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ON A SUMMABILITY OF OBSERVABLES IN GENERALIZED MEASURABLE SPACES

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It is well known (see [6]) that in measurable spaces the sum of any two measurable functions is measurable. An analogous property of observables in a generalized measurable space does not hold. We know by [2] that the sum of two compatible observables is an observable. The aim of the present paper is to introduce a sufficient condition for the sum of two observables having countable ranges to be an observable. This condition is more general than compatibility. The question of summability is important when investigating the additivity of the integral developed by S. Gudder in [2] and [3], see also [1], [4], [5].

Let X be a nonempty set. A σ-algebra $\mathscr{A}$ is a nonempty collection of subsets of X satisfying:

(i) If $A\in\mathscr{A}$, then the complement $A' = X - A \in \mathscr{A}$.

(ii) If $A_1, A_2, \ldots$ is a sequence of elements in $\mathscr{A}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathscr{A}$.

A σ-class $\mathscr{C}$ is a nonempty collection of subsets of X satisfying:

(i) If $A\in\mathscr{C}$, then the complement $A' = X - A \in \mathscr{C}$.

(ii) If $A_1, A_2, \ldots$ is a sequence of pairwise disjoint elements in $\mathscr{C}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathscr{C}$.

Evidently $\emptyset$, $X\in\mathscr{A}$, $\emptyset$, $X\in\mathscr{C}$. If A, $B\in\mathscr{A}$, then $A\cup B\in\mathscr{A}$, $A\cap B\in\mathscr{A}$. But σ-class $\mathscr{C}$ need not contain, with two arbitrary sets, their intersection or union. Likewise if A, $B\in\mathscr{C}$, $A\supset B$, then $A-B\in\mathscr{C}$.

A σ-algebra is a σ-class. One can easily obtain that the converse implication is false (see [2]). We see that a σ-class is a generalization of a σ-algebra.

A generalized measurable space is a pair $(X,\mathscr{C})$ where $\mathscr{C}$ is a σ-class of subsets of X.

In the sequel, $(X,\mathscr{C})$ will be a generalized measurable space.

We say that two sets A, $B\in\mathscr{C}$ are compatible (written $A\leftrightarrow B$) if $A\cap B\in\mathscr{C}$. A collection $\mathscr{A}$ of sets in $\mathscr{C}$ is said to be compatible provided that any finite intersection of sets in $\mathscr{A}$ belongs to $\mathscr{C}$. Let $\mathscr{A}$, $\mathscr{B}$ be two collections of sets in $\mathscr{C}$. We write $\mathscr{A}\leftrightarrow\mathscr{B}$ if $A\leftrightarrow B$ for every $A\in\mathscr{A}$ and for every $B\in\mathscr{B}$. It is shown in [2] that a σ-class is a σ-algebra if and only if all its elements are pairwise compatible.

An observable is a function $f\colon X \to R$ for which $f^{-1}(B) \in \mathscr{C}$ for every Borel set B in the real line R. We say that two observables f and g are compatible (written $f \leftrightarrow g$) if $f^{-1}(B) \cap g^{-1}(C) \in \mathscr{C}$ for all two Borel sets B, C. If f is an observable we define $A_f = \{f^{-1}(B)\colon B \text{ is a Borel set in } R\}$. Evidently $A_f \subset \mathscr{C}$. It is easily seen that A_f is a σ-algebra. Evidently, two observables f and g are compatible if and only if $A_f \leftrightarrow A_g$ and that holds if and only if the family $A_f \cup A_g$ is compatible (see [2]).

A characteristic observable is a function $\chi_A\colon X \to R$, $\chi_A(x) = 1$ if $x \in A$, $\chi_A(x) = 0$ if $x \notin A$, for $A \in \mathscr{C}$.

If f, g and $f + g$ are observables, the functions f and g are said to be summable.

