Mathematica Slovaca

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Mathematica Slovaca, Vol. 37 (1987), No. 1, 71--83

Persistent URL: http://dml.cz/dmlcz/136439

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JOZEF DUDEK

0. Introduction

Let $\mathfrak{A} = (A, F)$ be an algebra. By $p_n = p_n(\mathfrak{A})$ we shall denote the number of all essentially n-ary polynomials over an algebra $\mathfrak{A}$.

A groupoid $(G, \cdot)$ is called distributive if it satisfies $(xy)z = (xz)(yz)$ and $z(xy) = (zx)(zy)$ for all $x, y, z \in G$. Recall that an idempotent commutative groupoid $(G, \cdot)$ satisfying $(xy)y = x$ is called a Steiner quasigroup (e.g. see [1]).

For other definitions and notations used here we refer to [5].

In this paper we prove the following theorems:

Theorem 1. *Let $(G, \cdot)$ be a Steiner quasigroup. Then the following conditions are equivalent:*

(i_1) *$(G, \cdot)$ is distributive,*

(i_2) *$(G, \cdot)$ satisfies $(xy)z = ((xz)y)x$,*

(i_3) *$(G, \cdot)$ satisfies $(xz)(yz) = ((xz)y)x$,*

(i_4) *$(G, \cdot)$ satisfies $(((xz)y)((zy)x))((xy)z) = xy$,*

(i_5) *The polynomial $(((zx)y)((zy)x))((xy)z)$ is not essentially ternary over $(G, \cdot)$,*

(i_6) $p_3(G, \cdot) \leqslant 3$.

Theorem 2. *Let $(G, \cdot)$ be an idempotent commutative groupoid. Then the following conditions are equivalent:*

(j_1) *$(G, \cdot)$ is a distributive Steiner quasigroup,*

(j_2) *$(G, \cdot)$ satisfies $((xz)(yz))z = xy$,*

(j_3) *$(G, \cdot)$ satisfies $((xz)(yz))(xy) = z$,*

(j_4) *$(G, \cdot)$ satisfies $((zx)y)((zy)x) = z$,*

(j_5) *The polynomial $((xz)(yz))z$ is not essentially ternary in $(G, \cdot)$,*

(j_6) *The polynomial $((xz)(yz))(xy)$ is not essentially ternary in $(G, \cdot)$,*

(j_7) *The polynomial $((zx)y)(zy)x)$ is not essentially ternary in $(G, \cdot)$.*

Theorem 3. *Let $(G, \cdot)$ be an idempotent commutative groupoid. Then $p_3(G, \cdot)$*

$\leqslant 3$ *if and only if* $(G, \cdot)$ *is either a semilattice or a distributive Steiner quasigroup.*

Recall that an idempotent commutative semigroup is called a semilattice.

As a corollary from this theorem we get

Theorem 4. *Let* $(G, \cdot)$ *be an idempotent commutative groupoid with* card $G \geqslant 2$. *Then* $(G, \cdot)$ *is a distributive Steiner quasigroup if and only if* $p_3(G, \cdot) = 3$.

The proofs of Theorems 1, 2 and 3 are presented in the last section.

1. General remarks on idempotent commutative groupoids

In this section we prove several useful lemmas concerning mainly ternary polynomials over idempotent commutative groupoids. However, first we need further notations and definitions.

For a given groupoid $(G, \cdot)$ we write $x_1x_2\cdot\ldots\cdot x_{n-1}x_n$ instead of $(\ldots(x_1x_2)\cdot\ldots\cdot x_{n-1})\,x_n$ and xy^n stands for the polynomial $(\ldots(xy)\cdot\ldots\cdot y)\,y$ where x occurs once and y occurs n times ($n \geqslant 1$). The variety of idempotent and commutative groupoids $(G, \cdot)$ is denoted by $V(\cdot)$. For a fixed positive integer n, by $V_n(\cdot)$ we denote the subvariety of $V(\cdot)$ of all groupoids $(G, \cdot)$ which satisfy $xy^n = x$. It is clear that the variety $V_2(\cdot)$ coincides with the variety of all Steiner quasigroups. For $n \geqslant 3$, every member from $V_n(\cdot)$ will be called a generalized Steiner quasigroup.

Given two integers m and n, where $1 \leqslant m < n$, the symbol $V_{m,n}(\cdot)$ stands for the subvariety of $V(\cdot)$ of all groupoids $(G, \cdot)$ satisfying $xy^m = xy^n$. Members from $V_{1,2}(\cdot)$ will be called near-semilattices. It is clear that every semilattice is in $V_{m,n}(\cdot)$. Let us add that a near-semilattice i.e., a groupoid $(G, \cdot)$ satisfying $x^2 = x$, $xy = yx$ and $(xy)\,y = xy$ is called an upper bound algebra in [8].

Further, a groupoid $(G, \cdot)$ is said to be proper if the polynomial xy is essentially binary. In general. an algebra $(A, \{f_t\}_{t\in T})$ of type $(n_t)_{t\in T}$ is proper if the mapping $t \to n_t$ is one-to-one and f_t is essentially n_t-ary provided $n_t \geqslant 1$. It is easy to see that any idempotent commutative (symmetric) algebra $(A, +, \cdot)$ of type $(2, 2)$ is proper iff $x + y$ and xy are distinct on A.

Lemma 1.1. *Let* $(G, \cdot) \in V(\cdot)$. *Then* $(G, \cdot)$ *is proper if and only if* card $G \geqslant 2$.

Proof. If $(G, \cdot)$ is proper, then obviously card $G \geqslant 2$. Conversely, if xy is not essentially binary, then using the commutativity and the idempotency of xy we get a contradiction.

Lemma 1.2. *Let* $(G, \cdot)$ *be an idempotent groupoid with* card $G \geqslant 2$. *Then* $p_2(G, \cdot) = 1$ *if and only if* $(G, \cdot)$ *is either a Steiner quasigroup or a near-semilattice.*

Proof. If $(g, \cdot) \in V(\cdot)$ or $(G, \cdot) \in V_{1,2}(\cdot)$, then using the previous lemma we infer that $(G, \cdot)$ is proper. Now using, e.g. Marczewski's formula of a description of the set $\mathbf{A}^{(2)}(G, \cdot)$ (see [7]) one can prove that xy is the only essentially binary polynomial over the considered groupoid $(G, \cdot)$.

Conversely, assume that $p_2(G, \cdot) = 1$. Then we infer that $(G, \cdot)$ is a commutative groupoid and xy is the only essentially binary polynomial over $(G, \cdot)$. Consider now the polynomial xy^2. If xy^2 is essentially binary, then the above gives $xy = xy^2$. This proves that $(G, \cdot)$ is a near-semilattice. If xy^2 is not essentially binary, then applying Theorem 1 of [2] we infer that $(G, \cdot) \in V_2(\cdot)$. The proof is completed.

Before formulating the next lemma we need the definition of a linear polynomial (a good polynomial in [3]).

We say that a polynomial $f = f(x_1, \ldots, x_n)$ over $F = \{xy\}$ is a linear polynomial if all its variables are different. For example the polynomials: xy, $x_1x_2x_3$, $\ldots$, $x_1x_2 \cdot \ldots \cdot x_{n-1}x_n$, $x_1(x_2x_3)$ $x_1(x_2(\ldots(x_{n-1}x_n)\cdot \ldots \cdot))$, $(x_1x_2)(x_3x_4)$, and so on, are linear polynomials over any groupoid.

Lemma 1.3. *Let* $(G, \cdot)$ *be in* $V(\cdot)$. *Then the following conditions are equivalent*:

(c_1) $(G, \cdot)$ *is proper*

(c_2) card $G \geqslant 2$,

(c_3) *every n-ary linear polynomial over* $(G, \cdot)$ *is essentially n-ary*,

(c_4) *the polynomial* $g(x, y, z) = (xz)(yz)$ *is essentially ternary over* $(G, \cdot)$.

Proof. $(c_1) \Rightarrow (c_2)$ follows from Lemma 1.1.

$(c_2) \Rightarrow (c_3)$. The proof of this implication follows by induction on the arity of a linear polynomial. Indeed, for $n = 1$, the implication is obvious. For $n = 2$, it follows from Lemma 1.1 and the fact that xy is the only binary linear polynomial over $(G, \cdot)$. Further, observe that for every $n + 1$-ary linear polynomial $f = f(x_1, x_2, \ldots, x_n, x_{n+1})$ there exists an n-ary linear polynomial f_0 such that $f(x_1, x_2, \ldots, x_n, x_{n+1}) = f_0(x_ix_j, x_2, \ldots, x_n, x_{n+1})$ where $1 \leqslant i, j \leqslant n + 1$. Now using the idempotency, the commutativity and the inductive hypothesis we infer that f is essentially $n + 1$-ary.

$(c_3) \Rightarrow (c_4)$. By (c_3) we infer that the polynomial xy is essentially binary. Using this fact and the identities

$$xy = g(x, x, y) \quad \text{and} \quad g(x, y, z) = g(y, x, z)$$

we deduce that g is essentially ternary.

$(c_4) \Rightarrow (c_1)$. If $(G, \cdot)$ is improper, then xy is not essentially binary and hence g is also not essentially ternary, — a contradiction. This completes the proof of the lemma.

Lemma 1.4. *If* $(G, \cdot)$ *is proper and* $(G, \cdot) \in V(\cdot)$, *then the polynomial* $h(x, y, z) = ((xz)y)x$ *depends on both variables* y *and* z.

Proof. If h does not depend on y, then we get

$$((xz)\,y)\,x = ((xz)\,z)\,x = ((zx)\,x)\,x.$$

Putting $x = z$ we have $yx^2 = x$. This identity contradicts Theorem 1 of [2]. If h does not depend on z, then we have

$$((xz)\,y)\,x = ((xy)\,y)\,x = (xy)\,x.$$

Putting here $x = y$ we get $x = zx^3$, which again contradicts Theorem 1 of [2].

Lemma 1.5. *If $(G, \cdot) \in V(\cdot)$ and $h(x, y, z) = ((xz)\,y)\,x$ is not essentially ternary, then $(G, \cdot) \in V_4(\cdot)$. Moreover, there exist proper groupoids in $V_4(\cdot)$ for which h does not depend on x.*

Proof. If $(G, \cdot)$ is improper, then $(G, \cdot) \in V_4(\cdot)$. Let $(G, \cdot)$ be proper. Then using the previous lemma we get

$$((xz)\,y)\,x = zy^3 = yz^2.$$

Setting $y = z$ we obtain $xy^2x = y$. Using this identity and $xy^3 = yx^2$ we get $xy^4 = yx^2y = x$. Thus $(G, \cdot) \in V_4(\cdot)$. To prove the second assertion of the lemma it suffices to consider an affine groupoid $G(5)$, i.e., $G(5) = (G, 3x + 3y)$ where $(G, +)$ is an abelian group of exponent 5 (see the next section). It is easy to check that $G(5) \in V_4(\cdot)$ and $h(x, y, z) = 2y + z$. The proof is completed.

2. Affine groupoids

Let $(G, +)$ be an abelian group of an odd exponent n. Denote by $G(n)$ the groupoid $\left(G, \dfrac{n+1}{2}(x + y)\right)$. If p is prime, then $G(p)$ is called an affine groupoid (see [3]). Using the main result of [10] we have $G(n) = (G, I(G, +))$ where $I(\mathfrak{A})$ denote the full idempotent reduct of an algebra $\mathfrak{A}$. The last equality justifies the name "an affine groupoid" for the groupoid $G(p)$. We should mention here that $(A, F_1) = (A, F_2)$ means that the algebras (A, F_i) $(i = 1, 2)$ are polynomially equivalent, i.e. the sets $\mathbf{A}(F_1)$, $\mathbf{A}(F_1)$ of polynomials are equal.

Recall that a groupoid $(G, \cdot)$ is medial if $(G, \cdot)$ satisfies $(xy)\,(uv) = (xu)\,(yv)$ for all $x, y, u, v \in G$. Denote by $M_n(\cdot)$ the variety of idempotent commutative medial groupoids $(G, \cdot)$ satisfying $xy^n = x$.

Lemma 2.1. *$G(p) \in M_{p-1}(\cdot)$ for every prime $p \geqq 3$.*

Proof. It is clear that the groupoid $G(p) = (G, \cdot)$ where $xy = \dfrac{p+1}{2}(x+y)$ is idempotent, commutative and medial. Putting $s = \dfrac{p+1}{2}$ we have

$$xy^2 = s^2x + (s^2 + s)\, y \text{ and in general } xy^k = s^kx + \left(\sum_{i=1}^{k} s^i\right) y.$$

Hence for $k = p - 1$ we get $xy^{p-1} = s^{p-1}x + \left(\sum_{i=1}^{p-1} s^i\right) y$. To prove $xy^{p-1} = x$ it suffices to check the following congruences:

$$s^{p-1} = 1 \,(\mathrm{mod}\, p) \text{ and } \sum_{i=1}^{p-1} s^i = \frac{s(s^{p-1} - 1)}{s - 1} = 0 \,(\mathrm{mod}\, p).$$

Both congruences follow from the Fermat formula since s and p are relatively prime and $s - 1$ and s are relatively prime.

Note that $G(p)$ and in general any member from $V_n(.)$ is a quasigroup (see Theorem 2 of [2]). In particular, if $p = 3$ we infer that $G(3)$ is a (medial) Steiner quasigroup. We also have

Lemma 2.2. *If* $(G, +)$ *is a group* (*not necessary abelian*) *of exponent* 3, *then* $(G, .)$ *where* $xy = x + 2y + x$ *is a Steiner quasigroup.*

Proof. Clearly we have $x^2 = x$ and $xy = x + 2y + x = (x + (-y)) + x = -(y + (-x)) + x = 2(y + 2x) + x = y + 2x + y + 2x + x = y + 2x + y = yx$. Using this fact we have $xy^2 = (xy)\,y = yx + 2y + xy = y + 2x + y + 2y + x + 2y + x = x$. Thus $(G, \cdot) \in V_2(\cdot)$.

3. Ternary polynomials.

In this section we shall consider some special ternary polynomials over groupoids $(G, .)$ from $V(.)$ Namely, we shall deal with the following polynomials:

$$(*)\quad \begin{aligned} &f(x, y, z) = (xy)\, z,\ g(x, y, z) = (xz)\,(yz),\ h(x, y, z) = ((xz)\, y)\, x,\\ &a(x, y, z) = ((xz)\,(yz))\, z,\ b(x, y, z) = ((xz)\,(yz))\,(xy),\\ &c(x, y, z) = ((yz)\, x)\,((zx)\, y) \text{ and } d(x, y, z) = (((yz)\, x)\,((zx)\, y))\,((xy)\, z).\end{aligned}$$

To formulate further lemmas we need some more definitions.

