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# PRINCIPAL CONGRUENCE RELATIONS AND PRINCIPAL TOLERANCES ON VARIETIES OF LATTICES

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The symbol $\Theta(a, b)$ $(T(a, b))$ denotes the principal congruence relation (tolerance) generated by the pair $\langle a, b\rangle$, i.e. the least congruence relation (tolerance, respectively) containing $\langle a, b\rangle$. As shown in [2], [5], the equality $\Theta(a, b) = T(a, b)$ holds on any distributive lattice. The aim of this note is to prove that this relation equality characterizes the variety of all distributive lattices. Using this fact we present a single lattice term describing the principal congruence relations on distributive lattices.

**Theorem 1.** *Let $\boldsymbol{V}$ be a variety of lattices. The following conditions are equivalent:*

(1) *$\boldsymbol{V}$ is the variety of distributive lattices;*

(2) *the equality relation $\Theta(a, b) = T(a, b)$ holds for any $a, b \in L \in \boldsymbol{V}$.*

Proof. (1) $\Rightarrow$ (2): As remarked previously, this part of the proof can be found in [2] or in [5].

(2) $\Rightarrow$ (1): Suppose to the contrary that the variety of lattices $\boldsymbol{V}$ contains a nondistributive member. Then, by the well-known Birkhoff criterion, $\boldsymbol{V}$ contains either the diamond $M_3$ or the pentagon $N_5$, i.e. the five-element lattices depicted in Figure 1.

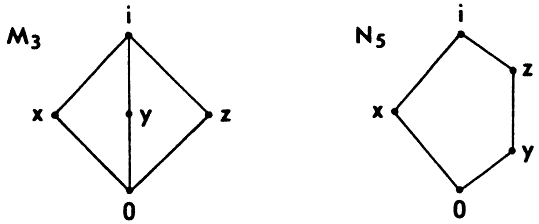


Fig. 1

Since $\boldsymbol{V}$ is closed under products and sublattices we infer that at least one of the lattices $M_3 \oplus 1$, $N_5 \oplus 1$, see Figure 2, belongs to $\boldsymbol{V}$.

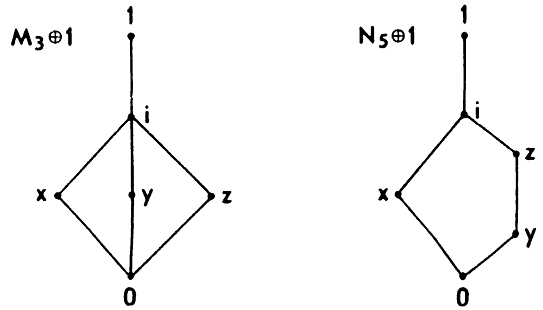

Fig. 2

Consider these two cases separately:

Case 1. Suppose that $M_3 \oplus 1 \in \boldsymbol{V}$. Then one can easily verify that $\Theta(y, 1) \neq T(y, 1)$, as Figure 3 illustrates.

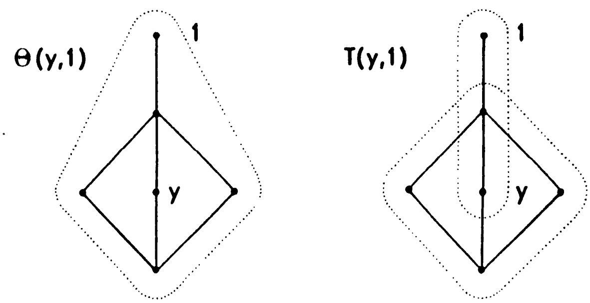

Fig. 3

Case 2. If $N_5 \oplus 1 \in \boldsymbol{V}$, then it is a routine to verify that $\Theta(z, 1) \neq T(z, 1)$, see Figure 4.

