Mathematica Slovaca

Jaromír Duda

Unified polynomials for congruence principality

Mathematica Slovaca, Vol. 37 (1987), No. 4, 407--416

Persistent URL: http://dml.cz/dmlcz/136457

Math. Slovaca 37, 1987, No. 4, 407—416

UNIFIED POLYNOMIALS FOR CONGRUENCE PRINCIPALITY

JAROMÍR DUDA

Varieties having *Principal Compact Congruences* (briefly *PCC varietis*) were independently studied in papers [2] and [15]. Varieties with *Principal Compact Blocks* (*PCB varieties*, for short) and varieties with *Principal Compactly Generated Congruences* (so called *PCGC varieties*) were investigated in a later paper [4]. While the aim of [2] and [15] is to prove that PCC varieties form a Mal'cev class we state in [4] that PCC varieties, PCB varieties, and PCGC varieties are definable by suitable polynomial pairs. Congruence distributive varieties having the *Principal Intersection Property* (briefly the *PIP*) were already characterized by a pair of so-called intersection polynomials in the well-known paper [1]. The aim of the present note is to show that any of the mentioned polynomial pairs arises from one unified polynomial whenever some additional condition is assumed. For varieties of rings having the PIP a more detailed description is achieved.

In the sequel the description of finitely generated congruences will be needed. From [2] and [13] we quote.

Lemma 1. *Let* $\mathbf{V}$ *be a congruence permutable variety,* $x, y, a_1, b_1, \ldots, a_n, b_n \in A \in \mathbf{V}$. *Then*

(i) $\Theta(\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle) = R(\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle)$ (*the symbol on the right-hand side denotes the compatible reflexive binary relation generated by pairs* $\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle \in A \times A$);

(ii) $\langle x, y\rangle \in \Theta\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle)$ *iff* $x = \alpha(a_1, \ldots, a_n)$, $y = \alpha(b_1, \ldots, b_n)$ *for some n-ary algebraic function* α *over* A.

The concept of a tolerance, see [3], enables to generalize permutable varieties to *Principal Tolerance Trivial* varieties (recall that a variety is *PTT* whenever $\Theta(a, b)$ is equal to the principal tolerance $T(a, b)$ for any $a, b \in A \in \mathbf{V}$, see [3] again). Principal congruences on algebras form *PTT* varieties have the following simple description:

Lemma 2. *Let* $\mathbf{V}$ *be a PTT variety,* $x, y, a, b \in A \in \mathbf{V}$. *Then* $\langle x, y\rangle \in \Theta(a, b)$ *iff* $x = \beta(a, b)$, $y = \beta(b, a)$ *for some binary algebraic function* β *over* A.

Now we shall turn to the varieties cited in the introduction. Apparently the congruence blocks on an algebra form an algebraic lattice. Finitely generated blocks are exactly the compact elements of this lattice and so they are named *compact blocks* in this paper.

Theorem 1. *Let $\mathbf{V}$ be a variety. The following conditions are equivalent:*

(1) *$\mathbf{V}$ is a permutable variety having PCB (= Principal Compact Blocks);*

(2) *There exists a quinary polynomial $\mathbf{r}$ and quaternary polynomials $\mathbf{f}$, $\mathbf{g}$ such that*

$$\begin{aligned} y &= \mathbf{f}(\mathbf{r}(x, y, x, y, z), x, y, z), \\ x &= \mathbf{f}(\mathbf{r}(y, z, x, y, z), x, y, z), \\ x &= \mathbf{g}(\mathbf{r}(x, y, x, y, z), x, y, z), \\ z &= \mathbf{g}(\mathbf{r}(y, z, x, y, z), x, y, z), \\ x &= \mathbf{r}(x, x, x, x, x) \end{aligned}$$

hold in $\mathbf{V}$;

(3) *$\mathbf{V}$ is a congruence permutablevariety having a quinary idempotent polynomial $\mathbf{r}$ such that*

$$\mathbf{r}(x, y, x, y, z) = \mathbf{r}(y, z, x, y, z) \quad \textit{implies} \quad x = y = z.$$