Let $f = f(x_1, \ldots, x_n)$ be a function on a set A. We say that f admits a permutation $\sigma \in S_n$ if $f(x_1, \ldots, x_n) = f(x_{\sigma 1}, \ldots, x_{\sigma n})$ for all $x_1, \ldots, x_n \in A$. We shall write $f^\sigma(x_1, \ldots, x_n)$ instead of $f(x_{\sigma 1}, \ldots, x_{\sigma n})$. By $G(f)$ we denote the subgroup of the group S_n of all admissible permutations of f (see [6]).

Let E be a set of identities. Then by E^* we denote the class of all algebras satisfying all identities of E. A permutation $\sigma \in S_n$ is said to be trivial for a polynomial $p = p(x_1, \ldots, x_n)$ with respect to E (or with respect to E^*) if the identity $p = p^\sigma$ is an identity in E^*.

Lemma 3.1. *Each polynomial from* $(*)$ *(except* h*) admits the transposition* (x, y) *of its variables.*

Proof. An easy consequence of the commutativity of xy.

Lemma 3.2. *If* $(G, \cdot) \in V(\cdot)$ *and* $(G, \cdot)$ *is proper, then* f, g *are essentially ternary,* h *depends on* y *and* z, a *depends on* x *and* y, b *and* c *depend on* z.

Proof. The first statement follows from Lemma 1.3, the second follows from Lemma 1.4. Using the previous lemma, we infer that a does not depend on x if and only if a does not depend on y. If $a(x, y, z) = z$, then we get $y = a(x, x, y) = ((xy)(xy))y = xy^2$. The identity $xy^2 = y$ contradicts Theorem 1 of [2]. Thus a depends on x and y. If b does not depend on z, then $b(x, x, y) = ((xy)(xy))x = yx^2$ does not depend on y which contradicts Theorem 1 of [2]. If c does not depend on z, then the polynomial $c(y, y, z)$ does not depend on z either. Hence we get $x = c(x, x, y) = ((xy)x)((yx)x) = yx^2$. The identity $yx^2 = x$ again contradicts Theorem 1 of [2].

Lemma 3.3. *If* $(G, \cdot) \in V(\cdot)$ *and* $(G, \cdot)$ *is not a semilattice, then* $G(f) = G(g) \cong S_2$.

Proof. The fact $G(f) \cong S_2$ follows from Lemma 3.1 and the nonassociativity of xy. The second statement follows from (iii) of Theorem 8 of [3].

Lemma 3.4. *If* $(G, \cdot) \in V(\cdot)$ and $d(x, y, z) = z$, *then* $(G, \cdot) \in V_4(\cdot)$. *Moreover, there exist groupoids (non-one-element) from* $V_4(\cdot)$ *for which* $d(x, y, z) = z$ *holds.*

Proof. Setting $x = y$ in the identity $d(x, y, z) = z$ we get

$$y = (((yz)y)((zy)y))(yz) = x(yz)^2$$

Putting $x = z$ in $d(x, y, z) = z$ we obtain $x = d(x, y, x) = (((yx)x)(xy))xy)x) = (x(xy)^2)(yx^2)$. Now we have $x = y(yx^2) = (yx^2)y = ((yx)x)y$. The identities $xy^2x = y$ and $x(xy)^2 = y$ imply $y = (yx^2)(yx^2y)^2 = (yx^2)x^2$. Thus $xy^4 = x$ holds in $(G, \cdot)$, i.e. $(G, \cdot) \in V_4(\cdot)$. To prove the remaining part of the assertion it suffices to consider an affine groupoid $G(5)$. Applying Lemma 2.1 we infer that $G(5) \in V_4(\cdot)$. One can also easy verify that $d(x, y, z) = z$ if $xy = 3x + 3y$ where $G(5) = (G, 3x + 3y)$. The proof of the lemma is completed.

Lemma 3.5. *If* $(G, \cdot) \in V_2(\cdot)$ *and* $(G, \cdot)$ *is proper, then the identity permutation and the transposition* (x, y) *are the only admissible permutations for the polynomials* f, g, a, b, c *and* d.

Proof. It is easy to see that if any of the above mentioned polynomials admits a nontrivial permutation of its variables, then using the commutativity of xy we infer that this polynomial is symmetric. If $p \in \{f, g, a, d\}$ and $p(x, y, z) = p(y, z, x)$, then putting $x = y$ we infer that $(G, \cdot)$ is improper, — a contradiction. If now $p \in \{b, c\}$ and $p(x, y, z) = p(y, z, x)$, then $p(y, y, z) = p(y, z, y)$. The last identity in both cases gives $y = z$, — a contradiction.

4. The polynomial h(x, y, z) = ((xz) y) x over Steiner quasigroups

In this section we characterize the distributive law for groupoids from $V_2(\cdot)$ using the polynomial h (see proposition 1).

Lemma 4.1. *If $(G, \cdot) \in V_2(\cdot)$ and $(G, \cdot)$ is proper, then the polynomial $h(x, y, z) = ((xz)\, y)\, x$ is essentially ternary.*

Proof. Using Lemma 3.2 we infer that h depends on y and z. If h does not depend on x, then $h(\ \ , y)$ does not depend on x either. This implies $y = h(x, y, y) = ((xy)\, y)\, x = xx = x$, — a contradiction.

