Mathematica Slovaca

Bohdan Zelinka

Connected domatic number of a graph

Mathematica Slovaca, Vol. 36 (1986), No. 4, 387--392

Persistent URL: http://dml.cz/dmlcz/136430

Math. Slovaca 36, 1986, No. 4, 387—392

CONNECTED DOMATIC NUMBER OF A GRAPH

BOHDAN ZELINKA

All graphs considered in this paper are finite graphs without loops and multiple edges.

The domatic number of a graph was defined by E. J. Cockayne and S. T. Hedetniemi [1]. Later some related concepts were introduced. The same authors together with R. M. Dawes [2] have introduced the total domatic number; R. Laskar and S. T. Hedetniemi [3] have introduced the connected domatic number.

A dominating set (or a total dominating set) in an undirected graph G is a subset D of the vertex set $V(G)$ of G with the property that to each vertex $x \in V(G) - D$ (or to each vertex $x \in V(G)$ respectively) there exists a vertex $y \in D$ adjacent to x. A connected dominating set of G is a dominating set of G with the property that the subgraph of G induced by it is connected.

A domatic (or total domatic, or connected domatic) partition of G is a partition of $V(G)$, all of whose classes are dominating (or total dominating, or connected dominating, respectively) sets of G. The maximum number of classes of a domatic (or total domatic, or connected domatic) partition of G is called the domatic (or total domatic, or connected domatic, respectively) number of G. The domatic number of G is denoted by $d(G)$, its total domatic number by $d_t(G)$, its connected domatic number by $d_c(G)$.

The connected domatic number of a graph is well defined only for connected graphs; in a disconnected graph there exists no connected dominating set and thus no connected domatic partition, while in every connected graph there exists at least one connected domatic partition, namely that which consists of one class. The connected domatic number of G is closely related to the vertex connectivity number of G.

If G is a connected graph, then a vertex cut of G is a subset R of $V(G)$ with the property that the subgraph of G induced by $V(G) - R$ is disconnected. If G is not a complete graph, then the vertex connectivity number $\varkappa(G)$ is the minimum cardinality of a vertex cut of G. If G is a complete graph (i. e. without vertex cuts) with n vertices, then we put $\varkappa(G) = n - 1$.

Lemma. *Let G be a connected graph which is not complete, let R be its vertex cut, let D be its connected dominating set. Then $D \cap R \neq \emptyset$.*

Proof. Let $C_1, \ldots, C_k$ be the connected components of the subgraph G' of G induced by the set $V(G)-R$; evidently $k \geqq 2$. Suppose that $D \cap R = \emptyset$. As the subgraph of G induced by D is connected, it is a subgraph of C_i for some $i \in \{1, \ldots, k\}$. Let $x \in C_j$ for $j \neq i$. Then $x \notin D$ and x is adjacent to no vertex of D, which is a contradiction. Hence $D \cap R \neq \emptyset$.

Theorem 1. *Let G be a connected graph which is not complete, let $d_c(G)$ be its connected domatic number, let $\varkappa(G)$ be its vertex connectivity number. Then*

$$d_c(G) \leqq \varkappa(G).$$

Proof. Let R be a vertex cut of G of the cardinality $\varkappa(G)$, let $\mathscr{D} = \{D_1, \ldots, D_d\}$ be a connected domatic partition of G having $d = d_c(G)$ classes. According to Lemma we have $D_i \cap R \neq \emptyset$ for $i = 1, \ldots, d$. As $D_1, \ldots, D_d$ are pairwise disjoint, the sets $D_1 \cap R, \ldots, D_d \cap R$ are also pairwise disjoint, therefore $d = d_c(G) \leqq \varkappa(G)$.

For complete graphs this assertion does not hold. For the complete graph K_n with n vertices we have

$$\varkappa(K_n) = n-1 < n = d_c(K_n).$$

Using this result, we can prove another theorem.