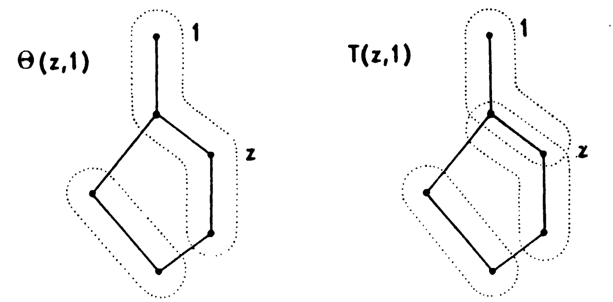

Fig. 4

Altogether we conclude that the equality $\Theta(a, b) = T(a, b)$ does not hold on $\boldsymbol{V}$ in any case. The proof is complete.

Many properties of a given variety can be derived from the form of its principal congruence relations. For these reasons every description of principal congruence relations is of some interest. From [3; Ex. 2.6] we quote the following description of principal congruence relations on distributive lattices:

$$\begin{aligned}
&\langle c, d\rangle \in \Theta(a, b) \quad \text{iff} \\
&c = \boldsymbol{p}_0(b, a, c, d) \\
&\boldsymbol{p}_0(a, a, c, d) = \boldsymbol{p}_1(a, a, c, d) \\
&\boldsymbol{p}_1(b, a, c, d) = \boldsymbol{p}_2(b, a, c, d) \\
&\boldsymbol{p}_2(a, a, c, d) = \boldsymbol{p}_3(a, a, c, d) \\
&d = \boldsymbol{p}_3(b, a, c, d),
\end{aligned}$$

where

$$\begin{aligned}
&\boldsymbol{p}_0(x_1, x_2, x_3, x_4) = [(x_1 \wedge x_2) \vee x_3] \wedge (x_3 \vee x_4), \\
&\boldsymbol{p}_1(x_1, x_2, x_3, x_4) = (x_1 \vee x_2 \vee x_3) \wedge (x_3 \vee x_4), \\
&\boldsymbol{p}_2(x_1, x_2, x_3, x_4) = [(x_1 \vee x_2) \wedge x_3] \vee x_4, \\
&\boldsymbol{p}_3(x_1, x_2, x_3, x_4) = (x_1 \wedge x_2 \wedge x_3) \vee x_4.
\end{aligned}$$

Making use of the equality $\Theta(a, b) = T(a, b)$ from Theorem 1 we state that one (sexenary) lattice term is enough for the description of congruence relations on distributive lattices.

**Theorem 2.** *Let* $\boldsymbol{V}$ *be a variety of lattices. The following conditions are equivalent:*

(1) $\boldsymbol{V}$ *is the variety of distributive lattices;*

(2) *for any* $a, b, c, d \in L \in \boldsymbol{V}$ *there holds*

$$\begin{gathered}
\langle c, d\rangle \in \Theta(a, b) \ \textit{iff} \\
c = \boldsymbol{p}(a, b, a, b, c, d) \\
d = \boldsymbol{p}(b, a, a, b, c, d),
\end{gathered}$$

where

$$\begin{gathered}
\boldsymbol{p}(x_1, x_2, x_3, x_4, x_5, x_6) = \boldsymbol{q}(\boldsymbol{r}(x_1, x_2, x_3, x_4, x_5, x_6), \\
\boldsymbol{r}(x_2, x_1, x_3, x_4, x_5, x_6), x_5, x_6), \\
\boldsymbol{q}(x_1, x_2, x_3, x_4) = (x_1 \vee x_3) \wedge (x_2 \vee x_4), \\
\boldsymbol{r}(x_1, x_2, x_3, x_4, x_5, x_6) = [(x_1 \wedge x_4) \vee (x_2 \wedge x_3) \vee (x_5 \wedge x_6)] \wedge (x_5 \vee x_6).
\end{gathered}$$