Proof. (1)$\Rightarrow$(2). Let $A = F_{\mathbf{V}}(x, y, z)$ be the free algebra over three generators x, y, and z *in* $\mathbf{V}$. By hypothesis the congruence block $[x, y, z] = [a, b]$ for some $a, b \in A$. Since $[x, y, z] = [x]\,\Theta(\langle x, y\rangle, \langle y, z\rangle)$ and $[a, b] = [a]\,\Theta(a, b)$, we have $\langle a, b\rangle \in \Theta(\langle x, y\rangle, \langle y, z\rangle) = R(\langle x, y\rangle, \langle y, z\rangle)$ and so

$$\begin{aligned} a &= \mathbf{r}(x, y, x, y, z), \\ b &= \mathbf{r}(y, z, x, y, z) \end{aligned}$$

for some quinary polynomial $\mathbf{r}$, see Lemma 1. Further $\langle y, x\rangle, \langle x, z\rangle \in \Theta(a, b) = R(a, b)$ yield

$$\begin{aligned} y &= \mathbf{f}(\mathbf{r}(x, y, x, y, z), x, y, z), \\ x &= \mathbf{f}(\mathbf{r}(y, z, x, y, z), x, y, z), \\ x &= \mathbf{g}(\mathbf{r}(x, y, x, y, z), x, y, z), \\ z &= \mathbf{g}(\mathbf{r}(y, z, x, y, z), x, y, z) \end{aligned}$$

for some quaternary polynomials $\mathbf{f}$, $\mathbf{g}$ of $\mathbf{V}$. Finally the identity $x = \mathbf{r}(x, x, x, x, x)$ follows directly from the fact that $\mathbf{r}(x, y, x, y, z) = a \in [x, y, z]$.

(2)$\Rightarrow$(3). The congruence permutability is ensured by the Mal'cev polynomial $\mathbf{m}(x, y, z) = \mathbf{f}(\mathbf{r}(y, z, x, z, z), x, z, z)$.

The implication $\mathbf{r}(x, y, x, y, z) = \mathbf{r}(y, z, x, y, z) \Rightarrow x = y = z$ is a consequence of the identities (2) from our Theorem 1.

(3)$\Rightarrow$(1). Apply [4, Thm 1] with $\mathbf{p}(x, y, z) = \mathbf{r}(x, y, x, y, z)$, $\mathbf{q}(x, y, z) = \mathbf{r}(y, z, x, y, z)$.

Example 1. (a) For Boolean algebras we propose the unified 5-ary polynomial $\boldsymbol{r}$ in the form:

$$\boldsymbol{r}(x_1, x_2, x_3, x_4, x_5) = [(x_1 \oplus x_4) \vee (x_2 \oplus x_5)] \oplus x_5,$$

where the symbol $\oplus$ denotes the symmetrical difference. Then

$$\begin{aligned} \boldsymbol{r}(x, x, x, x, x) &= [(x \oplus x) \vee (x \oplus x)] \oplus x = x, \\ \boldsymbol{r}(x, y, x, y, z) &= [(x \oplus y) \vee (y \oplus z)] \oplus z, \\ \boldsymbol{r}(y, z, x, y, z) &= [(y \oplus y) \vee (z \oplus z)] \oplus z = z, \end{aligned}$$

and so $\boldsymbol{r}(x, y, x, y, z) = \boldsymbol{r}(y, z, x, y, z)$ implies $x = y = z$, as required.

(b) Any variety of l-groups is permutable and has PCB. Take $\boldsymbol{r}(x_1, x_2, x_3, x_4, x_5) = |x_1 - x_4| + |x_2 - x_5| + x_5$ (here $|x|$ stands for $x \vee -x$). Then

$$\begin{aligned} \boldsymbol{r}(x, x, x, x, x) &= |x - x| + |x - x| + x = x, \\ \boldsymbol{r}(x, y, x, y, z) &= |x - y| + |y - z| + z, \quad \text{and} \\ \boldsymbol{r}(y, z, x, y, z) &= |y - y| + |z - z| + z = z. \end{aligned}$$

The equality $\boldsymbol{r}(x, y, x, y, z) = \boldsymbol{r}(y, z, x, y, z)$ evidently implies $x = y = z$.

(c) Let $\boldsymbol{V}$ be a discriminator variety (with the ternaty polynomial $\boldsymbol{t}$ which is a discriminator on any SI member of $\boldsymbol{V}$). Then the following identities hold in $\boldsymbol{V}$

(i) $x = \boldsymbol{t}(x, y, y) = \boldsymbol{t}(y, y, x) = \boldsymbol{t}(x, y, x)$

(ii) $\boldsymbol{t}(x, y, z) = \boldsymbol{t}(x, y, \boldsymbol{t}(y, x, z))$, see [10] or [5].