Theorem 2. *For an arbitrary positive integer q there exists a graph G such that*

$$d(G) - d_c(G) = q.$$

Proof. Let G' be a complete graph with vertices $u, v'_1, \ldots, v'_q$, let G'' be a complete graph with vertices $u, v''_1, \ldots, v''_q$. The vertices $u, v'_1, \ldots, v'_n, v''_1, \ldots, v''_n$ are pairwise distinct and the vertex u is common to both G' and G''. Let G be the union of G' and G''. The set $\{u\}$ is a vertex cut of G, therefore $\varkappa(G) = 1$ and by Theorem 1, $d_c(G) = 1$. The domatic number $d(G) = q+1$, because $\{\{u\}, \{v'_1, v''_1\}, \ldots, \{v'_q, v''_q\}\}$ is evidently a domatic partition of G.

Theorem 3. *Let $n \geqq 3$ be an integer. For each k such that $1 \leqq k \leqq n-2$ or $k = n$ there exists a graph G with n* vertices such that *$d_c(G) = k$. For $k = n-1$ such a graph does not exist.*

Proof. For $k = n$ the required graph is the complete graph K_n. For $k = 1$ it is a path with n vertices. For $2 \leqq k \leqq n-2$ it is obtained from a path with $n-k+1$ vertices and a complete graph with $k-1$ vertices by joining all vertices of one graph with all vertices of the other. The connected domatic partition with k classes consists then of $k-1$ one-element sets consisting of vertices of the complete graph and of the vertex set of the path. Now suppose that there exists a graph G with n vertices and with the connected domatic number $n-1$. Let $\mathscr{D}$ be a connected domatic partition of G with $n-1$ classes. Then exactly one class of $\mathscr{D}$ has the cardinality 2, all the others have the cardinality 1. Let $\{u, v\}$ be the class of $\mathscr{D}$ of the cardinality 2. Let $x \in V(G) - \{u, v\}$. As $\{x\}$ is a dominating set of G, the vertex x is saturated and u and v are adjacent to x; as x was chosen arbitrarily, these vertices are adjacent to all vertices of $V(G)-\{u, v\}$. As $\{u, v\}$ is a connected dominating set, they are adjacent also to each other; thus the graph G is complete and $d_c(G)=n$, which is a contradiction.

In the study of the connected domatic number an important role is played by the saturated vertices. We have written about them in the proof of Theorem 3. We remember that a saturated vertex of a graph G is a vertex which is adjacent to all other vertices of G. The one-element set is a connected dominating set in G if and only if its element is a saturated vertex of G.

Theorem 4. *Let G be a connected undirected graph, let n be the number of its vertices, let n_0 be the number of its saturated vertices. Then*

$$d_c(G) \leqq \frac{1}{2}(n+n_0).$$

Proof. A one-element subset of $V(G)$ is a dominating set if and only if its element is a saturated vertex. Thus in any connected domatic partition of G there are at most n_0 classes of the cardinality 1; other classes must have the cardinality at least 2. Hence there are at most $n_0+\frac{1}{2}(n-n_0)=\frac{1}{2}(n+n_0)$ classes of a connected domatic partition of G.

Theorem 5. *Let G be a connected undirected graph with at least three vertices. Let e be an edge of G which is not its bridge, let G' be the graph obtained from G by deleting e. If e joins two saturated vertices of G, then $d_c(G') \geqq d_c(G)-2$, otherwise $d_c(G') \geqq d_c(G)-1$.*

