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Zuzana Ladzianska

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M-POPRODUCT OF LATTICES

ZUZANA LADZIANSKA

The present paper generalizes the results of [3] concerning the free m-product of lattices. The notion of the poproduct of lattices was introduced and investigated in [4].

Throughout this paper, m is an infinite regular cardinal. A lattice L is m-complete (or L is an m-lattice) if for any nonempty $S \subseteq L$ with the cardinality $|S| < m$, the join and meet of S exist in L. The concepts of an m-sublattice, m-generated and an m-homomorphism are defined in the natural way.

Let R be a poset and let L_r, $r \in R$ be pairwise disjoint m-complete lattices. Let $Q = \bigcup(L_r;\ r \in R)$ be partially ordered in the following way:

for $a, b \in Q$ we put $a \leqq b$ if and only if one of the conditions (i) and (ii) holds:

(i) there is an $r \in R$ such that $a, b \in L_r$ and the relation $a \leqq b$ holds in L_r,
(ii) there are $p, r \in R$ such that $a \in L_p$, $b \in L_r$ and the relation $p < r$ holds in the poset R.

If f is a mapping from Q into a lattice M, then f_r denotes its restriction on L_r.

Definition 1. *Let R be a poset and let L_r, $r \in R$ and L be m-lattices. The lattice L is said to be the m-poproduct of the lattices L_r, $r \in R$ if:*

(i) *there is an isotone injection $i\colon Q \to L$ such that for each $r \in R$, i_r is an m-homomorphism,*
(ii) *if M is an m-lattice, then for every isotone mapping $f\colon Q \to M$ such that for each $r \in R$, f_r is an m-homomorphism, there exists uniquely an m-homomorphism $g\colon L \to M$ such that $g \circ i = f$.*

From the definition it follows that L is m-generated by the set $i(Q)$ (i.e., L is the smallest m-sublattice of l that contains $i(Q)$).

We shall identify the sets Q and $i(Q)$. Then we can say that $i\colon Q \to L$ is a canonical m-embedding. Q will be called a skeleton of L.

The m-poproduct of the m-lattices L_r, $r \in R$ will be denoted by $P_m(L_r;\ r \in R)$. From the definition it follows that an m-poproduct forms the free m-poproduct if and only if R is an antichain.

Let us denote by $W_m(Q)$ the set of lattice m-polynomials over Q. The concept of an m-polynomial is defined inductively as follows: $W_0(Q) = Q$, and for $m > 0$ the set $W_m(Q)$ consists of all elements of $\bigcup(W_n(Q) \mid n < m)$ together with all expressions of the form $\bigwedge S$ or $\bigwedge S$ (cosidered formally), where $S \subseteq \bigcup(W_n(Q) \mid n < m)$ and $0 < |S| < m$. The rank $l(a)$ of an m-polynomial a is the least ordinal n such that $a \in W_n(Q)$.

Denote by 0,1 two new elements, which do not belong to the skeleton Q and extend the partial ordering from the set Q to the set $Q \dot{\cup} \{0, 1\}$ ($\dot{\cup}$ denotes the disjoint union of sets) in the following way: for each $q \in Q$ the relation $0 < q < 1$ holds.

For each $a \in W_m(Q)$ and each $r \in R$ the upper r-cover $a^{(r)}$ and the lower r-cover $a_{(r)}$ are defined as follows:

Definition 2.

(i) *Let* $a \in L_p$.
If $p = r$, *then* $a_{(r)} = a^{(r)} = a$.
If $p \parallel r$, *then* $a_{(r)} = 0$, $a^{(r)} = 1$.
If $p < r$, *then* $a_{(r)} = 0$, $a^{(r)} = 0$.
If $p > r$, *then* $a_{(r)} = 1$, $a^{(r)} = 1$.

(ii) *If* $a = w(a_1, \ldots, a_n, \ldots)$, *then*
$a_{(r)} = w((a_1)_{(r)}, \ldots, (a_n)_{(r)}, \ldots)$,
$a^{(r)} = w((a_1)^{(r)}, \ldots, (a_n)^{(r)}, \ldots)$.

Note that $h_r(a) = a_{(r)}$, $h^r(a) = a^{(r)}$ are m-homomorphisms $W_m(Q) \to L_r \dot{\cup} \{0, 1\}$. A lower or upper cover that is distinct from both 0 and 1 is called proper.

Definition 3. *On the set* $W_m(Q)$ *we define the relation* $\subseteqq$ *in the following way: For* $a, b \in W_m(Q)$ *the relation* $a \subseteqq b$ *holds if it is a consequence of the following rules:*

(1) *there are* $p, r \in R$ $(p \leqq r)$ *such that* $a^{(p)}$, $b_{(r)}$ *are proper and* $a^{(p)} \leqq b_{(r)}$ *holds in* Q,
(2) $a = \bigwedge S$ *and* $s \subseteqq b$ *for some* $s \in S$,
(3) $a = \bigvee S$ *and* $s \subseteqq b$ *for all* $s \in S$,
(4) $b = \bigwedge T$ *and* $a \subseteqq t$ *for all* $t \in T$,
(5) $b = \bigvee T$ *and* $a \subseteqq t$ *for some* $t \in T$.

Theorem. *Let* L_r, $r \in R$ *be a family of* m-*lattices. Then the* m-*poproduct* $P_m(L_r; r \in R) = L$ *exists and* $L \cong W_m(Q)/\equiv$, *where* $a \equiv b$ *if and only if* $a \subseteqq b$ *and* $b \subseteqq a$.

Proof. Proof is similar to that of the corresponding theorem of [3]. First we need some auxiliary results.

Lemma 1. *Let $a \in W_m(Q)$. If $a_{(r)}$ is proper, then $a_{(r)} \subseteqq a$. If $a^{(r)}$ is proper, then $a \subseteqq a^{(r)}$.*

Proof. If $a \in L_r$, then $a_{(r)} = a = a^{(r)}$. Therefore $(a_{(r)})^{(r)} \leqq a_{(r)}$ and $a^{(r)} \leqq (a^{(r)})_{(r)}$ in Q. Now we can proceed by induction on the rank of $a \in W_m(Q)$.

Lemma 2. Let a, b, $c \in W_m(Q)$. