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The solution to the Gudder conjecture on an equivalence of the independence and the strong independence of observables on a logic of a separable Hilbert space, and its generalization to a logic of a Hilbert space whose dimension is a non-real measurable cardinal are given.

Further we bring a positive answer to a problem of the author: supposing a system of observables has a joint distribution in a state is then the state a valuation on the minimal sublogic generated by all ranges of the observables?

1. Definitions and preliminaries

Let L be a logic, that is, let L be a σ-lattice with the first and last elements 0 and 1, respectively, with an orthocomplementation $\perp$: $a \mapsto a^\perp$, $a, a^\perp \in L$, which satisfies: (i) $(a^\perp)^\perp = a$ for all $a \in L$; (ii) if $a < b$, then $b^\perp < a^\perp$; (iii) $a \vee a^\perp = 1$ for all $a \in L$; (iv) the orthomodular law holds in L: if $a < b$, then $b = a \vee (b \wedge a^\perp)$.

Two elements $a, b \in L$ are (i) orthogonal, and we write $a \perp b$, if $a < b^\perp$; (ii) compatible, and we write $a \leftrightarrow b$, if there are three mutually orthogonal elements a_1, b_1, c such that $a = a_1 \vee c$, $b = b_1 \vee c$. A logic is called separable if every system of mutually orthogonal nonzero elements is at most countable.

An observable is a map x: $B(R_1) \to L$ such that (i) $x(\emptyset) = 0$; (ii) if $E \cap F = \emptyset$, $E, F \in B(R_1)$, then $x(E) \perp y(F)$; (iii) $x\left(\bigcup_{i=1}^{\infty} E_i\right) = \bigvee_{i=1}^{\infty} x(E_i)$, $E_i \in B(R_1)$. The range of an observable x, $\mathcal{R}(x) = \{x(E)\colon E \in B(R_1)\}$, is a Boolean sub-$\sigma$-algebra of L. The spectrum of an observable x, $\sigma(x)$, is the smallest closed subset C of R_1 such that $x(C) = 1$. If $\sigma(x)$ is a compact, then x is said to be bounded. An observable x is purely atomic if (i) $\sigma(x) = \{\lambda_1, \lambda_2, \ldots\}$; (ii) $x(\{\lambda_i\})$ is an atom of L for any i. Two observables x and y are compatible if $x(E) \leftrightarrow y(F)$ for all $E, F \in B(R_1)$. There has been developed a calculus for compatible observables (see [13] and [15, Theorem 6. 17]). Therefore we may speak of a sum of compatible observables, of a scalar multiple of an observable, etc.

A state is a map m: $L \to [0, 1]$ such that (i) $m(1)=1$; (ii) $m\left(\bigvee_{i=1}^{\infty} a_i\right)=\sum_{i=1}^{\infty} m(a_i)$ if $a_i \perp a_j$, $i \neq j$.

If x is an observable and f a Borel function, then the mean value of $f \circ x$ in a state m is given by the formula $m(f \circ x)=\int_{-\infty}^{\infty} f(t)\, \mathrm{d} m_x(t)$ (provided the integral exists), where $m_x(E)=m(x(E))$, $E \in B(R_1)$.

We say that the observables $x_1, \ldots, x_n$ have a joint distribution in a state m if there is a probability measure $m_{x_1 \ldots x_x}$ on $B(R_n)$ such that

$$m_{x_1 \ldots x_n}(E_1 \times \ldots \times E_n)=m\left(\bigwedge_{i=1}^{n} x_i(E_i)\right), \tag{1.1}$$

$$E_i \in B(R_1), \quad i=1, \ldots, n.$$

The necessary and sufficient conditions for the existence of the joint distribution of observables in a given state may be found in [2, 4, 11, 12].

2. Joint measurability of Boolean subalgebras

One of important problems of the quantum logic theory is a determination of a joint distribution for noncompatible observables, as indicated, for example, in [10, Problem VII]. The answer to this question has been obtained in the above quoted papers [3, 4, 11, 12].

Here we shall investigate a similar problem for subalgebras of L. If x is an observable, then its range $\mathcal{R}(x)$ is a Boolean sub-σ-algebra of L with a countable set of generators. Conversely, for any Boolean sub-σ-algebra $\mathcal{A}$ of L with a countable set of generators there is an observable x such that $\mathcal{R}(x)=\mathcal{A}$ [15, Theorem 1,6]. Hence a study of a joint measurability (see the definition below) of Boolean sub-σ-algebras of L with countably many generators in a state m may be transformed to a study of a joint distribution of observables in a state. The general case of Boolean sub-σ-algebras will be solved in this section.

