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Mathematica Slovaca, Vol. 38 (1988), No. 1, 19--25

Persistent URL: http://dml.cz/dmlcz/136463

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THE DISTANCE BETWEEN VARIOUS ORIENTATIONS OF A GRAPH

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Various types of distances between isomorphism classes of graphs were studied by various authors; see [1]—[6]. Here we introduce the distance between isomorphism classes of mixed graphs which are obtained from the same undirected graph by orienting edges. This distance is a certain measure of how these graphs differ from each other. Therefore we shall start by some assertions concerning isomorphisms between different orientations of the same graph.

The considered graphs will be of two types: finite undirected graphs whose edges are coloured (arbitrarily) by two colours, green and red, and finite mixed graphs. (This concept includes also both directed and undirected graphs.) All graphs are without loops, any pair of vertices is joined by at most one edge.

Let H be a finite undirected graph whose edges are coloured green and red. The set of all green (or red) edges of H will be denoted by $E_g(H)$ (or $E_r(H)$ respectively). We admit the case when one of these sets is empty.

Consider the class $\mathcal{M}(H)$ of all mixed graphs which are obtained from H by assigning directions to all green edges (the red edges remain undirected). Further, by $\mathcal{M}^*(H)$ we denote the class of all isomorphism classes of $\mathcal{M}(H)$, i. e. such classes that two graphs from $\mathcal{M}(H)$ belong to the same class if and only if they are isomorphic.

If $\mathfrak{G}_1 \in \mathcal{M}^*(H)$, $\mathfrak{G}_2 \in \mathcal{M}^*(H)$, then the distance $d(\mathfrak{G}_1, \mathfrak{G}_2)$ of $\mathfrak{G}_1$ and $\mathfrak{G}_2$ is the minimum number of directed edges of a graph from $\mathfrak{G}_1$ whose directions must be reversed in order to obtain a graph from $\mathfrak{G}_2$. It is easy to prove that $\mathcal{M}^*(H)$ with this distance is a metric space.

For the sake of simplicity we shall sometimes speak about the distance between graphs instead of the distance between isomorphism classes of graphs; the distance $d(G_1, G_2)$ between the graphs G_1, G_2 from $\mathcal{M}(H)$ is the distance $d(\mathfrak{G}_1, \mathfrak{G}_2)$ between the classes $\mathfrak{G}_1$, $\mathfrak{G}_2$ from $\mathcal{M}^*(H)$ such that $G_1 \in \mathfrak{G}_1$, $G_2 \in \mathfrak{G}_2$. Note that two graphs having the distance zero need not be equal, but are isomorphic.

The case when all edges of H are red is trivial; then $\mathcal{M}(H) = \{H\}$.

Before the study of the distance we shall present some results on isomorphisms between graphs from $\mathcal{M}(H)$.

Let φ be an isomorphism of a graph $G_1 \in \mathcal{M}(H)$ onto a graph $G_2 \in \mathcal{M}(H)$. If u, v are two vertices of $V(H)$, then the pairs u, v and $\varphi(u)$, $\varphi(v)$ are either both joined by a green edge, or both joined by a red edge, or neither is joined. Therefore φ is also an automorphism of H which preserves the colours of edges. In the following we shall study the properties of these automorphisms from this viewpoint. The colour-preserving automorphisms of H form a group; we denote it by $CA(H)$.

If $\varphi \in CA(H)$, then it induces a permutation φ_v of the vertex set $V(H)$ of H and a permutation φ_e of the set $E_g(H)$. let e be a green edge of H which is contained in a cycle C of φ_e of the length k. This cycle consists of the edges e, $\varphi(e), \ldots, \varphi^{k-1}(e)$ and we have $\varphi^k(e) = e$. Thus e is fixed in φ^k. If both end vertices of e are fixed in φ^k, then we say that C is a cycle of the type I. If φ^k maps the end vertices of e mutually onto each other, then we say that C is a cycle of the type II.

Now let G_1, G_2 be two graphs from $\mathcal{M}(H)$. By $E^+(G_1, G_2)$ (or $E^-(G_1, G_2)$) we denote the set of green edges of H which have equal (or mutually opposite, respectively) directions in G_1 and G_2.

