})$ are different shortest $\overline{vx_1} - \overline{uy_{n-1}}$ paths in $L(G)$, and hence $L(G)$ is nongeodetic. □

Now we are ready to prove a characterization of graphs whose line graphs are geodetic.

Theorem 3. *Let G be a connected graph with at least one edge. Then the line graph L(G) is geodetic if and only if G is a tree or an odd cycle.*

Proof. If G is an odd cycle, $G = C_{2n+1}$ $(n \geq 1)$, then certainly $L(G) = C_{2n+1}$ is geodetic. If G is a tree, then every block of $L(G)$ is a complete graph and $L(G)$ is geodetic.

Conversely, suppose $L(G)$ is geodetic. By Lemma 2, G is geodetic. We claim that G is an odd cycle or a tree. For if not, then let C be any shortest cycle of G. From the choice of C it follows that $d_C(v, u) = d_G(v, u)$ for every two vertices v and u of C. Moreover, since G is geodetic, C has an odd length, say $2n + 1$.

Let $x_1, x_2, \ldots, x_{2n+1}$ be the consecutive vertices of C and let $x_0 \in V(G) - V(C)$ be a vertex adjacent to a vertex of C. Without loss of generality, we may assume that x_0 is adjacent to x_1. Let us now observe that $d_G(x_1, x_{n+1}) = d_G(x_1, x_{n+2}) = n$, $d_G(x_0, x_{n+1}) \geq n$, $d_G(x_0, x_{n+2}) \geq n$, and therefore we have $d_{L(G)}(\overline{x_0x_1}, \overline{x_{n+1}x_{n+2}}) = n + 1$ by Lemma 1. Thus $(\overline{x_0x_1}, \overline{x_1x_2}, \ldots, \overline{x_{n+1}x_{n+2}})$ and $(\overline{x_0x_1}, \overline{x_1x_{2n+1}}, \overline{x_{2n+1}x_{2n}}, \ldots, \overline{x_{n+2}x_{n+1}})$ are different shortest $\overline{x_0x_1} - \overline{x_{n+1}x_{n+2}}$ paths in $L(G)$, and hence $L(G)$ is nongeodetic. This contradicts our assumption. □

Corollary 2. *The middle graph $M(G)$ of a connected graph G is geodetic if and only if G is a tree.*

Proof. If G is a tree, then $G \circ K_1$ is a tree, and therefore $M(G) = L(G \circ K_1)$ is geodetic by Theorem 3. On the other hand, if G is not a tree, then $G \circ K_1$ is neither a tree nor an odd cycle, and therefore $M(G) = L(G \circ K_1)$ is nongeodetic by Theorem 3. □

Theorem 4. *The total graph $T(G)$ of a graph G is geodetic if and only if every connected component of G has at most one edge.*

The proof of Theorem 4 is similar to the proof of Theorem 2, so it will be omitted. We conclude this section with a necessary condition for a graph to be geodetic.

Corollary 3. *If a connected graph H is geodetic, then H is either a line graph of a tree or an odd cycle, or H contains $K_{1,3}$ as an induced subgraph and does not contain $K_4 - e$ as an induced subgraph.*

Proof. Assume that a graph H is connected and geodetic. Certainly, $K_4 - e$ is not an induced subgraph of H. Hence, if H is not a line graph, then from Beineke's Theorem [4] it follows that $K_{1,3}$ is an induced subgraph of H. If H is a line graph, then the result follows from Theorem 3. □

4. In this section we characterize graphs whose line graphs (middle graphs, total graphs, resp.) are strongly geodetic.

Theorem 5. *Let G be a nonempty graph without isolated vertices. Then the line graph $L(G)$ is strongly geodetic if and only if G is either an odd cycle, or a star, or a family of disjoint paths.*

Proof. It is obvious that $L(G)$ is strongly geodetic if G is either an odd cycle, or a star, or a family of disjoint paths.

Conversely, suppose $L(G)$ is strongly geodetic. If $L(G)$ is disconnected, then it is a forest. From this and the fact that $L(G)$ does not contain $K_{1,3}$ as an induced subgraph (see [4]), it follows that $L(G)$ is a family of disjoint paths. Hence G is a family of disjoint paths. Now assume that $L(G)$ is connected. Then G is connected and according to Theorem 3 the graph G is either an odd cycle or a tree. If G is a tree, then it is either a path or it contains a vertex of degree at least three. In the latter case $L(G)$ contains a triangle and therefore $L(G)$ is a complete graph. This combined with the assumption that G is a tree, implies that G is a star. □

The next two results follow immediately from Theorem 5 and Theorem 4, respectively.

Corollary 4. *The middle graph $M(G)$ of a graph G is strongly geodetic if and only if every connected component of G has at most one edge.* □

Corollary 5. *K_2 is the only nontrivial graph G such that $T(G)$ is strongly geodetic..* □

5. A graph H is said to be a line (middle, total, resp.) graph if $H = L(G)$ ($H = M(G)$, $H = T(G)$, resp.) for some graph G. Let $\mathscr{L}$, $\mathscr{M}$, $\mathscr{T}$, $\mathscr{W}$, $\mathscr{G}$, $\mathscr{S}$, denote the family of all nonempty line graphs, middle graphs, total graphs,