For any $a \in L$ we put $a^0=a^\perp$, $a^1=a$. Let $a \in L$. The nonempty subset $M \subset L$ is partially compatible with respect to a (in abbrevation, p.c. a) of (i) $M \leftrightarrow a$, that is, $b \leftrightarrow a$ for any $b \in M$; (ii) the set $M \wedge a=\{b \wedge a:\ b \in M\}$ is compatible in L, that is, $b_1 \wedge a \leftrightarrow b_2 \wedge a$ for any $b_1, b_2 \in M$. The condition (ii) may be equivalently expressed in the following way: $M \wedge a$ is the set of mutually compatible elements belonging to a logic $L_{(0,a)}=\{c \in L:\ c < a\}$ (here the orthocomplementation of c, c', is defined via $c'=a \wedge c^\perp$).

Let $F=\{a_1, \ldots, a_n\} \subset L$. According to [11] put

$$com\, F \equiv com\,(a_1, \ldots, a_n)=\bigvee_{d \in D^n} a_1^{d_1} \wedge \ldots \wedge a_n^{d_n}, \tag{2.1}$$

where $D=\{0, 1\}$, $d=(d_1, \ldots, d_n) \in D^n$.

The element $com\ F$ is called the commutator of the finite set F. For an arbitrary $\emptyset \neq M \subset L$ we define a commutator of M, com M, via

$$com\ M = \bigwedge \{com\ F\colon\ F\text{-finite subset of } M\} \tag{2.2}$$

(if it exists in L).

From [11, Theorem 3.10] it follows that if $com\ M$ exists, then M is p. c. $com\ M$. Especially, M is the set of mutually compatible elements iff $com\ M = 1$.

Let $\mathscr{A}_1, \ldots, \mathscr{A}_n$ be Boolean subalgebras of L, and let $\mathscr{A} = \bigcup_{i=1}^{n} \mathscr{A}_i$. For any n-tuple $(a_1, \ldots, a_n) \in \mathscr{A}_1 \times \ldots \times \mathscr{A}_n$ define

$$a^0(a_1, \ldots, a_n) = com\,(a_1, \ldots, a_n).$$

Put

$$a^0(\mathscr{A}_1, \ldots, \mathscr{A}_n) = \bigwedge \{a^0(a_1, \ldots, a_n)\colon\ a_i \in \mathscr{A}_i,\ \ i = 1, \ldots, n\} \tag{2.3}$$

if it exists in L.

We say that the Boolean subalgebras $\mathscr{A}_1, \ldots, \mathscr{A}_n$ are *jointly measurable* in a state m if

$$m(a^0(a_1, \ldots, a_n)) = 1 \ \text{ for any } \ a_i \in \mathscr{A}_i,\ \ i = 1, \ldots, n.$$

An arbitrary system of Boolean subalgebras of L, $\{\mathscr{A}_s\colon\ a \in S\}$, is jointly measurable in a state m if every finite subsystem of $\{\mathscr{A}_s\colon\ s \in S\}$ is jointly measurable.

In order to state a necessary and sufficient condition for a system of Boolean subalgebras to be jointly measurable in a state m we have to prove the following lemma.

Lemma 2.1. *Let $A_1 = \{a_1^1, \ldots, a_{k_1}^1\} \subset \mathscr{A}_1, \ldots, A_s = \{a_1^s, \ldots, a_{k_s}^s\} \subset \mathscr{A}_s$ be systems of orthogonal elements, $1 \leq s \leq n$. For any $i = 1, \ldots, s$, define*

$$a_{j_i t}^i = \begin{cases} a_{j_i}^i & \text{if } t = 0, \\[2ex] \bigwedge\limits_{j_i=1}^{k_i} a_{j_i}^{i\perp} & \text{if } t = 1. \end{cases}$$

Then for any $b_{s+1} \in \mathscr{A}_{s+1}, \ldots, b_n \in \mathscr{A}_n$ we have

$$\bigwedge_{j_1=1}^{k_1} \cdots \bigwedge_{j_s=1}^{k_s} com\,(a_{j_1}^1, \ldots, a_{j_s}^s, b_{s+1}, \ldots, b_n) =$$

$$= \bigvee_{d \in D^{n-s}} \ \bigvee_{\substack{t_1 + \ldots + t_s \leq s \\ t_i \in \{0,1\}}} \ \bigvee_{j_1=1}^{k_1} \cdots \bigvee_{j_s=1}^{k_s} a_{j_1 t_1}^1 \wedge \ldots \wedge a_{j_s k_s}^s \wedge b^d =$$

$$= com\,(A_1 \cup \ldots \cup A_s \cup B), \tag{2.4}$$

where $b^d = b_{s+1}^{d_{s+1}} \wedge \ldots \wedge b_n^{d_n}$, $d = (d_{s+1}, \ldots, d_n) \in D^{n-s}$, $B = \{b_{s+1}, \ldots, b_n\}$.

