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THE GREATEST ARCHIMEDEAN IDEAL IN A SEMIGROUP

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Let S be a semigroup. By an ideal we mean a non-empty two-sided ideal of S. An ideal $Q \subseteq S$ is prime if for any ideals A, B of S, $AB \subseteq Q$ implies $A \subseteq Q$ or $B \subseteq Q$. An ideal $P \subseteq S$ is completely prime if for any $a, b \in S$, $ab \in P$ implies $a \in P$ or $b \in P$. An ideal $A \in S$ is called completely semiprime if $a^n \in A$ for any positive integer n implies $a \in A$.

Denote by Q^* the intersection of all prime ideals of S and by P^* the intersection of all completely prime ideals of S (see [3]). It is known (see [4]) that P^* and Q^* may be empty.

An element $x \in S$ is nilpotent with respect to an ideal J if $x^n \in J$ for some positive integer n. An ideal I is called a nilideal with respect to an ideal J if any x of I is nilpotent with respect to J. Denote by $R^*(J)$ the Clifford radical with respect to J, i.e. the union of all nilideals of S with respect to J (see [5]).

A semigroup S is archimedean (see [2]) if for any $a, b \in S$ there exists a positive integer n such that $a^n \in SbS$. An ideal $I \subseteq S$ is called archimedean if the subsemigroup I is archimedean.

Denote by U^* the intersection of all prime ideals $Q \subseteq S$ with the property $R^*(Q) = Q$.

M. Satyanarayana [3, Theorem 10] proved that if Q^* is a completely semiprime ideal, then Q^* is the greatest archimedean ideal of the semigroup S.

By M. Satyanarayana [3, p. 291] "It is an open problem whether this result is true in arbitrary case".

In this note we have solved this problem. The result is as follows: If $U^* \neq \emptyset$, then it is the greatest archimedean ideal of S (Theorems 2 and 3). Concluding this result we give a new characterization of the Clifford radical $R^*(J)$ as an intersection of some (in general not all) prime ideals.

Theorem 1. *Let S be a semigroup and $U^* \neq \emptyset$. Then $R^*(U^*) = U^*$.*

Proof. Let $U^* = \bigcap_{\lambda \in \Lambda} Q_\lambda$. Evidently $U^* \subseteq R^*(U^*)$. Suppose that $U^* \subset R^*(U^*)$, $U^* \neq R^*(U^*)$. Let $x \in R^*(U^*) - U^*$. Since $x \notin U^*$, then $x \notin Q_\alpha$ for some $\alpha \in \Lambda$. Then the principal ideal $(x) \nsubseteq Q_\alpha = R^*(Q_\alpha)$, hence there exists $y \in (x)$ such that $y^n \notin Q_\alpha$ for all positive integers n. Since $Q_\alpha \supseteq U^*$, y is not nilpotent with respect to U^*, a contradiction to $y \in (x) \subseteqq R^*(U^*)$. Therefore $U^* = R^*(U^*)$ holds.

Lemma 1. *Let S be a semigroup, J an ideal, $H = \{x, x^2, x^3, \ldots\}$ a cyclic subsemigroup of S and let $H \cap J = \emptyset$. Then there exists a prime ideal Q containing J such that $H \cap Q = \emptyset$ and $R^*(Q) = Q$.*

Proof. Denote by $\boldsymbol{T}$ the set of all ideals which contain J and do not meet H. The set $\boldsymbol{T}$ is non-empty since it contains J. By Zorn's lemma there exists a maximal element $Q \in \boldsymbol{T}$.

We prove that Q is a prime ideal. Suppose that for some ideals $A \nsubseteq Q$ and $B \nsubseteq Q$ we have $AB \subseteq Q$. Then $x^r \in Q \cup A$, $x^s \in Q \cup B$ for some posi tive integers r, s. Since x^r, $x^s \notin Q$, we have $x^r \in A$, $x^s \in B$, thus $x^{r+s} \in AB \subseteq Q$, which contradicts $H \cap Q = \emptyset$. Therefore Q is a prime ideal.

We prove that $R^*(Q) = Q$. Evidently $Q \subseteq R^*(Q)$. Suppose that $Q \subset R^*(Q)$, $Q \neq R^*(Q)$. Then $x^m \in H \cap R^*(Q)$ for some positive integer m. However, $x^m \in R^*(Q)$ implies $(x^m)^n \in Q$ for some positive integer n. This is a contradiction to $Q \cap H = \emptyset$. Therefore $R^*(Q) = Q$.