Define $\boldsymbol{r}(x_1, x_2, x_3, x_4, x_5) = \boldsymbol{t}(x_1, x_3, \boldsymbol{t}(x_3, x_4, x_2))$. Then

$$\begin{aligned} \boldsymbol{r}(x, x, x, x, x) &= \boldsymbol{t}(x, x, \boldsymbol{t}(x, x, x)) = \boldsymbol{t}(x, x, x) = x, \\ \boldsymbol{r}(x, y, x, y, z) &= \boldsymbol{t}(x, x, \boldsymbol{t}(x, y, y)) = \boldsymbol{t}(x, y, y) = x, \quad \text{and} \\ \boldsymbol{r}(y, z, x, y, z) &= \boldsymbol{t}(y, x, \boldsymbol{t}(x, y, z)) = \boldsymbol{t}(y, x, z). \end{aligned}$$

Now let $\boldsymbol{r}(x, y, x, y, z) = \boldsymbol{r}(y, z, x, y, z)$, i.e. $x = \boldsymbol{t}(y, x, z)$. Then also $x = \boldsymbol{t}(x, x, z) = z$ and $x = \boldsymbol{t}(y, x, x) = y$. Then conclusion $x = y = z$ follows.

Theorem 2. *Let $\boldsymbol{V}$ be a variety. The following conditions are equivalent:*

(1) *$\boldsymbol{V}$ is a congruence permutable variety having PCC (= Principal Compact Congruences);*

(2) *There exist a sexnary polynomial $\boldsymbol{s}$ and quinary polynomials $\boldsymbol{f}$, $\boldsymbol{g}$ such that*

$$\begin{aligned} y &= \boldsymbol{f}(\boldsymbol{s}(x, u, x, y, u, v), x, y, u, v), \\ x &= \boldsymbol{f}(\boldsymbol{s}(y, v, x, y, u, v), x, y, u, v), \\ u &= \boldsymbol{g}(\boldsymbol{s}(x, u, x, y, u, v), x, y, u, v), \\ v &= \boldsymbol{g}(\boldsymbol{s}(y, v, x, y, u, v), x, y, u, v) \end{aligned}$$

hold in $\boldsymbol{V}$;

(3) *$\boldsymbol{V}$ is a congruence permutable variety having a sexnary polynomial $\boldsymbol{s}$ such that*

$$\mathbf{s}(x, u, x, y, u, v) = \mathbf{s}(y, v, x, y, u, v) \quad \textit{implies} \quad x = y \quad \textit{and} \quad u = v.$$

Proof. (1) ⇒ (2). Consider the congruence $\Theta(\langle x, y\rangle, \langle u, v\rangle)$ on $A = F_{\mathbf{V}}(x, y, u, v)$. By hypothesis $\Theta(\langle x, y\rangle, \langle u, v\rangle) = \Theta(a, b)$ for some elements $a, b \in A$. From $\langle a, b\rangle \in \Theta(\langle x, y\rangle, \langle u, v\rangle) = R(\langle x, y\rangle, \langle u, v\rangle)$ we find that

$$a = \mathbf{s}(x, u, x, y, u, v),$$
$$b = \mathbf{s}(y, v, x, y, u, v)$$

for suitable sexnary polynomial $\mathbf{s}$ of $\mathbf{V}$. On the other hand $\langle y, x\rangle, \langle u, v\rangle \in \Theta(a, b) = R(a, b)$ imply

$$y = \mathbf{f}(\mathbf{s}(x, u, x, y, u, v), x, y, u, v),$$
$$x = \mathbf{f}(\mathbf{s}(y, v, x, y, u, v), x, y, u, v),$$
$$u = \mathbf{g}(\mathbf{s}(x, u, x, y, u, v), x, y, u, v),$$
$$v = \mathbf{g}(\mathbf{s}(y, v, x, y, u, v), x, y, u, v)$$

for some quinary polynomials $\mathbf{f}$, $\mathbf{g}$ of $\mathbf{V}$.

(2) ⇒ (3). One easily sees that $\mathbf{m}(x, y, z) = \mathbf{f}(\mathbf{s}(y, z, x, z, z, z), x, z, z, z)$ is a Mal'cev polynomial.

The implication $\mathbf{s}(x, u, x, y, u, v) = \mathbf{s}(y, v, x, y, u, v) \Rightarrow x = y$ and $u = v$ follows directly from the identities (2).

(3) ⇒ (1). Apply [4, Thm 2] with $\mathbf{p}(x, y, u, v) = \mathbf{s}(x, u, x, y, u, v)$, $\mathbf{q}(x, y, u, v) = \mathbf{s}(y, v, x, y, u, v)$.