Proof. We shall prove (2.4) by induction over s and k_s.
(i) Let $s=1$. In this case (2.4) converts to the form

$$\bigwedge_{j=1}^{k} com\,(a_j^1, b_2, \ldots, b_n) = \bigvee_{d\in D^{n-1}} \left\{ \bigvee_{j=1}^{k} a_j^1 \wedge b^d \vee \bigwedge_{i=1}^{k} a_j^{1\perp} \wedge b^d \right\} =$$

$$= com\,(a_1^1, \ldots, a_k^1, b_2, \ldots, b_n). \tag{2.5}$$

It is evident that (2.5) holds for $k=1$. Suppose it holds for any $k_1 \leqslant k$. Then putting $com\,(a, b_2, \ldots, b_n) \equiv com\,(a, B)$ we have

$$\bigwedge_{j=1}^{k+1} com\,(a_j^1, B) = \bigwedge_{j=1}^{k} com\,(a_j^1, B) \wedge \bigwedge_{j=2}^{k+1} com\,(a_j^1, B) = \bigvee_{d\in D^{n-1}}$$

$$\left\{ \bigvee_{j=1}^{k} a_j^1 \wedge b^d \vee \bigwedge_{j=1}^{k} a_j^{1\perp} \wedge b^d \right\} \wedge \bigvee_{d\in D^{n-1}} \left\{ \bigvee_{j=2}^{k+1} a_j^1 \wedge b^d \vee \bigwedge_{j=2}^{k+1} a_j^{1\perp} \wedge b^d \right\}.$$

The elements in $M = \bigcup_{d\in D^{n-1}} \left\{ a_1^1 \wedge b^d, \ldots, a_{k+1}^1 \wedge b^d, \bigwedge_{j=1}^{k} a_j^{1\perp} \wedge b^d, \bigwedge_{j=2}^{k+1} a_j^{1\perp} \wedge b^d \right\}$ form the Foulis-Holland set. That is, from any triple of elements from M one of them is compatible with two others. We note that in M the only noncompatible pairs are $\bigwedge_{j=1}^{k} a_j^{1\perp} \wedge b^d$, $\bigwedge_{j=2}^{k+1} a_j^{1\perp} \wedge b^d$, $(d \in D^{n-1})$. Therefore the minimal sublattice of L generated by M is distributive (see [6]) and this proves (2.5).
(ii) The general case of (2.4) may be proved by analogical reasonings. This rather technical but essentially simple step is left to the reader.

Q.E.D.

Corollary 2.1.1. *Let $K_i = \{a_1^i, \ldots, a_{k_i}^i\} \subset \mathcal{A}_i$, $i=1, \ldots, k$, be finite decompositions of 1, that is, for any $i=1, \ldots, n$ we have $a_u^i \perp a_v^i$ for $1 \leqslant u, v \leqslant k_i$, and $\bigvee_{u=1}^{k_i} a_u^i = 1$. Then*

$$com\,(K_1 \cup \ldots \cup K_n) = \bigwedge_{a_i \in K_i} com\,(a_1, \ldots, a_n) \tag{2.6}$$

$$i=1, \ldots, n.$$

The element $a = \bigwedge \{a_t\colon t \in T\}$ is said to be *countably obtainable* from $\{a_t\colon t\in T\}$ if there is a countable subset $T_1 \subset T$ such that $a = \bigwedge \{a_t\colon t \in T_1\}$.

Theorem 2.2. *The commutator $com\,(\mathcal{A}_1, \ldots, \mathcal{A}_n) = com\left(\bigcup_{i=1}^{n} \mathcal{A}_i\right)$ exists iff $a^0(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ exists, and in that case $com\,(\mathcal{A}_1, \ldots, \mathcal{A}_n) = a^0(\mathcal{A}_1, \ldots, \mathcal{A}_n)$. Moreover, if $com\,(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ is countably obtainable from $\{com: R\text{-finile subset of } \mathcal{A}\}$, so is $a^0(\mathcal{A}), \ldots, \mathcal{A}_n)$ from $\{a^0(a_1, \ldots, a_n)\colon a_i \in \mathcal{A}_i,\ i=1, \ldots, n\}$.*