Lemma 4.2. *If $(G, \cdot) \in V_2(\cdot)$ and $(G, \cdot)$ is proper, then h does not admit any cycle of its variables and it does not admit the transpositions (x, z) and (y, z) of its variables.*

Proof. If $h(x, y, z) = h(y, z, x)$, then $((xz)\, y)x = ((yx)\, z)\, y$. Putting in to this identity $y = z$ we get $x = xx = ((xy)\, y)\, x = ((yx)\, y)\, y = xy$. Thus we have $xy = x$, — a contradiction. Let now $((xz)\, y)\, x = (zx)\, y)\, z$. As above we get $x = xy$, — a contradiction. Analogously one proves that h does not admit the transposition (y, z).

Lema 4.3. *Let $(G, \cdot) \in V_2(\cdot)$. Then $(G, \cdot)$ is distributive if and only if $(xy)\, z = ((xz)\, y)\, x$ holds in $(G, \cdot)$.*

Proof. If $(G, \cdot)$ is distributive, then $((xz)\, y)\, x = ((xz)\, x)\ (yx) = (xy)\, z$. Assuming $(xy)\, z = ((xz)\, y)\, x$ we get $((xz)\, y)x = ((yz)\, x)\, y$ since the polynomial $(xy)\, z$ admits the transposition (x, y). Using the last identity and putting yz for z in the first identity we get

$$(xy)\,(zy) = ((x(zy))\, y)\, x = (((yz)\, x)\, y)\, x = (((xz)\, y)\, x)\, x = (xz)\, y.$$

Thus $(G, \cdot)$ is distributive.

Lemma 4.4. *Let $(G, \cdot) \in V_2(\cdot)$. Then $(G, \cdot)$ is distributive if and only if $(xz)(yz) = ((xz)\, y)\, x$ holds in $(G, \cdot)$.*

Proof. If $(G, \cdot)$ is distributive, then

$$(xz)\,(yz) = ((xz)\, y)\,((xz)\, z) = ((xz)\, y)\, x.$$

Let now $(xz)\,(yz) = ((xz)\, y)\, x$ hold in $(\cdot G, \cdot)$. Then we have $((xz)\, y)\, x = ((yz)\, x)\, y$ since the polynomial $g(x, y, z) = (xz)\,(yz)$ admits the transposition (x, y) of its variables. Using this fact we get

$$(xy)\, z = (xy)\,((zy)\, y) = ((xy)\,(zy))\, x = ((zy)\, y)\, x)\,(zy) = (xz)\,(yz).$$

Hence $(G, \cdot)$ is distributive

Lemma 4.5. *If $(G, \cdot) \in V_2(\cdot)$, $(G, \cdot)$ is proper and the admissible group of h is nontrivial, then $(G, \cdot)$ satisfies $((xz)\, y)\, x = ((yz)\, x)\, y$.*

Proof. This fact follows immediately from Lemma 4.2. Let us add that we do not know if there exists any nondistributive Steiner quasigroup satisfying the above identity.

Lemma 4.6. *Let* $(G, \cdot)$ *be in* $V_2(\cdot)$. *Then* $(G, \cdot)$ *is distributive if and only if* $p_3(G, \cdot) \leqslant 3$.

Proof. Let $(G, \cdot)$ be a distributive Steiner quasigroup. If card $G = 1$, then $p_3(G, \cdot) = 0 < 3$. If card $G > 2$, then using Lemmas 3.2 and 3.4 we infer that the polynomials xyz, yzx and zxy are essentially ternary and different. Using Marczewski's description of the set $\mathbf{A}^{(3)}(G, \cdot)$ (see [7]) we infer that the polynomials xyz, yzx and zxy are the only essentially ternary polynomials over $(G, \cdot)$. Thus $p_3(G, \cdot) = 3$. Assume now that $(G, \cdot) \in V_2(\cdot)$ and $p_3(G, \cdot) \leqslant 3$. If $p_3(G, \cdot) = 0$, then by Lemma 1.3, the groupoid $(G, \cdot)$ is improper. Thus $(G, \cdot)$ is distributive as a one-element groupoid. Assume now that $G, \cdot)$ is nondistributive. Applying Lemma 3.5 we get card $G(f) = 2$. This implies that $p_3(G, \cdot) \leqslant 3$. Using once more Lemma 3.5 we deduce that $G(f) = G(g) \cong S_2$ and that the identity permutation and the transposition (x, y) are in the group $G(f) = G(g)$. It is obvious that (using Lemma 3.5) all the polynomials xyz, yzx, zxy, $(xz)(yz)$, $(xy)(zy)$ and $(yx)(zx)$ are pairwise distinct. Applying to these polynomials Lemma 1.3 we infer that they are all essentially ternary. Hence $p_3(G, \cdot) \geqslant 6$, which is impossible. Thus $(G, \cdot)$ is distributive, which finishes the proof of the lemma.