Examples 2. (a) l-groups constitute a permutable variety with PCC. Take $\mathbf{s}(x_1, x_2, x_3, x_4, x_5, x_6) = |x_1 - x_3| + |x_2 - x_5|$. Then $\mathbf{s}(x, u, x, y, u, v) = 0$ and $\mathbf{s}(y, v, x, y, u, v) = |y - x| + |v - u|$. Condition (3) from Theorem 2 is fulfilled.

(b) Any variety of Heyting algebras has permutable congruences and PCC. Condition (3) from Theorem 2 holds for

$$\mathbf{s}(x_1, x_2, x_3, x_4, x_5, x_6) = (x_1 \Leftrightarrow x_3) \wedge (x_2 \Leftrightarrow x_5)$$

(here $x \Leftrightarrow y$ abbreviates $(x \Rightarrow y) \wedge (y \Rightarrow x)$).

(c) Let $\mathbf{V}$ be a discriminator variety with the ternary discriminator $\mathbf{t}$ on SI members of $\mathbf{V}$. Put $\mathbf{s}(x_1, x_2, x_3, x_4, x_5, x_6) = \mathbf{t}(x_1, \mathbf{t}(x_3, x_1, x_4), x_2)$. Then

$$\mathbf{s}(x, u, x, y, u, v) = \mathbf{t}(x, \mathbf{t}(x, x, y), u) = \mathbf{t}(x, y, u) \quad \text{and}$$
$$\mathbf{s}(y, v, x, y, u, v) = \mathbf{t}(y, \mathbf{t}(x, y, y), v) = \mathbf{t}(y, x, v).$$

Apparently $\mathbf{t}(x, y, u) = \mathbf{t}(y, x, v)$ implies $x = y$ and $u = v$.

Theorem 3. *Let $\mathbf{V}$ be a variety. The following conditions are equivalent:*

(1) *$\mathbf{V}$ is a permutable variety having PCGC (= Principal Compactly Generated Congruences), i.e. any congruence $\Theta(a_1, \ldots, a_n) = \Theta(\{a_1, \ldots, a_n\} \times \{a_1, \ldots, a_n\})$, $a_1, \ldots, a_n \in A \in \mathbf{V}$, is principal;*

(2) *There exist a quinary polynomial* $\boldsymbol{t}$ *and quaternary polynomials* $\boldsymbol{f}$, $\boldsymbol{g}$ *such that*

$$\begin{aligned} y &= \boldsymbol{f}(\boldsymbol{t}(x, y, x, y, z), x, y, z), \\ x &= \boldsymbol{f}(\boldsymbol{t}(y, z, x, y, z), x, y, z), \\ x &= \boldsymbol{g}(\boldsymbol{t}(x, y, x, y, z), x, y, z), \\ z &= \boldsymbol{g}(\boldsymbol{t}(y, z, x, y, z), x, y, z) \end{aligned}$$

hold in $\boldsymbol{V}$;

(3) $\boldsymbol{V}$ *is a permutable variety having a quinary polynomial* $\boldsymbol{t}$ *such that*

$$\boldsymbol{t}(x, y, x, y, z) = \boldsymbol{t}(y, z, x, y, z) \quad \textit{implies} \quad x = y = z.$$

Proof. (1)$\Rightarrow$(2). Take $A = F_{\boldsymbol{V}}(x, y, z)$. Let $\Theta(x, y, z)$ be the congruence on A generated by the Cartesian square $\{x, y, z\} \times \{x, y, z\}$. Then $\Theta(x, y, z) = \Theta(a, b)$ for some $a, b \in A$, by (1). Further $\langle a, b\rangle \in \Theta(x, y, z) = \Theta(\langle x, y\rangle, \langle y, z\rangle) = R(\langle x, y\rangle, \langle y, z\rangle)$, by Lemma 1. Consequently

$$\begin{aligned} a &= \boldsymbol{t}(x, y, x, y, z) \\ b &= \boldsymbol{t}(y, z, x, y, z) \end{aligned}$$

for a suitable quinary polynomial $\boldsymbol{t}$ of $\boldsymbol{V}$. On the other hand $\langle y, x\rangle, \langle x, z\rangle \in \Theta(x, y, z) = \Theta(a, b) = R(a, b)$ yield

$$\begin{aligned} y &= \boldsymbol{f}(\boldsymbol{t}(x, y, x, y, z), x, y, z), \\ x &= \boldsymbol{f}(\boldsymbol{t}(y, z, x, y, z), x, y, z), \\ x &= \boldsymbol{g}(\boldsymbol{t}(x, y, x, y, z), x, y, z), \\ z &= \boldsymbol{g}(\boldsymbol{t}(y, z, x, y, z), x, y, z) \end{aligned}$$

for some quaternary polynomials $\boldsymbol{f}$, $\boldsymbol{g}$ of $\boldsymbol{V}$.