Proof. Let $a = com\,(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ exist in L. Then, for any $a_i \in \mathcal{A}_i$, $i=1, \ldots, n$, we have $a < a^0(a_1, \ldots, a_n)$. Let now $c < a^0(a_1, \ldots, a_n)$ for any $a_i \in \mathscr{A}_i$. Each finite subset $\emptyset \neq F \subset \mathscr{A}$ generates a finite decomposition of 1, K_i, in each $\mathscr{A}_i$ in the following way. If in F there are no elements from some $\mathscr{A}_i$, then we put $K_i = \{0, 1\}$, otherwise we put $K_i = \{a_1^{id_i} \wedge \ldots \wedge a_{k_i}^{id k_i}\colon\ a_j^i \in F \cap \mathscr{A}_i\}$. Then

$$com\ F = com\ (K_1 \cup \ldots \cup K_n).$$

From Corollary 2.1.1 we have

$$com\ F = \bigwedge_{\substack{a_i \in K_i \\ i = 1, \ldots, n}} com\ (a_1, \ldots, a_n). \tag{2.7}$$

Hence $a^0(\mathscr{A}_1, \ldots, \mathscr{A}_n)$ exists and equals a.

Conversely, let $a^0 = a^0(\mathscr{A}_1, \ldots, \mathscr{A}_n)$ exist. Then (2.7) implies $com\ F > a^0$ for any $F \subset \mathscr{A}$, and therefore $a = a^0$.

The last assertion is evident. Q.E.D.

Remark 1. Let $\{\mathscr{A}_s\colon\ s \in S\}$ be a system of Boolean subalgebras of a complete logic L. Put $\mathscr{A} = \bigcup_{s \in S} \mathscr{A}_s$. Then

$$com\ \mathscr{A} = \bigwedge_{\{s_{i1}, \ldots, s_{in}\}} com\ (\mathscr{A}_{s_{i1}}, \ldots, \mathscr{A}_{s_{in}}). \tag{2.8}$$

Theorem 2.3. Let $\mathscr{A}_1, \ldots, \mathscr{A}_n$ *be Boolean subalgebras of* L.

(i) *If* $a^0(\mathscr{A}_1, \ldots, \mathscr{A}_n)$ *is countably obtainable from* $\{com\ F\colon\ F \subset \mathscr{A}, F$ — *finite*$\}$, *then the following conditions are equivalent:* (a) $\mathscr{A}_1, \ldots, \mathscr{A}_n$ *are jointly measurable in a state* m; (b) $m(com\ F) = 1$ *for any finite* $F \subset \mathscr{A}$; (c) $m(a^0(\mathscr{A}_1, \ldots, \mathscr{A}_n)) = 1$.

(ii) *Let* $\{\mathscr{A}_s\colon\ s \in S\}$ *be a system of Boolean subalgebras of a separable logic* L. *Then the following conditions are equivalent:* (a) $\{\mathscr{A}_s\colon\ s \in S\}$ *is jointly measurable in a state* m; (b) $m(a^0(\mathscr{A}_{s_1}, \ldots, \mathscr{A}_{s_n})) = 1$ *for any finite subset* $\{s_{i_1}, \ldots, s_{i_n}\} \subset S$; (c) $m\left(com \bigcup_{s \in S} \mathscr{A}_s\right) = 1$.

Proof. To prove Theorem 2.3 it suffices to show that if $\{a_i^k\}_{k=1}$ is a sequence of elements from $\mathscr{A}_i$, $i = 1, \ldots, n$, then

$$m\left(\bigwedge_{k=1}^{j} a^0(a_1^k, \ldots, a_n^k)\right) = 1 \quad \text{for any } j = 1, 2, \ldots .$$

Indeed, let j be given. For any $i = 1, \ldots, n$ there is an observable x_i so that $\mathscr{R}(x_i) = \tilde{\mathscr{A}}_i$, where $\tilde{\mathscr{A}}_i$ is the minimal subalgebra of L generated by $\{a\ {}_i^1 \ldots, a\ \}$. For the observables $x_1, \ldots, x_n$ there is a joint distribution in a state m [3, 4]. Theorefore

$m\left(\bigwedge_{k=1}^{j} a^0(a_1^k, \ldots, a_n^k)\right)=1$. The continuity of a state m from above implies that $m\left(\bigwedge_{k=1}^{\infty} a^0(a_1^k, \ldots, a_n^k)\right)=\lim_{j\to\infty} m\left(\bigwedge_{k=1}^{j} a^0(a_1^k, \ldots, a_n^k)\right)=1$. Hence $m(a^0)=1$. This observation and (2.7) complete the proof of the first part of Theorem 2.3. Analogically we proceed for the second part, too.

Q.E.D.

Remark 2. If a state m on a logic L has the property: "if $m(a_t)=1$ for any $t \in T$, and $a=\bigwedge_{t\in T} a_t$ exists, then

$$m(a)=1, \tag{2.9}$$

then the part (i) in Theorem 2.3 holds without the assumption on countable obtainability of $a^0(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ provided $com\,(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ exists.