We may now state the first main result of this paper:

Proposition 1. *Let* $(G, \cdot)$ *be a Steiner quasigroup. Then the following conditions are equivalent*

(h_1) $(G, \cdot)$ *is distributive*

(h_2) $(G, \cdot)$ *satisfies* $(xy)z = (xz)y)x$

(h_3) $(G, \cdot)$ *satisfies* $(xz)yz) = ((xz)y)x$

(h_4) $p_3(G, \cdot) \leqslant 3$.

Proof. The proof follows from Lemmas 4.3, 4.4 and 4.6.

5. The polynomial a(x, y, z) = ((xz) (yz)) z over groupoids from V(·)

In this section we characterize distributive Steiner quasigroups by means of the polynomial a. Before formulating the main result of this section (Proposition 2) we need three lemmas.

Lemma 5.1. *Let* $(G, \cdot) \in V(\cdot)$. *Then* $a(x, y, z) = ((xz)(yz))z$ is *not essentially ternary if and only if* $a(x, y, z) = xy$.

Proof. If $(G, \cdot)$ is improper, then the assertion is obvious. Let now $(G, \cdot)$ be proper. If a is not essentially ternary, then applying Lemma 3.2 we infer that a does not depend on z. Hence $a(x, x, y) = xy^2$ does not depend on y. This gives $xy^2 = x$. Using this identity we have $a(x, y, z) = a(x, y, y) = ((xy)(yy))y = xy^3 = xy$. Thus $a(x, y, z) = xy$. The converse is obvious.

Lemma 5.2. *Let $(G, \cdot) \in V(\cdot)$. Then $(G, \cdot)$ is a distributive Steiner quasigroup if and only if $a(x, y, z) = xy$.*

Proof. If $(G, \cdot)$ is a distributive Steiner quasigroup, then

$$a(x, y, z) = ((xz)(yz))z = (xz^2)(yz^2) = xy.$$

Let now $a(x, y, z) = xy$. Then $y = yy = a(y, y, z) = yz^2$. This shows that $(G, \cdot) \in V_2(\cdot)$. Further, we have $(xy)z = (a(x, y, z))z = (((xz)(yz))z)z = (xz)yz)$. Hence $(G, \cdot)$ is distributive.

Lemma 5.3. *Let $(G, \cdot) \in V(\cdot)$. Then $(G, \cdot)$ a distributive Steiner quasigroup if and only if $a(x, y, z) = ((xz)(yz))z$ is not essentially ternary.*

Proof. Immediately follows from the last two lemmas.

Combining these three lemmas we get

Proposition 2. *Let $(G, \cdot) \in V(\cdot)$. Then the following conditions are equivalent:*

(a_1) $(G, \cdot)$ is a distributive Steiner quasigroup,

(a_2) $(G, \cdot)$ satisfies the identity $((xz)(yz)z = xy$,

(a_3) The polynomial $a(x, y, z) = ((xz)(yz)z$ is not esentially ternary.

6. The polynomial b(x, y, z) = ((xz) (yz)) (xy) and the distributivity of groupoids from V(·)

We start with

Lemma 6.1. *Let $(G, \cdot) \in V(\cdot)$. Then the polynomial b is not essentially ternary if and only if $b(x, y, z) = z$.*

Proof. The proof of this lemma is similar to that of Lemma 5.1 and follows from Lemmas 3.1 and 3.2.

Lemma 6.2. *Let $(G, \cdot) \in V(\cdot)$. Then $(G, \cdot)$ is a distributive Steiner quasigroup if and only if $(G, \cdot)$ satisfies $(bx, y, z) = z$.*

Proof. If $(G, \cdot)$ is a distributive Steiner quasigroup, then

$$b(x, y, z) = ((xz)(yz))(xy) = ((xy)z)(xy) = z(xy)^2 = z.$$

Supposing $b(x, y, z) = z$. Then we have $y = b(x, x, y) = ((xy)(xy))(xx) = yx^2$. Thus $(G, \cdot)$ is a Steiner quasigroup. To prove the distributive law for $(G, \cdot) \in V_2(\cdot)$ we use the identity $b(x, y, z) = z$. We have

$$(xy)\,z = z(xy) = (b(x, y, z))\,(xy) = ((xz)\,(yz))\,(xy)^2 = (xz)\,(yz).$$

This finishes the proof of the lemma.

From the preceding two lemmas we get

Lemma 6.3. *Let* $(G, \cdot) \in V(\cdot)$. *Then* $(G, \cdot)$ *is a distributive Steiner quasigroup if and only if the polynomial* $b(x, y, z) = ((xz)\,(yz))\,(xy)$ *is not essentially ternary.*

As a corollary from the last three lemmas we get

Proposition 3. *Let* $(G, \cdot) \in V(\cdot)$. *Then the following conditions are equivalent*:

(b_1) $(G, \cdot)$ *is a distributive Steiner quasigroup,*
(b_2) *the groupoid* $(G, \cdot)$ *satisfies* $((xz)\,(yz))\,(xy) = z$,
(b_3) *the polynomial* $b(x, y, z) = ((xz)\,(yz))\,(xy)$ *is not essentially ternary.*

7. The polynomial c(x, y, z) = ((zx) y) ((zy) x) over groupoids from V(·)

In this section as in the previous ones we characterize, by means of the polynomial c, those groupoids from $V(\cdot)$ which are distributive Steiner quasigroups.