Theorem 1. (i) If $A, B \in \mathscr{C}$, then $A \leftrightarrow B$ if and only if there exists an observable f such that $A, B \in A_f$.
(ii) If f and g are observables and $f \leftrightarrow g$, then $f + g$ and $f.g$ are observables.
(iii) The σ-class $\mathscr{C}$ is a σ-algebra if and only if the sum of any two observables is an observable.

The proof can be found in [2].

One can show that the sum of two noncompatible characteristic observables is never an observable. However, there are noncompatible observables whose sum an observable (see example 1).

Theorem 2. Let f and g be observables with countable ranges. If $f^{-1}(B) \cap g^{-1}(B) \in \mathscr{C}$ for every Borel set B in the real line R, then $f + g$ is an observable.

Proof. We shall prove that if $f + g$ is not an observable, then there exists a Borel set B satisfying $f^{-1}(B) \cap g^{-1}(B) \notin \mathscr{C}$. We shall use the following proposition: Let $B_1, B_2, \ldots$ be a sequence of pairwise disjoint Borel sets, let $h\colon X \to R$ be a function. If $h^{-1}(B)_i \in \mathscr{C}$ for all i = 1, 2, ..., then $h^{-1}\left(\bigcup_{i=1}^{\infty} B_i\right) \in \mathscr{C}$ (1). This is because $h^{-1}\left(\bigcup_{i=1}^{\infty} B_i\right) = \bigcup_{i=1}^{\infty} h^{-1}(B_i)$. Thus let $f + g$ not be an observable. Then there is a Borel set C satisfying $(f + g)^{-1}(C) \notin \mathscr{C}$. Denote by D the intersection $C \cap (f + g)(X)$. D is a countable set, hence it is a Borel set and $(f + g)^{-1}(D) = (f + g)^{-1}(C) \notin \mathscr{C}$. Then, however, by (1) there exists a real number $k \in D$ satisfying $(f + g)^{-1}(\{k\}) \notin \mathscr{C}$ (2). Observe that $(f + g)^{-1}(\{k\}) = \bigcup_{m_i + n_i = k} (f^{-1}(\{m_i\}) \cap g^{-1}(\{n_i\}))$ (3) where $m_i \in f(X)$, $n_i \in g(X)$ and the union on the right-hand side of equality is at most countable. If $f^{-1}(\{m_i\}) \cap g^{-1}(\{n_i\}) \in \mathscr{C}$ for every pair (m_i, n_i) such that $m_i + n_i = k$, then by (1) $(f + g)^{-1}(\{k\}) \in \mathscr{C}$, because the sets $f^{-1}(\{m_i\}) \cap g^{-1}(\{n_i\})$ are pairwise disjoint. Thus there is at least one pair (m_i, n_i) such that $m_i + n_i = k$, $f^{-1}(\{m_i\}) \cap g^{-1}(\{n_i\}) \notin \mathscr{C}$ (4). If $m_i = n_i$, then it suffices to put $B = \{m_i\}$ and we obtain $f^{-1}(B) \cap g^{-1}(B) \notin \mathscr{C}$.