From the results of [12] it follows that the commutator of a sequence of separable subalgebras of L exists on every logic.

3. Independence and strong independence

We say that observables $x_1, \ldots, x_n$ are independent in a state m if

$$m\left(\bigwedge_{i=1}^{n} x_i(E_i)\right)=\prod_{i=1}^{n} m(x_i(E_i)), \tag{3.1}$$

$$E_i \in B(R_1), \quad i=1, \ldots, n.$$

Following S. Gudder [8] we say that bounded observables $x_1, \ldots, x_n$ on a sum logic [7] are strongly independent in a state m if for any bounded Borel functions $f_1, \ldots, f_n$

$$m_{f_1\circ x_1+\ldots+f_n\circ x_n}=m_{f_1\circ x_1}*\ldots*m_{f_n\circ x_n}, \tag{3.2}$$

where the sign $*$ denotes the convolution.

The system of observables $\{x_s\colon s \in S\}$ is independent (strongly independent) in a state m if any finite subsystem of $\{x_s\colon s\in S\}$ is independent (strongly independent) in a state m.

For compatible observables these notions are equivalent. Gudder [8, Theorem 4.5] has proved that the strong independence of observables in a state m implies the independence. The converse proposition has been proved only for special question observables on the logic $L(H)$ of all closed subspaces of a separable Hilbert space H in a pure state [8, Theorem 4.6]. We extend the validity of the above converse implication on a logic $L(H)$ for all states and observables.

Comparing (3.1) and (1.1) we see that the observables $x_1, \ldots, x_n$ are independent in a state m iff there is a joint distribution in a state m, $m_{x_1 \ldots x_n}$, so that $m_{x_1 \ldots x_n}(E_1 \times \ldots \times E_n) = \prod_{i=1}^{n} m(x_i(E_i))$, $E_i \in B(R_1)$, $i = 1, \ldots, n$.

Theorem 3.1. *Let $x_1, \ldots, x_n$ be purely atomic observables on a logic L and let m be a state. Let $\sigma(x_i) = \{\lambda_1^i, \lambda_1^i, \ldots\}$. Put*

$$M(x_1, \ldots, x_n) = \{a \in L:\ \text{there are } i_1, \ldots, i_n \text{ so that}$$
$$a = x_1(\{\lambda_{i_1}^1\}) = \ldots = x_n(\{\lambda_{i_n}^n\})\},$$
$$M_m(x_1, \ldots, x_n) = \{a:\ a \in M(x_1, \ldots, x_n):\ m(a) \neq 0\},$$
$$a^\sigma = \bigvee\{a:\ a \in M(x_1, \ldots, x_n)\},$$
$$a_m = \bigvee\{a:\ a \in M_m(x_1, \ldots, x_n)\}.$$

Then the following conditions: (a) $x_1, \ldots, x_n$ *have a joint distribution in a state m*; (b) $m(a^\sigma) = 1$; (c) $m(a_m) = 1$; *are equivalent.*

Proof. From the results of paper [12, Corollary 2.1] we have that $a^\sigma = a^0(\mathcal{R}(x_1), \ldots, \mathcal{R}(x_n))$, and Theorem 2.3 yields the statement of Theorem 3.1.

Q.E.D.

Theorem 3.2. *Let $\{x_s\colon s \in S\}$ be a system of purely atomic observables on a logic L. Then*

(i) *$\{x_s\colon s \in S\}$ has a joint distribution in a state m* iff any pair of observables *$\{x_s, x_t\}$, $s, t \in T$, has a joint distribution.*

(ii) *$\{x_s\colon s \in S\}$ are independent in a state m iff any pair of observables $\{x_s, x_t\}$, $s \neq t$, is independent in m.*

Proof. For any finite system $x_1, \ldots, x_n$ we may verify that $M_m(x_1, \ldots, x_n) = \bigcap_{i=1}^{n} M_m(x_i)$.

Theorem 3.1 proves the part (i).

For (ii) we may observe that for any i there is a unique λ_i such that $m(x_i(\{\lambda_i\})) = 1$. Therefore the independence of any pair of the observables implies independence of the entire system $\{x_s\colon s \in S\}$. Q.E.D.