Lemma 2. *Let S be a semigroup, $x \in U^*$ and A be any ideal of S. Then $x^n \in A$ for some positive integer n.*

Proof. If $A = S$, then the statement holds. Suppose therefore that A is a proper ideal of S and $x^n \notin A$ for all positive integers n. By Lemma 1 there exists a prime ideal $Q = R^*(Q)$ such that $x \notin Q$. This contradicts $x \in U^*$. Thus for any proper ideal A we have $x^n \in A$ for some positive integer n.

Theorem 2. *Let S be a semigroup and $U^* \neq \emptyset$. Then U^* is an archimedean ideal.*

Proof. Let x, $y \in U^*$. Then by Lemma 2, $x^n \in (y) = S^1 y S^1$ for some positive integer n. From this we have obtain that $x^{n+2} \in x S^1 y S^1 x \subseteq U^* y U^*$. Thus U^* is an archimedean semigroup.

Theorem 3. *Let S be a semigroup $U^* \neq \emptyset$, and let A be an ideal of S. Then A in an archimedean ideal if and only if $A \subseteq U^*$.*

Proof. Let A be an archimedean ideal of S. Suppose $A \nsubseteq U^*$. By Theorem 1, $U^* = R^*(U^*)$ therefore $A \nsubseteq R^*(U^*)$. Then we obtain that A is not a nilideal with respect to the ideal U^*. Therefore there exists an element $y \in A$ such that $y^n \notin U^*$ for all positive integers n. Let $a \in A \cap U^*$. Then $AaA \subseteq (a) \subseteq U^*$ and thus $y^n \notin AaA$ for any positive integer n. This contradicts the assumption that A is an archimedean semigroup. Therefore for any archimedean ideal A of S we have $A \subseteq U^*$.

Conversely, suppose that A is an ideal of S and $A \subseteq U^*$. Then for any $x, y \in A$ we have by Theorem 2, $x^n \in U^* y U^*$ for some positive integer n. Then $x^{n+2} \in x U^* y U^* x \subseteq AyA$, which means that A is an archimedean semigroup.

We now give a new characterization of the Clifford radical $R^*(J)$.

Theorem 4. *Let S be a semigroup, J an ideal,* $\{Q_\lambda / \lambda \in \Lambda\}$ be *the set of all prime ideals of S containing J with the property $R^*(Q_\lambda) = Q_\lambda$. Then $R^*(J) = \bigcap_{\lambda \in \Lambda} Q_\lambda$.*

Proof. By the assumption $J \subseteq Q_\lambda$ implies $R^*(J) \subseteq R^*(Q_\lambda) = Q_\lambda$, hence

$$R^*(J) \subseteq \bigcap_{\lambda \in \Lambda} Q_\lambda.$$

Conversely, we show that $\bigcap_{\lambda \in \Lambda} Q_\lambda \subseteq R^*(J)$. If $R^*(J) = S$, then $\bigcap_{\lambda \in \Lambda} Q_\lambda \subseteq S$ holds. Suppose therefore that $R^*(J) \neq S$. It is sufficient to show that for any $x \notin R^*(J)$ there exists an $\alpha \in \Lambda$ such that $x \notin Q_\alpha$. Let $x \notin R^*(J)$. Then the principal ideal $(x) \nsubseteq R^*(J)$ and so there exists $y \in (x)$ such that $y^n \notin J$ for all positive integers n. Denote $H = \{y, y^2, y^3, \ldots\}$. We have $H \cap J = \emptyset$. By Lemma 1 there exists a prime ideal Q_α such that $H \cap Q_\alpha = \emptyset$, $Q_\alpha \supseteq J$ and $Q_\alpha = R^*(Q_\alpha)$. We have $x \notin Q_\alpha$ since $x \in Q_\alpha$ would imply $(x) \subseteq Q_\alpha$, hence $y \in Q_\alpha$, a contradiction with $H \cap Q_\alpha = \emptyset$.

Next we shall show some relations concerning radicals and the sets Q^*, U^*, P^*.

Let S be a semigroup with an ideal J. The McCoy radical $M(J)$ with respect to J is the intersection of all prime ideals of S containing J. The Luh radical $C(J)$ with respect to J is the intersection of all completely prime ideals containing J.