Theorem 1. *Let G_1, G_2 be two isomorphic graphs from $\mathcal{M}(H)$, let φ be an isomorphic mapping of G_1 onto G_2. Let C be a cycle of φ of the type* I. *Then the number of edges from $E^-(G_1, G_2)$ in C is even.*

Proof. Let k be the length of C. Then we denote $e_0 = e$ and $e_i = \varphi^i(e)$ for $i = 1, \ldots, k-1$. For each $i = 0, 1, \ldots, k-1$ let u_i (or v_i) be the initial (or terminal, respectively) vertex of e_i in G_1. Now let j be an integer, $0 \leqq j \leqq k-2$. If $e_{j+1} \in E^+(G_1, G_2)$, then also in G_2 its initial vertex is u_{j+1} and its terminal vertex is v_{j+1}. The edge e_{j+1} is the image of e_j in φ. If we consider φ as a mapping of G_1 onto G_2, then φ maps the edge e_j of G_1 directed from u_j to v_j onto the edge e_{j+1} of G_2 directed from u_{j+1} to v_{j+1}. Therefore $\varphi(u_j) = u_{j+1}$, $\varphi(v_j) = v_{j+1}$. If $e_{j+1} \in E^-(G_1, G_2)$, then in G_2 its initial vertex is v_{j+1} and its terminal vertex is u_{j+1}. Then $\varphi(u_j) = v_{j+1}$, $\varphi(v_j) = u_{j+1}$. By induction we obtain that $\varphi^j(u_0) = u_j$ and $\varphi^j(v_0) = v_j$ if and only if the number of edges from $E^-(G_1, G_2)$ among the edges $e_1, \ldots, e_j$ is even. This holds al so for $j = k$. As C is of the type I, we have $\varphi^k(u_0) = u_0$, $\varphi^k(v_0) = v_0$ and the number of edges from $E^-(G_1, G_2)$ must be even. □

Theorem 2. *Let G_1, G_2 be two isomorphic graphs from $\mathcal{M}(H)$, let φ be an isomorphic mapping of G_1 onto G_2. Let C be a cycle of φ of the type* II. *Then the number of edges from $E^-(G_1, G_2)$ in C is odd.*

Proof is analogous to the proof of Theorem 1. □

If G is a mixed graph, then by $G(\leftarrow)$ we shall denote the graph obtained from G by reversing the directions of all directed edges.

Theorem 3. *Let $G \in \mathcal{M}(H)$, let $G \cong G(\leftarrow)$. Then any isomorphic mapping φ of G onto $G(\leftarrow)$ has the following properties:*

(i) *If two vertices u, v belong both to cycles of φ_v of odd lengths, then they are either non-adjacent, or joined by a red edge.*

(ii) *If two vertices u, v belong to the same cycle of φ_v of the length k divisible by 4 and $v = \varphi^{k/2}(u)$ holds, then they are either non-adjacent, or joined by a red edge.*

Proof. Suppose that u, v belong both to cycles of φ_v of odd lengths and they are joined by a green edge e. Then e is in a cycle C of φ_e whose length is the least common multiple of the lengths of cycles to which u and v belong; this implies that it is also odd. Evidently all green edges are in $E^-(G, G(\leftarrow))$, therefore C contains an odd number of edges from $E^-(G, G(\leftarrow))$. According to Theorem 1 the cycle C cannot be of the type I. But neither C can be of the type II; in this case both u and v would belong to the same cycle of φ_v with the length equal to the length of C multiplied by 2, i.e. with an even length. This is a contradiction. Now suppose that u and v belong to the same cycle of φ_v of the length k divisible by 4, $v = \varphi^{k/2}(u)$ holds and u, v are joined by a green edge e. Then $\varphi^{k/2}(e) = e$ and e belongs to a cycle C of φ_e of the length $k/2$. As k is divisible by 4, the number $k/2$ is even and C contains an even number of edges from $E^-(G, G(\leftarrow))$, which contradicts Theorem 2. □

If H is a complete graph, all of whose edges are green, then $\mathcal{M}(H)$ is the class of all tournaments on $V(H)$.