(2)$\Rightarrow$(3). It is enough to verify that the ternary polynomial $\boldsymbol{m}(x, y, z) = \boldsymbol{f}(\boldsymbol{t}(y, z, x, y, z), x, y, z)$ is a Mal'cev polynomial.

(3)$\Rightarrow$(1). Put $\boldsymbol{p}(x, y, z) = \boldsymbol{t}(x, y, x, y, z)$ and $\boldsymbol{q}(x, y, z) = \boldsymbol{t}(y, z, x, y, z)$. [4, Thm 3] completes the proof.

Remark 1. Since PCGC varieties include PCB varieties as well as PCC varieties any variety already presented in Examples 1 and Examples 2 can be used to demonstrate the unified polynomial $\boldsymbol{t}$ from Theorem 3.

Proposition 1 ([1; Thm 2.8, Thm 2.9]). *Let* $\boldsymbol{V}$ *be a congruence distributive variety. The following conditions are equivalent*:

(1) $\boldsymbol{V}$ *has the PIP (= Principal Intersection Property), i.e. the congruence* $\Theta(a_1, b_1) \wedge \Theta(a_2, b_2)$ *is principal for any* $a_1, b_1, a_2, b_2 \in A \in \boldsymbol{V}$;

(2) *There exist quaternary intersection polynomials* $\boldsymbol{D}_0$, $\boldsymbol{D}_1$ *such that*

$$\Theta(a_1, b_1) \wedge \Theta(a_2, b_2) = \Theta(\boldsymbol{D}_0(a_1, b_1, a_2, b_2), \boldsymbol{D}_1(a_1, b_1, a_2, b_2))$$

for any $a_1, b_1, a_2, b_2 \in A \in \boldsymbol{V}$;

(3) *There exist quaternary polynomials* $\boldsymbol{D}_0$, $\boldsymbol{D}_1$ *such that* $\boldsymbol{D}_0(x, y, u, v) = \boldsymbol{D}_1(x, y, u, v)$ *iff* $x = y$ *or* $u = v$ *holds on any* SI *member of* $\boldsymbol{V}$.

Theorem 4. *Let* $\boldsymbol{V}$ *be an arithemetical variety. The following conditions are equivalent:*

(1) $\boldsymbol{V}$ *has the PIP;*

(2) *There exists a quinary polynomial* $\boldsymbol{i}$ *such that*
$\boldsymbol{i}(x, x, y, u, u) = \boldsymbol{i}(y, x, y, u, u)$ *holds on any* $A \in \boldsymbol{V}$, *and*
$\boldsymbol{i}(x, x, y, u, v) = \boldsymbol{i}(y, x, y, u, v)$ *implies* $x = y$ *or* $u = v$ *on any* SI *member of* $\boldsymbol{V}$;

(3) *There exists a quinary polynomial* $\boldsymbol{i}$ *such that*
$\boldsymbol{i}(x, x, y, u, v) = \boldsymbol{i}(y, x, y, u, v)$ *iff* $x = y$ *or* $u = v$ *holds on any* SI *member of* $\boldsymbol{V}$.

Proof. (1) $\Rightarrow$ (2). By Proposition 1 (2) we have 4-ary polynomials $\boldsymbol{D}_0$, $\boldsymbol{D}_1$ such that $\Theta(x, y) \wedge \Theta(u, v) = \Theta(\boldsymbol{D}_0(x, y, u, v), \boldsymbol{D}_1(x, y, u, v))$ for any $x, y, u, v \in A \in \boldsymbol{V}$. From $\langle \boldsymbol{D}_0(x, y, u, v), \boldsymbol{D}_1(x, y, u, v)\rangle \in \Theta(x, y) = R(x, y)$ we find that

$$\boldsymbol{D}_0(x, y, u, v) = \boldsymbol{i}(x, x, y, u, v) \quad \text{and}$$
$$\boldsymbol{D}_1(x, y, u, v) = \boldsymbol{i}(y, x, y, u, v)$$