Lemma 7.1. *Let* $(G, \cdot) \in V(\cdot)$. *Then the polynomial* c *is not essentially ternary if and only if* $c(x, y, z) = z$.

Proof. It follows from Lemmas 3.1 and 3.2.

Lemma 7.2. *Let* $(G, \cdot) \in V(\cdot)$. *Then* $(G, \cdot)$ *is a distributive Steiner quasigroup if and only if* $(G, \cdot)$ *satisfies* $c(x, y, z) = z$.

Proof. If $(G, \cdot)$ is a distributive Steiner quasigroup, then we have

$$c(x, y, z) = ((zx)\,y)\,((zy)\,x) = ((zy)\,(xy))\,((zy)\,x) = (zy)\,((xy)\,x) = zy^2 = z.$$

Let now $((yz)\,x)\,((xz)\,y) = z$. Putting into this identity $x = y$ we get $z = zy^2$. Thus $(G, \cdot)$ is a Steiner quasigroup. Using this fact we obtain

$$((zx)\,y)\,z = (c(x, y, z))\,((zx)\,y) = ((yz)\,x)\,((zx)\,y)^2 = (yz)\,x.$$

Hence $(G, \cdot)$ satisfies $(xy)\,z = ((xz)\,y)\,x$ and $(G, \cdot) \in V_2(\cdot)$. Using Proposition 1 we satisfy our requirement.

Analogously to the previous sections we have

Lemma 7.3. *Let* $(G, \cdot) \in V(\cdot)$. *Then* $(G, \cdot)$ *is a distributive Steiner quasigroup if and only if the polynomial* c *is not essentially ternary.*

From the above three lemmas we get

Proposition 4. *Let* $(G, \cdot) \in V(\cdot)$. *Then the following conditions are equivalent*:
(c_1) $(G, \cdot)$ *is a distributive Steiner quasigroup,*

(c_2) *$(G, \cdot)$ satisfies $((zx)\,y)\,((zy)\,x) = z$,*
(c_3) *the polynomial $c(x, y, z) = ((zx)\,y)\,((zy)\,x)$ is not essentially ternary.*

8. The polynomial d(x, v, z) = (((yz) x) ((zx) y)) (xy) z) over Steiner quasigroups

Lemma 8.1. *Let $(G, \cdot) \in V_2(\cdot)$. Then the polynomial d is not essentially ternary if and only if $d(x, y, z) = xy$.*

Proof. If $d(x, y, z) = xy$, then d is obviously not essentially ternary. Conversely, assume that d is not essentially ternary over $(G, \cdot)$. Using Lemma 3.1 we infer that d admits the transposition (x, y) of its variables. If $(G, \cdot)$ is a one-element groupoid, then simply $d(x, y, z) = xy$. If again card $G \geqslant 2$, then by Lemma 1.3 the groupoid $(G, \cdot)$ is proper and therefore the assumption implies that $d(x, y, z) = z$ or $d(x, y, z) = d^*(x, y)$ where d^* is essentially binary. In the first case we have $xy = (xy)\,z^2 = z((xy)\,z) = (d(x, y, z))\,((xy)\,z) = c(x, y, z)$. Hence we get $y = yy = c(y, y, z) = ((yz)\,y)\,((zy)\,y) = zy^2 = z$, which is impossible. If the second case, holds, then we have

$$d^*(x, y) = d(x, y, y) = (((yy)\,x)\,((yx)\,y))\,((xy)\,y) = (yx^2)\,(xy^2) = xy.$$

Let us add here that the fact that $d^*(x, y) = xy$ also follows from Lemma 1.2. The proof of the lemma is completed.

Let us mention (see Lemma 3.4) that there exist proper groupoids $(G, \cdot)$ from $V(\cdot)$ for which $d(x, y, z) = z$ but such groupoids do not belong to the variety $V_2(\cdot)$. However, as Lemma 3.4 shows they belong to the variety $V_4(\cdot)$. It is clear that $V_2(\cdot) \subset V_4(\cdot)$.

Lemma 8.2. *Let $(G, \cdot) \in V_2(\cdot)$. Then $(G, \cdot)$ is distributive if and only if $(G, \cdot)$ satisfies the identity $d(x, y, z) = xy$.*

Proof. If $(G, \cdot)$ is a distributive Steiner quasigroup, then $d(x, y, z) = (((yz)\;x)\;((zx)\;y))\;((xy)\;z) = ((xy)\;(xz))\;((xy)\;(yz))\;((xy)\;z) = ((xy)\;(xz)\;(yz)))\;((xy)\;z) = ((xy)\;((xy)\;z))\;((xy)\;z) = xy$. Hence $d(x, y, z) = xy$.
Let now $d(x, y, z) = xy$. Then we have $z = z(xy)^2 = (xy)\;((xy)\,z) = (d(x,y,z))\,((xy)\,z) = ((yz)\,x)\,((zx)\,y) = c(x,y,z)$. Thus we get $c(x,y,z) = z$. Applying Proposition 4 we deduce that $(G, \cdot)$ is distributive.