Proof. (1)$\Rightarrow$(2): (i) Suppose that $\langle c, d\rangle \in \Theta(a, b)$ holds for $a, b, c, d \in L \in \boldsymbol{V}$. Then

$$(*) \qquad a \wedge b \wedge c = a \wedge b \wedge d,$$

$$(**) \qquad a \vee b \vee c = a \vee b \vee d,$$

see [4; Thm 3, p. 74]. We want to prove that these equalities together with the assumption of distributivity give $c = \boldsymbol{p}(a, b, a, b, c, d)$ and $d = \boldsymbol{p}(b, a, a, b, c, d)$. To do this compute:

$$\boldsymbol{r}(a, b, a, b, c, d) =$$

$$= [(a \wedge b) \vee (b \wedge a) \vee (c \wedge d)] \wedge (c \vee d) = [(a \wedge b) \vee (c \wedge d)] \wedge (c \vee d) =$$

$$= [(a \wedge b) \wedge (c \vee d)] \vee [(c \wedge d) \wedge (c \vee d)], \text{ by distributivity,}$$

$$= (a \wedge b \wedge c) \vee (a \wedge b \wedge d) \vee (c \wedge d), \text{ by distributivity,}$$

$$= (a \wedge b \wedge c) \vee (c \wedge d), \text{ by } (*),$$

$$= [(a \wedge b) \vee d] \wedge c, \text{ by distributivity.}$$

(Clearly the last result can be expressed also in the form

$$\boldsymbol{r}(a, b, a, b, c, d) = [(a \wedge b) \vee c] \wedge d).$$

Further

$$\boldsymbol{r}(b, a, a, b, c, d) =$$

$$= [(b \wedge b) \vee (a \wedge a) \vee (c \wedge d)] \wedge (c \vee d) = [(a \vee b) \vee (c \wedge d)] \wedge (c \vee d) =$$

$$= [(a \vee b \vee c) \wedge (a \vee b \vee d)] \wedge (c \vee d), \text{ by distributivity,}$$

$$= (a \vee b \vee c) \wedge (c \vee d), \text{ by } (**),$$

$$= [(a \vee b \vee c) \wedge c] \vee [(a \vee b \vee d) \wedge d], \text{ by distributivity,}$$

$$= c \vee d.$$

Then

$$\begin{aligned} \boldsymbol{p}(a, b, a, b, c, d) &= \boldsymbol{q}([(a \wedge b) \vee d] \wedge c, c \vee d, c, d) = \\ &= [[[(a \wedge b) \vee d] \wedge c] \vee c] \wedge [(c \vee d) \vee d] = \\ &= c \wedge (c \vee d) = c, \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{p}(b, a, a, b, c, d) &= \boldsymbol{q}(c \vee d, [(a \wedge b) \vee c] \wedge d, c, d) = \\ &= [(c \vee d) \vee c] \wedge [[[(a \wedge b) \vee c] \wedge d] \vee d] = \\ &= (c \vee d) \wedge d = d, \end{aligned}$$

as claimed.

(ii) The converse implication is trivial since

$\langle \boldsymbol{p}(a, b, a, b, c, d), \boldsymbol{p}(b, a, a, b, c, d)\rangle \in \Theta(a, b)$ holds for any lattice term $\boldsymbol{p}$ applied to the elements $a, b, c, d$.

(2) ⇒ (1): Since $\langle \boldsymbol{p}(a, b, a, b, c, d), \boldsymbol{p}(b, a, a, b, c, d)\rangle \in T(a, b)$ we have verified the inclusion $\Theta(a, b) \subseteq T(a, b)$. Theorem 1 completes the proof.

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## ГЛАВНЫЕ КОНГРУЭНЦИИ И ГЛАВНЫЕ ТОЛЕРАНЦИИ В МНОГООБРАЗИЯХ РЕШЕТОК


Jaromír Duda



Резюме

Показано, что главные конгруэнции и главные толеранции совпадают только вслучае многообразия дистрибутивных решеток.