Theorem 4. *let G be a tournament such that $G \cong G(\leftarrow)$, let φ be an isomorphism of G onto $G(\leftarrow)$. If G has an even number of vertices, then all cycles of φ_v have lengths congruent to 2 modulo 4. If G has an odd number of vertices, then φ_v has one cycle consisting of a fixed vertex and all other cycles of φ_v have lengths congruent to 2 modulo 4.*

Proof. In the case of tournaments any two vertices are joined by a green edge. Hence by Theorem 3 no two of them belong to cycles of φ_v of odd lengths. This implies that there is at most one cycle of φ_v of an odd length and this cycle (if it exists) has the length 1. Other cycles of φ_v must have even lengths, i.e. lengths congruent to 0 or 2 modulo 4. Suppose that there exists a cycle C of φ_v of the length $k \equiv 0 \pmod 4$. Then $k/2$ is even and there exists an edge joining a vertex u of C with $\varphi^{k/2}(u)$, which again contradicts Theorem 3. The number of vertices belonging to cycles of φ_v of even lengths is evidently fixed; hence a fixed vertex exists if and only if the number of vertices of G is odd. □

Theorem 5. *Let a finite undirected graph H with edges coloured in green and red be given. Let φ be a colour-preserving automorphism of G. Let E^- be a subset of $E_g(H)$ with the property that in each cycle of φ_e of the type I there is an even number of edges from E^- and in each cycle of φ_e of the type II there is an odd* *number of edges from E^-. Then there exist graphs G_1, G_2 from $\mathcal{M}(H)$ such that φ is an isomorphism of G_1 onto G_2 and $E^-(G_1, G_2) = E^-$.*

Proof. We shall prove the assertion by the construction of the graphs G_1, G_2. Let C be a cycle of φ_e, let k be its length. Choose and edge e_0 in C and direct it arbitrarily. For $i = 1, \ldots, k-1$ denote $e_i = \varphi^i(e_0)$. Now proceed by induction. Let $1 \leqq i \leqq k-1$ and let e_{i-1} be yet directed. Let u_{i-1} be its initial vertex and let v_{i-1} be its terminal vertex. If $e_i \in E^-$, direct it from $\varphi(v_{i-1})$ to $\varphi(u_{i-1})$; if $e_i \notin E^-$, direct it from $\varphi(u_{i-1})$ to $\varphi(v_{i-1})$. We do this with each cycle of φ_e; the mixed graph thus obtained will be G_1. The graph G_2 will be obtained from G_1 by reversing the directions of all edges from E^-. It is easy to prove that the graphs G_1, G_2 are the required graphs. □

In the case when $E^- = \emptyset$, we have $G_1 = G_2$ and φ is an automorphism of G_1. If $E^- = E_g(H)$, then $G_2 = G_1(\leftarrow)$. This yields us two corollaries.

Corollary 1. *Let a finite undirected graph H with edges coloured in green and red be given. Let φ be a colour-preserving automorphism of G. The graph $G \in \mathcal{M}(H)$ with the property that φ is an automorphism of G exists if and only if all cycles of φ_e are of the type* I.

Corollary 2. *Let a finite undirected graph H with edges coloured in green and red be given. Let φ be a colour-preserving automorphism of G. The graph $G \in \mathcal{M}(H)$ such that φ_e is an isomorphism of G onto $G(\leftarrow)$ exists if and only if all cycles of φ_e of the type* I *have even lengths and all cycles of φ_e of the type* II *have odd lengths.*

The necessity of the conditions follows from Theorem 1 and Theorem 2. We have yet another corollary.

Corollary 3. *For a finite undirected graph H with edges coloured in green and red the following assertions are equivalent:*

(i) *All green edges of H are fixed in all colour-preserving automorphisms of H.*

(ii) *Any two distinct graphs from $\mathcal{M}(H)$ are non-isomorphic.*

Now we may return to the distance. By $\mathcal{D}(H)$ we denote the graph whose vertex set is $\mathcal{M}^*(H)$ and in which two vertices $\mathfrak{G}_1$, $\mathfrak{G}_2$ are adjacent if and only if $d(\mathfrak{G}_1, \mathfrak{G}_2) = 1$.