If H is a Hilbert space over real or complex numbers, not necessarily separable, and W is a von Neumann operator, that is, W is a Hermitean, nonnegative operator of trace class with $trW = 1$ (for details of the trace class see [14]), then $m_W(M) = tr(WP^M)$, $M \in L(H)$, is a state on $L(H)$. Here P^M denotes the orthoprojector onto the subspace M. It is known that to any bounded observable x there is a unique Hermitean operator A_x and conversely. For any $f \in H$, $\|f\| = 1$, we define an operator $f \otimes \bar{f}$: $x \mapsto (f, x)f$, $x \in H$. Using results of [9, 3], we have:

Theorem 3.3. (i) *The Boolean subalgebras* $\mathscr{A}_1, \ldots, \mathscr{A}_n$ *of* $L(H)$ *are jointly measurable in a state* m_W, *where* $W = \sum_{a \in I} \lambda_a f_a \otimes \bar{f}_a$, $f_a \perp f_b$, $a \neq b$, $\|f_a\| = 1$, $a, b \in I$, *is a von Neumann operator on H, iff one of the following conditions is satisfied:*
(a) *for any* f_a *with* $\lambda_a \neq 0$ *we have*

$$P^{M_1} \ldots P^{M_n} f_a = P^{M_{i1}} \ldots P^{M_{in}} f_a,$$

where $(i_1, \ldots, i_n)$ *is any permutation of* $(1, \ldots, n)$ *and* $M_i \in \mathscr{A}_i$, $i = 1, \ldots, n$;
(b) $P^{M_1} \ldots P^{M_n} W = P^{M_{i1}} \ldots P^{M_{in}} W$ *for any* $M_i \in \mathscr{A}_i$, $i = 1, \ldots, n$.

(ii) *The bounded observables* $x_1, \ldots, x_n$ *have a joint distribution in a state* m_W *iff one condition is satisfied:* (a) *for any* f_a *with* $\lambda_a \neq 0$ *we have*

$$A_{x_1} \ldots A_{x_n} f_a = A_{x_{i1}} \ldots A_{x_{in}} f_a\ ,$$

where $(i_1, \ldots, i_n)$ *is any permutation of* $(1, \ldots, n)$;

M_{i_1} M_{i_n} M_{i_1} M_{i_n} x_{i_1} x_{i_n} x_{i_1} x_{i_n}

(b) $A_{x_1} \ldots A_{x_n} W = A_{x_{i1}} \ldots A_{x_{in}} W$
for any permutation $(i_1, \ldots, i_n)$ *of* $(1, \ldots, n)$.

Moreover, if H_0 *is the commutator of* $\mathscr{A}_1, \ldots, \mathscr{A}_n$, *then* $\mathscr{A}_1, \ldots, \mathscr{A}_n$ *are jointly measurable in* m_W *iff* $f_a \in H_0$ *for any* $a \in I$ *with* $\lambda_a \neq 0$.

Proof. The proof follows from the observation that if a von Neumann operator is of the form $W = \sum_{a \in I} \lambda_a f_a \otimes \bar{f}_a$, then there is a countable subset $D \subset I$ such that $\lambda_a = 0$ if $a \in I - D$. The rest of the proof follows from results of papers [9, Theorem 3.7] and [3, Theorem 14]. Q.E.D.

We shall say that a subspace H_0 reduces an observable x on $L(H)$ if $P^{x(E)} P^{H_0} = P^{H_0} p^{x(E)}$ for any $E \in B(R_1)$.

We recall that a cardinal I is said to be non-real measurable if there exists a positive measure $\mu \neq 0$ on the power set of I with $\mu(\{a\}) = 0$ for each $a \in I$.

Theorem 3.4. *Let* $L(H)$ *be a logic of a Hilbert space H (real or complex) whose dimension is a non-real measurable cardinal number. Then the independence and strong independence of bounded observables in any state are equivalent.*

Proof. Let a finite system of bounded observables $x_1, \ldots, x_n$ be independent in a state m on $L(H)$. The completeness of the lattice $L(H)$ provides an existence of a commutator $H_0 = com\ (\mathscr{R}(x_1), \ldots, \mathscr{R}(x_n))$. If H is a separable Hilbert space, then the commutator is countable obtainable, so that, due to Theorem 2.3 (i), $m(H_0) = 1$. If H is nonseparably, then according to a generalization of the Gleason theorem to a nonseparable Hilbert space logic with a non-real measurable cardinal I we have ([1, 5]) that there is a unique von Neumann operator $W = \sum_{a \in I} \lambda_a f_a \otimes \bar{f}_a$,

$f_a \perp f_b$, $\|f_a\|=1$, $f_a \in H$, such that $m(M)=tr(WP^M)$, $M \in L(H)$. Theorem 2.3 (i) and Remark 2 imply that $m(H_0)=1$. Hence the restriction of m to $L(H_0)$, m_0, is a state, too.

