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NEIGHBOURHOOD REALIZATIONS OF COMPLEMENTS OF INTERSECTION GRAPHS

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This paper is a continuation of the study of a problem proposed by A. A. Zykov [1] at the symposium on graph theory in Smolenice in 1963. We consider undirected graphs without loops and multiple edges.

Let G be an undirected graph, let x be its vertex. By $N_G(x)$ we denote the subgraph of G induced by the set of all vertices which are adjacent to x in G. This subgraph is called the neighbourhood of x in G.

Let H be an undirected graph. If there exists a graph G such that $N_G(x) \cong H$ for each vertex x of G, then G is called a neighbourhood realization of H and H is said to be neighbourhood-realizable.

The problem of Zykov is the problem to characterize neighbourhood-realizable graphs. This problem has not yet been solved completely, but it was studied by many authors. From the papers concerning this topic we quote [4] and a survey paper [6].

Let $\mathscr{S}$ be family of sets. The intersection graph $G(\mathscr{S})$ of $\mathscr{S}$ is the graph whose vertex set is $\mathscr{S}$ and in which two vertices are adjacent if and only if they have a non-empty intersection. These graphs were also studied by many authors, eg. [2], [3].

This paper will concern complements of intersection graphs. A complement $\bar{G}(\mathscr{S})$ of an intersection graph $G(\mathscr{S})$ has the same vertex set as $G(\mathscr{S})$ and two vertices are adjacent in it if and only if they are disjoint. We shall investigate the neighbourhood realizability of such graps.

Theorem 1. *Let k, m be positive integers, let $k < m$. Let M be a set of cardinality m, let $\mathscr{S}$ be the family of all subsets of M which have the cardinality k. Then $\bar{G}(\mathscr{S})$ is neighbourhood-realizable.*

Proof. Let M_0 be a set of cardinality $m + k$, let $\mathscr{S}_0$ be the family of all subsets of M_0 which have the cardinality k. Let X be a vertex of $\bar{G}(\mathscr{S}_0)$, i.e $X \subset M_0$, $|X| = k$. The neighbourhood of X in $\bar{G}(\mathscr{S}_0)$ is the subfamily of $\mathscr{S}_0$ formed by all sets disjoint with X; in other words, it is the family of all subsets of $M_0 - X$ which have the cardinality k. As $|M_0 - X| = m$, this graph is isomorphic to $\bar{G}(\mathscr{S})$. As X was chosen arbitrarily, the graph $\bar{G}(\mathscr{S}_0)$ is a neighbourhood realization of $\bar{G}(\mathscr{S})$. □

Note that for $m = 2k + 1$ the graph $\bar{G}(\mathscr{S})$ is called the odd graph O_k; see, eg. [5].

Theorem 2. *Let k, m be positive integers, let $k < m$. Let P be a path of length m, let $\mathscr{S}$ be the family of edge sets of all paths of length k which are contained in P. Then $\bar{G}(\mathscr{S})$ is neighbourhood-realizable.*

Proof. Let C be a circuit of length $m + k$, let $\mathscr{S}_0$ be the family of edge sets of all paths of length k which are contained in C. Let X be a vertex of $\bar{G}(\mathscr{S}_0)$, i.e. the edge set of a path P_0 of length k contained in C. The neighbourhood of X in $\bar{G}(\mathscr{S}_0)$ is the subfamily of $\mathscr{S}_0$ formed by the edge sets of all paths of length k contained in the path P_1 in C connecting the end vertices of P_0 and distinct from P_0. The length of P_1 is m, therefore this graph is isomorphic to $\bar{G}(\mathscr{S})$. □

There exist geometrical analogies of the graph described in Theorem 2.

Theorem 3. *Let k, m be positive real numbers, let $k < m$. Let L be an interval of length m on the real line, let $\mathscr{S}$ be the family of all open intervals of length k which are contained in L. Then $\bar{G}(\mathscr{S})$ is neighbourhood-realizable.*

Proof. Let C be a circle of length $m + k$, let $\mathscr{S}_0$ be the family of sets of inner points of all arcs of length k of the circle C. Analogously as above we prove that $\bar{G}(\mathscr{S}_0)$ is a neighbourhood realization of $\bar{G}(\mathscr{S})$. □

The graph described in Theorem 3 has an uncountable vertex set. Quite analogously we may describe a similar graph with a countable set of vertices.

Theorem 4. *Let k, m be positive real numbers, let $k < m$. Let L be an interval of length m on the real line, let $\mathscr{S}$ be the family of all open intervals of length k with rational endpoints which are contained in L. Than $\bar{G}(\mathscr{S})$ is neighbourhood-realizable.*

Now we have a case when we do not mention lengths of intervals.

Theorem 5. *Let L be an open interval, let $\mathscr{S}$ be the family of all open intervals contained in L. Then $\bar{G}(\mathscr{S})$ is neighbourhood-realizable.*

Proof is analogous to the preceding ones. The neighbourhood realization of $\bar{G}(\mathscr{S})$ is the graph $\bar{G}(\mathscr{S}_0)$, where $\mathscr{S}_0$ is the family of sets of inner points of all arcs of a circle. □

Not that in this case the lengths of the intervals are not substantial. We need not even distinguish bounded intervals (a, b) and unbounded intervals (a, ∞), $(-\infty, b)$, $(-\infty, \infty)$, because in all the cases the graphs $\bar{G}(\mathscr{S})$ are isomorphic. Namely between any two open intervals there exists a bijection preserving the ordering; this bijection induces an isomorphism between the corresponding graphs.

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ОСУЩЕСТВЛЕНИЯ ОКРЕСНОСТЯМИ ДОПОЛНЕНИЙ ГРАФОВ ПЕРЕСЕЧЕНИЙ

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Резюме

Граф H называется осуществимым окресностями, если существует граф G, обладающий тем свойством, что для каждой вершины графа G подграф графа G, порожденный множеством всех вершин, смежных с x, изоморфен графу G. Граф пересечений $G(\mathscr{S})$ семейства множеств $\mathscr{S}$ есть граф, множеством вершин которого является $\mathscr{S}$ и в котором две вершины смжны тогда и только тогда, когда их пересечение непусто. Показано несколько примеров дополнений графов пересечеий, которые осуществимы окресностями.