Theorem 6. *The distance of any two vertices $\mathfrak{G}_1$, $\mathfrak{G}_2$ in $\mathcal{D}(H)$ in the graph-theoretical sense is equal to $d(\mathfrak{G}_1, \mathfrak{G}_2)$.*

Proof. Let $\mathfrak{G}$, $\mathfrak{G}'$ be two classes from $\mathcal{M}^*(H)$, let $d(\mathfrak{G}, \mathfrak{G}') = k$. Let $G \in \mathfrak{G}$, $G' \in \mathfrak{G}'$ and let G, G' be such that there exist exactly k edges of H whose directions in G and in G' are opposite. Let these edges be $e_1, \ldots, e_k$. For $i = 1, \ldots, k$ let G_i be the graph obtained from G by reversing the directions of edges $e_1, \ldots, e_i$ and let $\mathfrak{G}_i$ be the class from $\mathcal{M}^*(H)$ such that $G_i \in \mathfrak{G}_i$. Evidently the classes $\mathfrak{G}, \mathfrak{G}_1, \mathfrak{G}_2, \ldots, \mathfrak{G}_k = \mathfrak{G}'$ are vertices of a path of the length k from $\mathfrak{G}$ to $\mathfrak{G}'$ in $\mathcal{D}(H)$. On the other hand, if there exists a path of the length k it is evident that a graph from $\mathfrak{G}'$ can be obtained from a graph from $\mathfrak{G}$ by reversing the directions of at most k edges. This implies the assertion. □

Theorem 7. *Let H be a finite undirected graph whose edges are coloured in green and red, let all green edges of H be fixed in all automorphisms from $CA(H)$. Let k be the number of green edges of H. Then $\mathscr{D}(H)$ is the graph of the k-dimensional cube.*

Proof. Denote the green edges of H by $e_1, \ldots, e_k$. Choose a graph $G_0 \in \mathscr{M}(H)$. Then to each graph $G \in \mathscr{M}(H)$ assign a k-dimensional vector $\mathbf{v}(G) = (a_1, \ldots, a_k)$, where for each $i = 1, \ldots, k$ the number $a_i = 1$ if e_i has opposite directions in G_0 and G, and $a_i = 0$ otherwise. Evidently to each k-dimensional vector $\mathbf{v}$ whose coordinates are equal to 0 or 1 there exists $G \in \mathscr{M}(H)$ such that $\mathbf{v} = \mathbf{v}(G)$. The graph of the k-dimensional cube is the graph whose vertex set is the set of all such vectors and in which two vertices are adjacent if and only if they differ in exactly one coordinate. As each class of $\mathscr{M}^*(H)$ contains only one graph from $\mathscr{M}(H)$, evidently to each vertex of the graph of the k-dimensional cube a class of $\mathscr{M}^*(H)$ can be assigned and thus $\mathscr{D}(H)$ is obtained. □

In the sequel the distance between G and $G(\leftarrow)$ will be important.

Theorem 8. *Let $\mathfrak{G} \in \mathscr{M}^*(H)$, let $G \in \mathfrak{G}$. Then the maximum of distances $d(\mathfrak{G}, \mathfrak{G}')$ for all $\mathfrak{G}' \in \mathscr{M}^*(H)$ is at most $\frac{1}{2}|E_g| + \frac{1}{2}d(G, G(\leftarrow))$.*

Proof. Let $G' \in \mathfrak{G}'$, let F be the set of green edges of H which have opposite directions in G and G'. Let E_0 be the subset of the set E_g of the maximum cardinality with the property that by reversing directions of all edges of E_0 again a graph from $\mathfrak{G}$ is obtained. Evidently $|E_0| = |E_g| - d(G, G(\leftarrow))$. Let $E_1 = E_0 \cup F$, $E_2 = F - E_0$; then $|E_2| \leqq d(G, G(\leftarrow))$. If $|E_1| \leqq \frac{1}{2}|E_g| - \frac{1}{2}d(G, G(\leftarrow))$, then $d(\mathfrak{G}, \mathfrak{G}') \leqq |E_1| + |E_2| \leqq \frac{1}{2}|E_g| + \frac{1}{2}d(G, G(\leftarrow))$. If $|E_1| \geqq \frac{1}{2}|E_g| - \frac{1}{2}d(G, G(\leftarrow))$, we consider the graph G'' obtained from G by reversing the directions of all edges of E_0; we have $G'' \in \mathfrak{G}$. By reversing the directions of all edges of $(E_0 - E_1) \cup E_2$ we obtain $G'' \in \mathfrak{G}$ from $G' \in \mathfrak{G}'$. Hence $|(E_0 - E_1) \cup E_2| \geqq d(\mathfrak{G}, \mathfrak{G}')$. We have $|E_0 - E_1| = |E_g| - d(G, G(\leftarrow)) - |E_1|$ and thus $d(\mathfrak{G}, \mathfrak{G}') \leqq |(E_0 - E_1| \cup E_2| = |E_g| - d(G, G(\leftarrow)) - |E_1| + |E_2| \leqq |E_g| - d(G, G(\leftarrow)) - \frac{1}{2}|E_g| + \frac{1}{2}d(G, G(\leftarrow)) + d(G, G(\leftarrow)) = \frac{1}{2}|E_g| + \frac{1}{2}d(G, G(\leftarrow))$. □