for some quinary polynomial $\boldsymbol{i}$ of $\boldsymbol{V}$, see Lemma 1. The identity $\boldsymbol{i}(x, x, y, u, u) = \boldsymbol{i}(y, x, y, u, u)$ is immediate. If A is a SI algebra in $\boldsymbol{V}$ and $\boldsymbol{i}(x, x, y, u, v) = \boldsymbol{i}(y, x, y, u, v)$, then $\Theta(x, y) \wedge \Theta(u, v) = \Theta(\boldsymbol{i}(x, x, y, u, u), \boldsymbol{i}(y, x, y, u, u))$ and so $\Theta(x, y) = \omega$ or $\Theta(u, v) = \omega$ hold. Consequently $x = y$ or $u = v$, which was to be proved.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1). Put $\boldsymbol{D}_0(x_1, x_2, x_3, x_4) = \boldsymbol{i}(x_1, x_1, x_2, x_3, x_4)$ and $\boldsymbol{D}_1(x_1, x_2, x_3, x_4) = \boldsymbol{i}(x_2, x_1, x_2, x_3, x_4)$. Then $\boldsymbol{D}_0$, $\boldsymbol{D}_1$ are polynomials mentioned in Proposition 1 (3). The proof is complete.

Examples 3. (a) Heyting algebras constitute an arithmetical variety having the PIP; Define $\boldsymbol{i}(x_1, x_2, x_3, x_4, x_5) = (x_1 \Leftrightarrow x_3) \vee (x_4 \Leftrightarrow x_5)$. Consider the equality $\boldsymbol{i}(x, x, y, u, v) = \boldsymbol{i}(y, x, y, u, v)$ onan arbitrary SI Heyting algebra A. Then $\boldsymbol{i}(x, x, y, u, v) = \boldsymbol{i}(y, x, y, u, v)$ iff $(x \Leftrightarrow y) \vee (u \Leftrightarrow v) = 1$ iff $x \Leftrightarrow y = 1$ or $u \Leftrightarrow v = 1$ (by hypothesis A has the least nontrivial filter) if $x = y$ of $u = v$, as required.

(b) Any discriminator variety $\boldsymbol{V}$ is arithmetical and has the PIP; Take $\boldsymbol{i}(x_1, x_2, x_3, x_4, x_5) = \boldsymbol{t}(\boldsymbol{t}(x_1, x_3, x_4), \boldsymbol{t}(x_1, x_3, x_5), x_5)$. Then $\boldsymbol{i}(x_1, x_2, x_3, x_4, x_5) = \boldsymbol{n}(x_1, x_3, x_4, x_5)$ where $\boldsymbol{n}$ denotes the so-called normal transform $\boldsymbol{n}(x, x, u, v) = u$ and $\boldsymbol{n}(x, y, u, v) = v$ for $x \neq v$. Then $v = \boldsymbol{n}(x, y, u, v) = u$, a contradiction.

(c) It follows directly from the preceding example (b) that any variety $\boldsymbol{V}$ of arithmetical rings has the PIP. This fact can be made more distinct by taking $\boldsymbol{i}(x_1, x_2, x_3, x_4, x_5) = (x_1 - x_3)(x_4 - x_5)$. Then $\boldsymbol{i}(x, x, y, u, v) = \boldsymbol{i}(y, x, y, u, v)$ iff $(x - y)(u - v) = 0$ iff $x = y$ or $u = v$, since any SI member from $\boldsymbol{V}$ is a finite field, see (11).

Something more can be stated for varieties of rings. First two auxiliary results:

Proposition 2 ([14] and [9]). *Any variety of arithmetical rings is a variety of commutative rings.*

Proof. Combine the following two facts:

(i) Any variety of arithmetical rings satisfies the identity $x^n = x$ for some $n > 1$, see [14, p. 38].

(ii) Let R be a ring in which for every $x \in R$ there exists an integer $n(x) > 1$ such that $x^{n(x)} = x$. Then R is commutative. See [9; Chap. 10.1, Thm 1].

Lemma 3. *Let R be a ring such that r divides r for every $r \in R$. Then $\Theta(0, x) \supseteq \Theta(0, y)$ iff x divides y for any $x, y \in R$.*

Proof. The equivalence $\Theta(0, x) \supseteq \Theta(0, y)$ iff $(x) \supseteq (y)$ ((x) denotes the ideal generated by an element x) is well known from the ring theory. The rest of the proof is evident.