Proof. Let u, v be the end vertices of e. Let $\mathscr{D}$ be a connected domatic partition of G having $d_c(G)$ classes. Suppose that $\{u\} \in \mathscr{D}$, $\{v\} \in \mathscr{D}$; this is possible only if both u and v are saturated vertices of G. In G' the vertices u and v are not saturated and thus $\{u\}$ and $\{v\}$ are not dominating sets in G'. Let $D \in \mathscr{D}-\{\{u\}, \{v\}\}$. The set $D \cup \{u, v\}$ is evidently a connected dominating set in G'. Thus if we omit D, $\{u\}$, $\{v\}$ from $\mathscr{D}$ and add $D \cup \{u, v\}$ to it, we obtain a connected domatic partition of G' with $d_c(G)-2$ classes and $d_c(G') \geqq d_c(G)-2$. Now suppose that $\{u\} \notin \mathscr{D}$; then there exists $D_1 \in \mathscr{D}$ such that $u \in D_1$ and $|D_1| \geqq 2$. Let D_2 be the class of $\mathscr{D}$ which contains v. If $D_1 \neq D_2$, let $w \in D_1$, $w \neq u$. Then w is adjacent to a vertex $x \in D_2$, because D_2 is a dominating set. The set $D_1 \cup D_2$ is a connected dominating set in G' and thus if we omit D_1 and D_2 from $\mathscr{D}$ and add $D_1 \cup D_2$ to it, we obtain a connected domatic partition of G' with $d_c(G)-1$ classes. Hence $d_c(G') \geqq d_c(G)-1$. If $D_1=D_2$, let $D_3 \in \mathscr{D}$, $D_3 \neq D_1$. As D_3 is a connected dominating set, there exist vertices x_1, x_2 such that u is adjacent to x_1 and v is adjacent to x_2; the vertices x_1, x_2 may coincide. (We suppose that $|\mathscr{D}| \geqq 2$; the case $|\mathscr{D}|=1$ is trivial.) Then $D_1 \cup D_3$ is a connected dominating set. If we omit D_1 and D_3 from $\mathscr{D}$ and add $D_1 \cup D_3$ to it, we obtain a connected domatic partition of G' with $d_c(G)-1$ classes and the assertion is true.

An example of a graph whose connected domatic number decreases by two by deleting one edge (moreover an arbitrary edge) is a complete graph with at least three vertices.

Now we shall define a connectively domatically critical graph. We say that a graph G is connectively domatically critical if $d_c(G') < d_c(G)$ for each graph G' obtained from G by deleting an edge.

Theorem 6. *Let G be a connectively domatically critical graph, let $d_c(G) = d$. Then the vertex set of G is the union of pairwise disjoint sets $D_1, \ldots, D_d$ such that*
(i) *the subgraph G_i of G induced by D_i is a tree for each $i = 1, \ldots, d$;*
(ii) *the subgraph G_{ij} of G with the vertex set $D_i \cup D_j$ and with the edge set consisting of all edges joining a vertex of D_i with a vertex of D_j is a forest, each of whose connected components is a star or a complete graph with two vertices for any i, j from the set $\{1, \ldots, d\}$, $i \neq j$.*

Proof. Let $\mathscr{D} = \{D_1, \ldots, D_d\}$ be a domatic partition of G with d classes. Then for each $i = 1, \ldots, d$ the subgraph G_i of G induced by D_i must be connected. If we delete any edge from G_i, the sets $D_1, \ldots, D_d$ remain dominating sets; hence it is necessary that after this deleting the graph G_i would be disconnected. This is possible if and only if G is a tree. The graph G_{ij} for any i, j from $\{1, \ldots, d\}$, $i \neq j$, has evidently no isolated vertex. Let e be an edge of G_{ij}, let v_i, v_j be its and vertices, $v_i \in D_i$, $v_j \in D_j$. The edge e must have the property that in the graph G' obtained from G by deleting e either D_i, or D_j is not a dominating set; otherwise we should have $d_c(G') = d = d(G)$ and G would not be critical. Hence in G' either v_i is not adjacent to a vertex of D_j, or v_j is not adjacent to a vertex of D_i. This implies that at least one of the end vertices of e has degree 1 in G_{ij}. As each edge of G_{ij} must have this property, each connected component of G_{ij} is a star or a complete graph with two vertices.

Remarks. 1. A graph consisting of one vertex is also considered a tree.

2. An example of a graph fulfilling the assumptions of Theorem 6 is any Cartesian product of a complete graph with a tree.

Theorem 7. *Let G be a connectively domatically critical graph with $d_c(G) = d$. If G is regular of degree $d-1$, then $G \cong K_d$. If G is regular of degree d, then $G_i \cong K_2$ for each $i \in \{1, \ldots, d\}$ and G_{ij} consists of two connected components isomorphic to K_2 for any i, j from $\{1, \ldots, d\}$, $i \neq j$.*