If S is a semigroup with a zero 0, then $M(0) = Q^*$, $C(0) = P^*$ and by Theorem 4, $R^*(0) = U^*$.

The following examples show that there are semigroups with $Q^* \neq U^*$ and $U^* \neq P^*$.

Example 1. Let S_1 be a semigroup generated by a set $\{0, a_1, a_2, \ldots, a_n, \ldots\}$ subject to the generating relations $0 . x = x . 0 = x^2$ for any $x \in S$. Then $M(0) = 0$ and $R^*(0) = S_1$ (see [1, p. 232]). Thus in S_1 we have $0 = Q^* \neq U^* = S_1$.

Example 2. Let $S_2 = \{0, e_{11}, e_{12}, e_{21}, e_{22}\}$ be a semigroup with the multiplication $e_{ik} . e_{kn} = e_{in}$, $e_{ik} . e_{jn} = 0 . e_{ik} = e_{ik} . 0 = 0$ for $i, j, k, n \in \{1, 2\}, j \neq k$. Then $U^* = R^*(0) = 0$, $P^* = C(0) = S_2$, thus $U^* \neq P^*$. Evidently U^* is not a completely semiprime ideal of S_2 since $e_{12}^2 \in U^*$, $e_{12} \notin U^*$.

We note that if P^* is non-empty then it is a completely semiprime ideal of S.

Theorem 5. *Let S be a semigroup. Then $Q^* \subseteq U^* \subseteq P^*$. If $U^* \neq \emptyset$ and $U^* \neq P^*$, then U^* is not a completely semiprime ideal of S.*

Proof. Let $\boldsymbol{U} = \{Q_\lambda / \lambda \in \Lambda\}$ be the set of all prime ideals of S with the property $R^*(Q_\lambda) = Q_\lambda$. Since $\boldsymbol{U}$ is a subset of the set of all prime ideals of S we have $Q^* \subseteq U^*$.

Let $\boldsymbol{P} = \{P_\lambda \mid \lambda \in \Lambda_1\}$ be the set of all completely prime ideals of S. Evidently a completely prime ideal is prime. The inclusions $P_\lambda \subseteq R^*(P_\lambda) \subseteq C(P_\lambda)$ (see [5, Lemma 19]) and the equality $C(P_\lambda) = P_\lambda$ imply $R^*(P_\lambda) = P_\lambda$, for any $\lambda \in \Lambda_1$. Therefore $\boldsymbol{P}$ is the set of all such prime ideals of S which have the property $R^*(P_\lambda) = P_\lambda$ and are completely prime. Hence $\boldsymbol{P} \subseteq \boldsymbol{U}$ and so $U^* \subseteq P^*$.

An ideal $A \subset P^*$, $A \neq P^*$ cannot be completely semiprime, since the assumption that A is completely semiprime would imply $A = \bigcap_{\lambda \in \Lambda_2} P_\lambda$ (see [2, Theorem II. 3.7]) where $\Lambda_2 \subseteq \Lambda_1$ hence $P^* \subseteq A$, a contradiction.

Thus if $U^* \neq \emptyset$ and $U^* \neq P^*$, then U^* is not a completely semiprime ideal of S.

REFERENCES

[1] BOSÁK, J.: О радикалах полугрупп. Mat. fyz. čas. 12, 1962, 230—234.

[2] PETRICH, M.: Introduction to Semigroups, Ch. Merrill, Columbus, Ohio, 1973.

[3] SATYANARAYANA, M.: On generalized kernels I. Semigroup Forum 12, 1976, 283 292.

[4] SCHWARZ, Š.: Prime ideals and maximal ideals in semigroups. Czech. Math. J. 19 (94), 1969, 72 79.

[5] ŠULKA, R.: О нильпотентных элементах, идеалах и радикалах полугруппы. Mat. fyz. čas. 13, 1963, 209—222.

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НАЙБОЛЬШИЙ АРХИМЕДОВ ИДЕАЛ В ПОЛУГРУППЕ

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Резюме

Пусть U^*-пересечение всех простых идеалов Q полугруппы S, обладающих свойством $R^*(Q) = Q$, где $R^*(Q)$-радикал Клиффорда относительно Q.

Доказано, что если $U^* \neq \emptyset$, то U^* является найбольшим архимедовым идеалом полугруппы.