Analogously to the previous sections one gets

Lemma 8.3. *Let $(G, \cdot) \in V_2(\cdot)$. Then $(G, \cdot)$ is distributive if and only if the polynomial $d(x, y, z)$ is not essentially ternary.*

Summarizing we get

Proposition 5. *Let $(G, \cdot) \in V_2(\cdot)$. Then the following conditions are equivalent:*
(d_1) *$(G, \cdot)$ is distributive,*

(d_2) *$(G, \cdot)$ satisfies $(((yz)x)(((zx)y))((xy)z) = z$,*
(d_3) *the polynomial $d(x, y, z) = (((yz)x)((zx)y)(xy)z)$ is not essentially ternary.*

9. Proofs of Theorems

Theorem 1 follows from Propositions 1 and 5, Theorem 2 follows from Propositions 2, 3 and 4. Now we give the proof of Theorem 3. If card $G = 1$, then $(G, \cdot)$ as a one-element groupoid is simultaneously a semilattice and also a distributive Steiner quasigroup. Let now card $G \geqslant 2$. Using Lemma 1.3 we infer that $(G, \cdot)$ is proper and the following polynomials xyz, yzx, zxy are essentially ternary in $(G, \cdot)$. If xy is associative, then $(G, \cdot)$ is a semilattice and the assertion follows. If xy is not associative, then the polynomials xyz, yzx and zxy are different (see Lemmas 3.1 and 3.3). Since in our groupoid $(G, \cdot)$ we have $p_3(G, \cdot) = 3$ we infer that the polynomials xyz, yzx and zxy are the only essentially ternary polynomials over $(G, \cdot)$. Applying now Lemma 1.3 we deduce that the polynomial $g(x, y, z) = (xz)(yz)$ is essentially ternary. Again by Lemma 3.3 we get $G(g) \cong S_2$. If $f \neq g$, then applying Lemmas 3.1—3.3 we infer that the following polynomials:

$$xyz,\ yzx,\ zxy,\ (xz)(yz),\ (xy)(zy) \text{ and } (yx)(zx)$$

are essentially ternary and pairwise distinct. This gives $p_3(G, \cdot) \geqslant 6$, — a contradiction. Thus we have proved that $(G, \cdot)$ is distributive. Consider now the polynomial $x \circ y = xy^2$. Using Theorem 1 of [2] we infer that $xy^2 \neq y$. Thus if $x \circ y$ is not essentially binary, then $(G, \cdot)$ is a distributive Steiner quasigroup. If again $x \circ y$ is essentially binary and $x \circ y = xy$, then the application of Theorem 8 of [3] proves that $(G, \cdot)$ is a semilattice. This contradicts $p_3(G, \cdot) = 3$. If $x \circ y$ is essentially binary and $x \circ y \neq xy$, then using Theorem 1 of [6] we infer that $p_3(G, \cdot) = p_3(G, \cdot, \circ) \geqslant p_2(G, \cdot, \circ) + 2 - 1 \geqslant 4$ provided $x \circ y$ is noncommutative. Hence $p_3(G, \cdot) \geqslant 4$, — a contradiction. If $x \circ y$ is commutative and $x \circ y \neq xy$, then applying Lemma 4 of [9] we obtain $p_3(G, \cdot) \geqslant 8$, which again contradicts $p_3(G, \cdot) = 3$. This completes the proof of Theorem 3.

REFERENCES

[1] BELOUSOV, W.: Introduction into the Theory of Quasigroups and Loops. Moscow, 1967 (in Russian).
[2] DUDEK, J.: On binary polynomials in idempotent commutative groupoids. Fund. Math., 120, 1984, 187 191.
[3] DUDEK, J.: Varieties of idempotent commutative groupoids. Ibidem.
[4] DUDEK, J.: On bisemilattices I. Colloq. Math., 47, 1982, 1—5.
[5] GRÄTZER, G.: Universal Algebra. Springer-Verlag 1979.
[6] GRÄTZER, G.—PŁONKA, J.: On the number of polynomials of an idempotent algeba I. Pacific J. Math. 32, 1970, 697—709.

[7] MARCZEWSKI, E.: Independence and homomorphisms in abstract algebras. Fund. Math., 50, 1961, 45—61.
[8] PARK, R. E.: A four-element algebra whose identities are not finitely based. Algebra Universalis 11, 1980, 225—260.
[9] PŁONKA, J.: On the number of independent elements in finite abstract algebras with two binary symmetrical operations, Colloq. Math., 19, 1968, 9—21.
[10] PŁONKA, J.: On the arity of idempotent reducts of groups. Colloq. Math., 21, 1970, 35—37.

Received April 26, 1985

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ОБ КВАЗИГРУППАХ ШТЕЙНЕРА

Jozef Dudek

Резюме

В этой работе рассматриваются квазигруппы Штейнера. В частности, доказаны необходимые и достаточные условия для того, чтобы идемпотентный коммутативный группоид был дистрибутивной квазигруппой Штейнера.