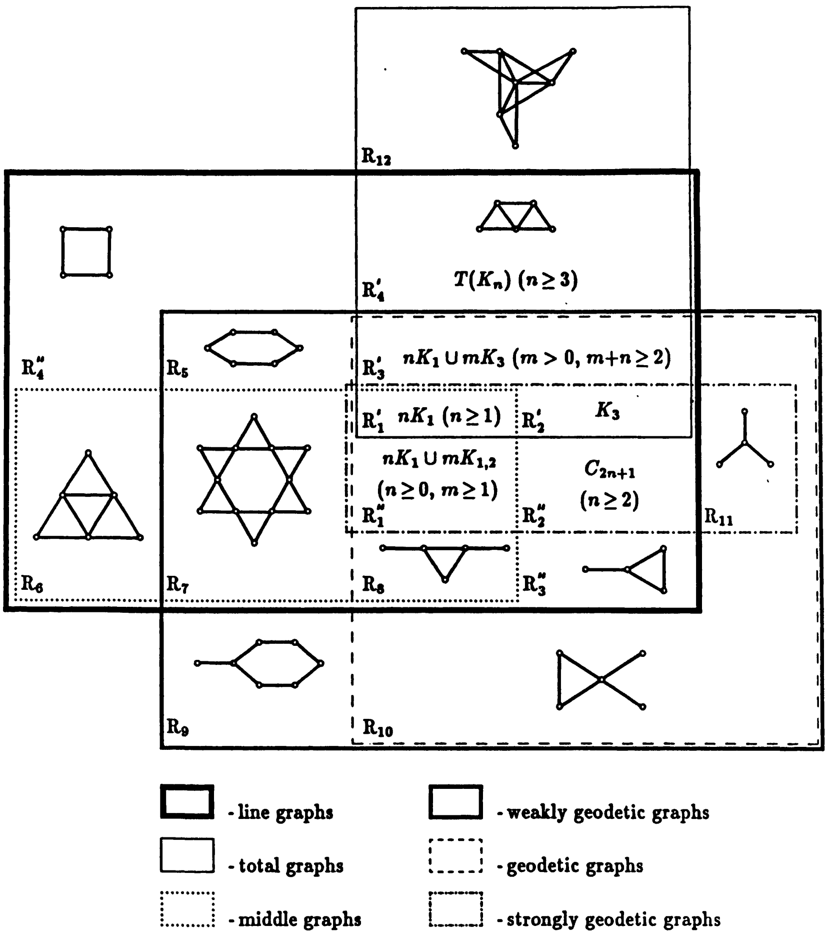

Fig. 2

weakly geodetic graphs, geodetic graphs, and strongly geodetic graphs, respectively. The set $\mathscr{L} \cup \mathscr{T} \cup \mathscr{W}$ can be partitioned into twelve subsets $\mathbf{R}_i$, where $\mathbf{R}_1 = \mathscr{M} \cap \mathscr{S}$, $\mathbf{R}_2 = \mathscr{L} \cap \mathscr{S} - \mathscr{M}$, $\mathbf{R}_3 = (\mathscr{L} \cap \mathscr{G}) - (\mathscr{M} \cup \mathscr{S})$, $\mathbf{R}_4 = \mathscr{L} - (\mathscr{M} \cup \mathscr{W})$, $\mathbf{R}_5 = (\mathscr{L} \cap \mathscr{W}) - (\mathscr{M} \cup \mathscr{G})$, $\mathbf{R}_6 = \mathscr{M} - \mathscr{W}$, $\mathbf{R}_7 = (\mathscr{M} \cap \mathscr{W}) - \mathscr{G}$, $\mathbf{R}_8 = (\mathscr{M} \cap \mathscr{G}) - \mathscr{S}$, $\mathbf{R}_9 = \mathscr{W} - (\mathscr{L} \cup \mathscr{G})$, $\mathbf{R}_{10} = \mathscr{G} - (\mathscr{L} \cup \mathscr{S})$, $\mathbf{R}_{11} = \mathscr{S} - \mathscr{L}$, $\mathbf{R}_{12} = \mathscr{T} - (\mathscr{L} \cup \mathscr{W})$. It follows from Theorem 2 that $\mathscr{T} \cap (\mathscr{W} - \mathscr{L}) = \emptyset$ and therefore $\mathscr{T}$ is disjoint from $\mathbf{R}_9$, $\mathbf{R}_{10}$, $\mathbf{R}_{11}$. Hence $\mathscr{T} - \mathbf{R}_{12} = \mathscr{T} \cap \mathscr{L}$. Since $\mathscr{T} \cap \mathscr{M} = \{nK_1 : n = 1, 2, \ldots\}$ (see [2]), $\mathscr{T}$ is disjoint from $\mathbf{R}_5$, $\mathbf{R}_6$, $\mathbf{R}_7$, $\mathbf{R}_8$, and hence $\mathscr{T} \cap \mathscr{L} \subset \mathbf{R}_1 \cup \mathbf{R}_2 \cup \mathbf{R}_3 \cup \mathbf{R}_4$. For $i \in \{1, 2, 3, 4\}$, let $\mathbf{R}_i' = \mathbf{R}_i \cap \mathscr{T}$ and $\mathbf{R}_i'' = \mathbf{R}_i - \mathscr{T}$. Fig 2 shows the relationships of the classes of line graphs, middle graphs and total graphs to the classes of weakly geodetic, geodetic and strongly geodetic graphs. An example is known for every region. From the results of Sections 2—4 and the fact that every connected graph belonging to $\mathscr{T} \cap \mathscr{L}$ is isomorphic to $T(K_{1,2})$ or $T(K_n) = L(K_{n+1})$ (see [7]) it follows that elements of the regions $\mathbf{R}_1'$, $\mathbf{R}_1''$, $\mathbf{R}_2'$, $\mathbf{R}_2''$, $\mathbf{R}_3'$, $\mathbf{R}_4'$ can be explicitly listed.

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ГЕОДЕЗИЧЕСКИЕ РЕБЕРНЫЕ, СРЕДНИЕ И ТОТАЛЬНЫЕ ГРАФЫ

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Резюме

В статье даётся характеристика тех графов, которых реберные, средние и тотальные графы являются слабо геодезическими, геодезическими и строго геодезическими.