Theorem 5. *Let $\boldsymbol{V}$ be a variety of rings. The following conditions are equivalent:*

(1) *$\boldsymbol{V}$ has the PIP;*

(2) *$\boldsymbol{V}$ is congruence distributive;*

(3) *For any $x, y \in R \in \boldsymbol{V}$ there hold*

 (a) *x divides x;*

 (b) *xy is the least common multiple of x and y.*

Proof. (1)$\Rightarrow$(2). Consider the principal congruences $\Theta(0, x)$, $\Theta(0, y)$ on the free ring $R = F_{\boldsymbol{V}}(x, y, z)$ with free generators x, y, and z. By hypothesis $\Theta(0, x) \wedge \Theta(0, y) = \Theta(\boldsymbol{p}_1(x, y, z), \boldsymbol{p}_2(x, y, z)) = \Theta(0, \boldsymbol{p}(x, y, z))$ where $\boldsymbol{p}(x, y, z) = \boldsymbol{p}_2(x, y, z) - \boldsymbol{p}_1(x, y, z)$. The ternary polynomial $\boldsymbol{p}$ evidently satisfies the identities $\boldsymbol{p}(x, 0, z) = \boldsymbol{p}(0, y, z) = 0$, which means that $\boldsymbol{p}$ is a commutator polynomial in the sense of [8]. Hence $\Theta(0, x) \wedge \Theta(0, y) = \Theta(0, \boldsymbol{p}(x, y, z)) \subseteq [\Theta(0, x), \Theta(0, y)]$ holds. Since the opposite inclusion is a general consequence of the commutator theory we have proved the equality $[\Theta(0, x), \Theta(0, y)] = \Theta(0, x) \wedge \Theta(0, y)$. Now take an arbitrary congruence $\Psi \in \operatorname{Con} R$ and elements $a, b \in R \in \boldsymbol{V}$. Then

$$[\Psi, \Theta(a, b)] = \left[\bigvee_{\langle u, v\rangle \in \Psi} \Theta(u, v), \Theta(a, b)\right] = \bigvee_{\langle u, v\rangle \in \Psi} [\Theta(u, v), \Theta(a, b)],$$

By the additivity of commutator, see e.g. [7]. Further

$$\bigvee_{\langle u, v\rangle \in \Psi} [\Theta(u, v), \Theta(a, b)] = \bigvee_{\langle u, v\rangle \in \Psi} (\Theta(u, v) \wedge \Theta(a, b))$$

and it is a routine to verify the equality

$$\bigvee_{\langle u, v\rangle \in \Psi} (\Theta(u, v) \wedge \Theta(a, b)) = \Psi \wedge \Theta(a, b).$$

So we find that $[\Psi, \Theta(a, b)] = \Psi \wedge \Theta(a, b)$; applying the same arguments to the second variable one easily sees that $[\Psi, \Phi] = \Psi \wedge \Phi$ for any congruences Ψ, Φ on $R = F_{\mathbf{V}}(x, y, z)$. Combining this fact with the above mentioned additivity of commutator we conclude that $F_{\mathbf{V}}(x, y, z)$ has distributive congruences. Apparently the same holds for any ring $R \in \mathbf{V}$.

(2) ⇒ (3). We have $x^n = x$, $n > 1$, by the Werner characterization of arithmetical varieties of rings, see the proof of Proposition 2. Hence x divides x.

The congruence distributivity of $\mathbf{V}$ implies $[\Theta(0, x), \Theta(0, y)] = \Theta(0, x) \wedge \wedge \Theta(0, y)$ for any $x, y \in R \in \mathbf{V}$, see [7] again. Simultaneously $[\Theta(0, x), \Theta(0, y)] = = \Theta(0, xy)$, by Proposition 2. Fruther let x and y divide an element $t \in R$. Then $\Theta(0, t) \subseteq \Theta(0, x) \wedge \Theta(0, y)$ and so $\Theta(0, t) \subseteq \Theta(0, xy)$. Consequently xy divides t, by Lemma 3.

(3) ⇒ (1). Let $x, y \in R \in \mathbf{V}$. We have to prove that $\Theta(0, x) \wedge \Theta(0, y) = = \Theta(0, xy)$. The inclusion $\Theta(0, x) \wedge \Theta(0, y) \supseteq \Theta(0, xy)$ is evident. Conversely let $\langle a, b\rangle \in \Theta(0, x) \wedge \Theta(0, y)$. Then $\Theta(0, b - a) = \Theta(a, b) \subseteq \Theta(0, x) \wedge \Theta(0, y)$, which means that x and y divide $b - a$, by Lemma 3. From (3b) we infer that xy divides $b - a$. Summarizing $\langle a, b\rangle \in \Theta(a, b) = \Theta(0, b - a) \subseteq \Theta(0, xy)$ and so $\Theta(0, x) \wedge \Theta(0, y) \subseteq \Theta(0, xy)$. The proof is complete.