The commutator H_0 reduces the observables $x_1, \ldots, x_n$ so that $x_{i0}(E)=x_i(E)\wedge H_0$, $E \in B(R_1)$, $i=1, \ldots, n$, are mutually compatible observables on a logic $L(H_0)$. Moreover, $m\left(\bigwedge_{i=1}^{n} x_i(E_i)\right)=m_0\left(\bigwedge_{i=1}^{n} x_{i0}(E_i)\right)$, $E_i \in B(R_1)$.

If H_0 reduces x, then it reduces $f \circ x$ for any Borel function f. Let bounded observable x correspond to a Hermitean operator A_x. Then H_0 reduces x iff $A_x P^{H_0}=P^{H_0}A_x$. So that H_0 reduces $f_1 \circ x_1+\ldots+f_n \circ x_n$ for any bounded functions $f_1, \ldots, f_n$ and $(f_1 \circ x_1+\ldots+f_n \circ x_n)_0=f_1 \circ x_{10}+\ldots+f_n \circ x_{n0}$.

It can be easily checked that

$$m((f_1 \circ x_1+\ldots+f_n \circ x_n)(E)))=m((f_1 \circ x_1+\ldots+f_n \circ x_n)(E\wedge H_0)))=$$
$$=m_0((f_1 \circ x_{10}+\ldots+f_n \circ x_{n0})(E)).$$

Hence

$$m_{f_1 \circ x_1+\ldots+f_n \circ x_n}=m_{0\,(f_1 \circ x_{10}+\ldots+f_n \circ x_{n0})}=$$
$$=m_{0\,f_1 \circ x_{10}} * \ldots * m_{0\,f_n \circ x_{n0}}=m_{f_1 \circ x_1} * \ldots * m_{f_n \circ x_n}.$$

Q.E.D.

This Theorem is a consequence of the correspondence between the joint distribution of observables defined by (1.1) and the so-called type II joint distribution on the logic $L(H)$.

We recall that the bounded observables $x_1, \ldots, x_n$ on a sum logic have a type II joint distribution in a state m [15] if for any $a_1, \ldots, a_n \in R_1$ there is a probability measure $\mu^{a_1 \cdots a_n}$ on $B(R_n)$ such that

$$\mu^{a_1 \cdots a_n}(\{(t_1, \ldots, t_n):\ a_1t_1+\ldots+a_nt_n \in E\})=$$
$$=m((a_1x_1+\ldots+a_nx_n)(E)),\ E \in B(R_1).$$

By analogical reasonings as those used in the proof of Theorem 3.4 we may prove the following theorem.

Theorem 3.5. *Let $L(H)$ be a logic of a Hilbert space H (real or complex) whose dimension is a non-real measurable cardinal. Then if the bounded observables $x_1, \ldots, x_n$ have a joint distribution in a state m, they have a type* II *joint distribution, and they are identical.*

Remark 3. Theorem 3.5 may be proved for the case of unbounded observables, too, see [3, Theorem 20].

The general relationship between the joint distribution of observables in a state by (1.1) (type I joint distribution, too) and a type II joint distribution is still unknown. A very special case of independence may be given for logics with a distributive Segal product $x \circ y$ of two bounded observables x and y defined in [7]:

$$x \circ y = 1/2((x+y)^2 - x^2 - y^2). \tag{3.3}$$

For any $a \in L$ we define a question observable x_a: $x_a(\{0\}) = a^{\perp}$, $x_a(\{1\}) = a$.

Lemma 3.6. *If the Segal product is distributive, then*

$$(x_a + x_b)(\{1\}) = (a \wedge b^{\perp}) \vee (a^{\perp} \wedge b). \tag{3.4}$$

Proof. The distributivity of the product (3.3) implies

$$(x_a + x_b)^2 + (x_a - x_b)^2 = 2x_a^2 + 2x_b^2 = 2x_a + 2x_b,$$

$$(x_a - x_b)^2 = 2(x_a + x_b) - (x_a + x_b)^2.$$

But

$$(x_a - x_b)^2(\{1\}) = [2(x_a + x_b) - (x_a + x_b)^2](\{1\}).$$

Hence

$$(x_a - x_b)(\{-1,1\}) = f \circ (x_a + x_b)(\{1\}) = (a^{\perp} \wedge b) \vee (b^{\perp} \wedge a) =$$
$$= (x_a + x_b)(\{1\}), \text{ where } f(t) = 2t - t^2.$$

The latter equality follows from [7, Corollary 6.3].

Q.E.D.

Corollary 3.6.1. *Two observables x and y on a sum logic with the distributive Segal product are independent in a state m iff*

$$m_{\chi_E \circ x + \chi_F \circ y} = m_{\chi_E \circ x} * m_{\chi_F \circ y} \tag{3.5}$$

for any E, F $B(R_1)$. (χ_A is a characteristic function of a set $A \in B(R_1)$).