Let $m_i \neq n_i$. $f^{-1}(\{m_i, n_i\}) \cap g^{-1}(\{m_i, n_i\}) = (f^{-1}(\{m_i\}) \cup f^{-1}(\{n_i\})) \cap (g^{-1}(\{m_i\}) \cup \cup g^{-1}(\{n_i\})) = (f^{-1}(\{m_i\}) \cap g^{-1}(\{m_i\})) \cup (f^{-1}(\{m_i\}) \cap g^{-1}(\{n_i\})) \cup (g^{-1}(\{m_i\}) \cap \cap f^{-1}(\{n_i\})) \cup (f^{-1}(\{n_i\}) \cap g^{-1}(\{n_i\}))$ (5). It suffices to show that at least one of the sets $f^{-1}(\{m_i, n_i\}) \cap g^{-1}(\{m_i, n_i\}), f^{-1}(\{m_i\}) \cap g^{-1}(\{m_i\}), f^{-1}(\{n_i\}) \cap g^{-1}(\{n_i\})$ does not belong to $\mathscr{C}$. Let all those sets be in $\mathscr{C}$. Since the sets on the right-hand side of equality (5) are pairwise disjoint, it follows that $(f^{-1}(\{m_i, n_i\}) \cap g^{-1}(\{m_i, n_i\})) - - ((f^{-1}(\{m_i\}) \cap g^{-1}(\{m_i\})) \cup (f^{-1}(\{n_i\}) \cap g^{-1}(\{n_i\}))) = (f^{-1}(\{m_i\}) \cap g^{-1}(\{n_i\})) \cup \cup (f^{-1}(\{n_i\}) \cap g^{-1}(\{m_i\})) \in \mathscr{C}$ (6). Either none of the sets $f^{-1}(\{m_i\}) \cap g^{-1}(\{n_i\})$, $f^{-1}(\{n_i\}) \cap g^{-1}(\{m_i\})$ belongs to $\mathscr{C}$ or both are in $\mathscr{C}$. The second alternative is in conflict with (4). Thus $f^{-1}(\{m_i\}) \cap g^{-1}(\{n_i\}) \notin \mathscr{C}$ and also $f^{-1}(\{n_i\}) \cap g^{-1}(\{m_i\}) \notin \mathscr{C}$ for every pair (m_i, n_i) satisfying (4). Since we suppose $m_i \neq n_i$ for every pair (m_i, n_i) satisfying (4), we can divide the union in equality (3) into the union of those sets $f^{-1}(\{m_i\}) \cap g^{-1}(\{n_i\})$ which are in $\mathscr{C}$ and the union of those sets $f^{-1}(\{m_i\}) \cap \cap g^{-1}(\{n_i\})$ which are not in $\mathscr{C}$. Since the pair (m_i, n_i) satisfies (4) if and only if (n_i, m_i) satisfies (4) we can write the second part of the union as a union of sets $(f^{-1}(\{m_i\}) \cap g_{-1}(\{n_i\})) \cup (f^{-1}(\{n_i\}) \cap g^{-1}(\{m_i\}))$, but these sets are in $\mathscr{C}$ by (6). We obtain $(f+g)^{-1}(\{k\}) \in \mathscr{C}$ and it is a conflict with (2). Hence either $f^{-1}(\{m_i, n_i\}) \cap \cap g^{-1}(\{m_i, n_i\}) \notin \mathscr{C}$ or $f^{-1}(\{m_i\}) \cap g^{-1}(\{m_i\}) \notin \mathscr{C}$ or $f^{-1}(\{n_i\}) \cap g^{-1}(\{n_i\}) \notin \mathscr{C}$. Then it suffices to denote $B = \{m_i, n_i\}$ or $B = \{m_i\}$ or $B = \{n_i\}$, respectively.

The following example shows that the converse assertion to Theorem 2 does not hold.

Example 1. Let $X = \{1, 2, \ldots, 8\}$, let $\mathscr{C}$ be a family of those subsets of X which have an even number of elements. Evidently $\mathscr{C}$ is a σ-class. We define the functions $f\colon X \to R$, $g\colon X \to R$ according to the following table:

	1	2	3	4	5	6	7	8
f	1	1	2	2	0	1	0	1
g	2	1	2	0	3	3	1	0
$f+g$	3	2	4	2	3	4	1	1

Evidently, f, g and $f+g$ are observables, but $f^{-1}(\{1\}) \cap g^{-1}(\{1\}) = \{2\} \notin \mathscr{C}$.

It is an open problem whether Theorem 2 remains valid for observables with uncountable ranges.

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О СУММИРУЕМОСТИ ИЗМЕРИМЫХ ФУНКЦИЙ В ОБОБЩЕННЫХ ИЗМЕРИМЫХ ПРОСТРАНСТВАХ

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Резюме

В статье изучается проблема измеримости суммы двух измеримых функций в обобщенном измеримом пространстве. Приведено достаточное условие измеримости суммы двух измеримых функций со счетными областами значений. то условие более общее, чем условие совместимости. В статье показано, что приведенное условие не является необходимым.