A unified intersection polynomial can be derived also for congruence distributive PTT varieties. In this way we obtain a somewhat stronger version of Theorem 4:

Theorem 6. *Let* $\mathbf{V}$ *be a congruence distributive PTT variety. The following conditions are equivalent*:

(1) $\mathbf{V}$ *has the PIP*;

(2) *There exists a sexnary polynomial* $\mathbf{k}$ *such that* $\mathbf{k}(x, y, x, y, u, u) = = \mathbf{k}(y, x, x, y, u, u)$ *holds on any* $A \in \mathbf{V}$, *and* $\mathbf{k}(x, y, x, y, u, v) = \mathbf{k}(y, x, x, y, u, v)$ *implies* $x = y$ *or* $u = v$ *on any* SI *member of* $\mathbf{V}$;

(3) *There exists a sexnary polynomial* $\mathbf{k}$ *such that* $\mathbf{k}(x, y, x, y, u, v) = = \mathbf{k}(y, x, x, y, u, v)$ *iff* $x = y$ *or* $u = v$ *holds on any* SI *member of* $\mathbf{V}$.

Proof. (1) ⇒ (2) proceeds along the same line as the proof of Theorem 4, only Lemma 1 is replaced by Lemma 2.

(2) ⇒ (3) is evident.

(3) ⇒ (1). Put

$$\boldsymbol{D}_0(x_1, x_2, x_3, x_4) = \boldsymbol{k}(x_1, x_2, x_1, x_2, x_3, x_4) \quad \text{and}$$
$$\boldsymbol{D}_1(x_1, x_2, x_3, x_4) = \boldsymbol{k}(x_2, x_1, x_1, x_2, x_3, x_4).$$

Proposition 1 completes the proof.

Example 4. It is already known that distributive lattices form a PTT variety, see [3]. Now we state that this congruence distributive variety satisfies the PIP: Define $\boldsymbol{k}(x_1, x_2, x_3, x_4, x_5, x_6) = \boldsymbol{m}(x_1, x_5, x_6)$ where $\boldsymbol{m}$ denotes the median polynomial $\boldsymbol{m}(a, b, c) = (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$. Then it is an easy exercise to verify that condition (3) from Theorem 6 is fulfilled.

REFERENCES

[1] BAKER, K. A.: Primitive satisfaction and equational problems for lattices and other algebras. Trans. Amer. Math. Soc. 190, 1974, 125—150.

[2] CHAJDA, I.: A Mal'cev condition for congruence principal permutable varieties. Algebra Univ. 19, 1984, 337—340.

[3] CHAJDA, I.—ZELINKA, B.: Minimal compatible tolerances on lattices. Czech. Math. J. 27, 1977, 452—459.

[4] DUDA, J.: Polynomial pairs characterizing principality. Proceedings of the Colloquium held in Szeged 1983, Coll. Math. Soc. J. Bolyai 43, 109—122.

[5] FRIED, E.—PIXLEY, A. F.: The dual discriminator function in universal algebra. Acta Sci. Math. (Szeged) 41, 1979, 83—100.

[6] GRÄTZER, G.: Universal Algebra, 2nd edition, Springer Verlag, Berlin—Heidelberg—New York, 1979.

[7] GUMM, H.-P.: Geometrical methods in congruence modular algebras. Memoirs of the Amer. Math. Soc. 286, 1983. Providence, 1983.

[8] GUMM, H.-P.—URSINI, A.: Ideals in universal algebras. Algebra Univ. 19, 1984, 45—54.

[9] JACOBSON, N.: Structure of Rings. Amer. Math. Soc. Colloq. Publications, Providence, 1968.

[10] McKENZIE, R.: On spectra and the negative solution of the decision problem for identities having a finite non-trivial model. J. Symb. Logic, 40, 1975, 186—195.

[11] MICHLER, G.—WILLE, R.: Die primitiven Klassen arithmetischer Ringe. Math. Zeitschrift 113, 1970, 369—372.

[12] QUACKENBUSH, R. W.: Varieties with n-principal compact congruences. Algebra Univ. 14, 1982, 292—296.

[13] WERNER, H.: A Mal'cev condition for admissible relations. Algebra Univ. 3, 1973, 263.

[14] WERNER, H.: Discriminator Algebras. Akademie Verlag, Berlin, 1978.

[15] ZLATOŠ, P.: A Mal'cev condition for compact congruences to be principal. Acta Sci. Math. (Szeged) 43, 1981, 383—387.

Received March 28, 1986

Kroftova 21
616 00 Brno 16

СОЕДИНЕННЫЕ ТЕРМЫ ДЛЯ ГЛАВНЫХ КОНГРУЭНЦИЙ

Jaromír Duda

Резюме

Известно, что конечно порожденные конгруэнции многообразий алгебр главны, если существует подходящая пара термов. Статья заменяет упомянутую пару одним термом.