Proof. If $x \in V(G)$, then $x \in D_i$ for some $i \in \{1, \ldots, d\}$. In each G_{ij} for $j \in \{1, \ldots, d\}$, $j \neq i$, there is at least one edge incident with x. Thus if G is regular of degree $d-1$, then all edges of G belong to the graphs G_{ij} and none of them is in any G_i. As the graphs G_i are connected, they must consist of one vertex and $G \cong K_d$. If G is regular of degree d, then, as at least $d-1$ edges incident with x belong to the graphs G_{ij}, the degree of x in G_i is at most one. As x was chosen arbitrarily, none of the graphs G_i can have a vertex of a degree greater than one. Thus each G_i is either isomorphic to K_2, or consists of one vertex. If all graphs G_i

consist of one vertex, then G is a complete graph and is regular of degree $d-1$, which is a contradiction. If only one of the graphs G_i is isomorphic to K_2, then $G \cong K_{d+1}$ and $d_c(G)=d+1$, which is a contradiction. Thus among the graphs $G_1, \ldots, G_d$ there are at least two which are isomorphic to K_2. Without loss of generality let G_1, G_2 be such graphs. If $d=2$, then the assertion is true. If $d \geqq 3$, suppose without loss of generality that G_3 consists of one vertex. Then the (unique) vertex u of G_3 is a saturated vertex of G. As $|D_1|=|D_2|=2$ and $|D_i| \geqq 1$ for each i, the graph G has at least $d+2$ vertices and the degree of u is at least $d+1$, which is a contradiction. Hence all graphs G_i are isomorphic to K_2. Then each G_{ij} is a graph with four vertices and regular of degree 1; hence it consists of two disjoint copies of K_2.

Concluding this paper we shall mention open problems from [3], where two results of F. Jaegar and C. Payan [4] for domination numbers and domatic numbers of graphs are quoted; it is conjectured that they hold also for the connected domination number (the minimum cardinality of a connected dominating set in G) and the connected domatic number of G. These results are the following:

$$\gamma(G) \cdot \gamma(\bar{G}) \leqq n,$$

$$\gamma(G) \leqq d(\bar{G}),$$

where $\gamma(G)$ is the domination number of G, the symbol $\bar{G}$ denotes the complement of G and n is the number of vertices of G.

We shall show a counterexample disproving this conjecture. The symbol $\gamma_c(G)$ will denote the connected domination number of G.

Theorem 8. *There exists a graph G such that*

$$\gamma_c(G) \cdot \gamma_c(\bar{G}) > n,$$

$$\gamma_c(G) > d_c(\bar{G}).$$

Proof. Let n be an even integer, $n \geqq 6$. Let C_n be the circuit of the length n. Then $\gamma_c(C_n)=n-1$, $\gamma_c(\bar{C}_n)=2$, $d_c(C_n)=1$, $d_c(\bar{C}_n)=n/2$. We see that both the graphs C_n and $\bar{C}_n$ have the required properties.

REFERENCES

[1] COCKAYNE, E. J.—HEDETNIEMI, S. T.: Towards a theory of domination in graphs. Networks 7, 1977, 247—261.

[2] COCKAYNE, E. J.—DAWES, R.—HEDETNIEMI, S. T.: Total domination in graphs. Networks 10, 1980, 211—219.

[3] LASKAR, R.—HEDETNIEMI, S. T.: Connected Domination in Graphs. Tech. Report 414, Clemson Univ., Clemson, South Carolina, March 1983.

[4] JAEGAR, F.—PAYAN, C.: Relations du type Nordhaus-Gaddum pour le numbre d'absorption d'un graphe simple. C. R. Acad. Sci. Paris, Series A 274, 1972, 728—730.

Received October 15, 1984

Katedra tváření a plastů
Vysoké školy strojní a textilní
Studentská 1292
461 17 Liberec 1

СВЯЗНОЕ ДОМАТИЧЕСКОЕ ЧИСЛО ГРАФА

Bohdan Zelinka

Резюме

Доминирующее множество в графе G есть подмножество D множества $V(G)$ вершин графа G, обладающее тем свойством, что для каждой вершины $x \in V(G) - D$ существует вершина $y \in D$, смежная с x. Если подграф графа G, порожденный этим множеством D, является связным, то D называется связным доминирующим множеством в G. Разбиение множества $V(G)$, все классы которого являются связными доминирующими множествами в Γ, называется связным доматическим разбиением графа. максимальное число классов такого разбиения называется связным доматическим числом графа G и обозначается через $d_c(G)$. В статье исследованы свойства этого числа.