Proof. In [7, Theorem 6.2, Corollary 6.3] one proves that

$$(x_a + x_b)(\{0\}) = a^{\perp} \wedge b^{\perp},$$
$$(x_a + x_b)(\{2\}) = a \wedge b,$$

for any a, $b \in L$. If for E, $F \in B(R_1)$ we put $a = x(E)$, $b = y(F)$, then $x_a = \chi_E \circ x$, $x_b = \chi_F \circ y$. An easy calculation shows that

$$m_{x_a} * m_{x_b}(\{0\}) = m(a^{\perp} \wedge b^{\perp}),$$

$$m_{x_a} * m_{x_b}(\{1\}) = m(a^{\perp} \wedge b) + m(a \wedge b^{\perp}),$$

$$m_{x_a} * m_{x_b}(\{2\}) = m(a \wedge b).$$

Using these equalities and Lemma 3.6 we prove (3.5). Q.E.D.

4. Valuation property

In [2, p. 349] the author posed the following question: If $x_1, \ldots, x_n$ have a joint distribution in a state m, does m have the property of a valuation on the minimal sublogic L_0 generated by all ranges $\mathcal{R}(x_i)$? The question is whether we have

$$m(a \vee b) + m(a \wedge b) = m(a) + m(b) \tag{4.1}$$

for any $a, b \in L_0$.

Here we show that the answer is in the affirmative.

Let $a \in L$ and $\emptyset \neq M \subset L$ be p.c. a. By Zorn's lemma there is a maximal set Q_a p.c. a and such that $M \subset Q_a \subset L$. Denote by $L_0(M)$ the minimal sublogic of L generated by M.

Lemma 4.1. *Let M be p.c. a and let com M exist. Then*

(i) $a < com\ M$. (4.2)

(ii) *If* $a = com\ M$, *then*

$$a = com\ Q_a = com\ L_0(M) = com\ M. \tag{4.3}$$

Proof. Since for any finite subset $F \subset M$ we have $com\ F \wedge a = a$ we obtain (4.2) immediately.

According to Theorem 3.8 from [11], we see that Q_a is a sublogic of L, so that $M \subset L_0(M) \subset Q_a(M)$. From (4.2) we have that for any finite $F \subset Q_a(M)$, $a < com\ F$. Let now $c < com\ F$, then $c < com\ M$. Therefore (4.3) holds.

Q.E.D.

Theorem 4.2. *Let $\{\mathcal{A}_s\colon s \in S\}$ be a system of Boolean subalgebras of L jointly measurable in a state m. Let either com $\mathcal{A}$ be countably obtainable from $\{com\ F\colon F \subset \mathcal{A}\}$ or let com $\mathcal{A}$ exist and for the state m the condition* (2.9) *hold. Then the restrictions of m to $Q_{com\,A}$ and $L_0(\mathcal{A})$, respectively, have the valuation property* (4.1).

Proof. Let $a = com\ \mathcal{A}$. Using Remark 3.9 from [11], we have that $Q_a \wedge a$ is a Boolean sub-σ-algebra of logic $L_{(0,a)} = \{b \in L\colon b < a\}$, and the restriction of m to $L_{(0,a)}$, m_0, is a state. Theorem 2.3 and Remark 2 imply that $m(b) = m(b \wedge a) = m_0(b \wedge a)$ for any $b \in Q_a$. For mutually compatible elements from $Q_a \wedge a$ the property (4.1) holds, and therefore it holds for all elements from $L_0(\mathcal{A})$ and Q_a, respectively.

Q.E.D.

Remark 4. In the process of investigating the properties of the above constructions some open problem arose:

1. Does pairwise joint measurability in a state of Boolean subalgebras imply a joint measurability of a given system of Boolean subalgebras?

2. Especially, have we

$$com\ (a, b, c) = com\ (a, b) \wedge com\ (a, c) \wedge com\ (b, c)$$

for any $a, b, c \in L$?

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О ДВУХ ПРОБЛЕМАХ КВАНТОВЫХ ЛОГИК

Anatolij Dvurečenskij

Резюме

Дается решение предположения Гаддера о равносильности независимости и сильной независимости наблюдаемых на логике сепарабельного пространства Гильберта, а также обобщение предположения на логику пространства Гильберта, размерность которого неизмеримое кардинальное число.

Кроме того решается одна проблема автора заметки: Если система наблюдаемых имеет совместное распределение в состоянии, будет-ли состояние оценкой на минимальной логике, генерированной всеми областями значений наблюдаемых?