Corollary 4. *The diameter of $\mathscr{D}(H)$ is at most*

$$\tfrac{1}{2}|E_g| + \tfrac{1}{2}\max\{d(G, G(\leftarrow)) | G \in \mathscr{M}(H)\}.$$

Corollary 5. The radius of $\mathscr{D}(H)$ is at most

$$\tfrac{1}{2}|E_g| + \tfrac{1}{2}\min\{d(G, G(\leftarrow)) | G \in \mathscr{M}(H)\}.$$

In the case of the graph H from Theorem 6 we have $d(G, G(\leftarrow)) = |E_g|$ for each $G \in \mathscr{M}(H)$. Both the radius and the diameter of $\mathscr{D}(H)$ are equal to $|E_g|$.

We shall describe an example of a graph H with the property that $G \cong G(\leftarrow)$ for each $g \in \mathscr{M}(H)$. Let the vertex set of H be $\{u_1, \ldots, u_k, v_1, \ldots, v_k\}$. Let the green edges of H be all edges $u_i v_i$ for $i = 1, \ldots, k$ and let the red edges of H be all edges $i_i u_j$ and $v_i v_j$ for $1 \leqq i \leqq k$, $1 \leqq j \leqq k$, $i \neq j$. The diameter of $\mathscr{D}(H)$ is then $[k/2]$ and its radius is $[k/4]$.

Finally we shall prove a theorem on tournaments.

Theorem 9. *Let H be a complete graph with n vertices, all of whose edges are green. Let $G \in \mathscr{M}(H)$. Then $d(G, G(\leftarrow)) \leqq \frac{1}{2}n(n-2)$ for n even and $d(G, G(\leftarrow)) \leqq \frac{1}{2}(n-1)^2$ for n odd.*

Proof. We choose a certain partition $\mathscr{P}$ of $V(H)$. If n is even, then each class of $\mathscr{P}$ consists of two vertices; if n is odd, then one class consists of one vertex and all the others of two vertices. Then there exists an automorphism φ of H such that each class of $\mathscr{P}$ is a cycle of φ_v. Then each cycle of φ_e either has the length 1 and the type II, or has the length 2 and the type I. According to Corollary 2 there exists a graph $G' \in \mathscr{M}(H)$ such that $G' \cong G'(\leftarrow)$ and φ maps G' onto $G'(\leftarrow)$. Moreover, we may construuct G' in such a way that we choose the direction of one edge in any cycle of φ_e. Hence we choose these directions to be the same as in G. The number of cycles of φ_e is $\frac{1}{4}n^2$ for n even and $\frac{1}{4}(n^2-1)$ for n odd. Hence $d(G, G') \leqq \frac{1}{2}n(n-1) - \frac{1}{4}n^2 = \frac{1}{4}n(n-2)$ for n even and $d(G, G') \leqq \frac{1}{4}(n-1)^2$ for n odd. Analogously $d(G(\leftarrow), G'(\leftarrow)) = d(G(\leftarrow), G') \leqq \frac{1}{4}n(n-2)$ and $d(G(\leftarrow), G'(\leftarrow)) = d(G(\leftarrow), G') \leqq \frac{1}{4}n - 1)^2$ for n odd. According to the triangle inequality $d(G, G(\leftarrow)) \leqq d(G, G') + d(G(\leftarrow), G')$ and this is at most $\frac{1}{2}m(n-2$ for n even and $\frac{1}{2}(n-1)^2$ for n odd. □

We have spoken about mixed graphs, but all results can be easily transferred to directed graphs in which pairs of oppositely directed edges joining the same pair of vertices are admitted. It suffices to replace each undirected edge by such a pair.

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Received June 6, 1986

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РАССТОЯНИЕ МЕЖДУ РАЗЛИЧНЫМИ ОРИЕНТАЦИЯМИ ГРАФА

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Резюме

В статье исследованы смешанные графы, которые получены из заданного неориентированного графа через введение ориентации на заданном подмножестве его множества ребер. Вводится и изучается некоторое расстояние между такими графами.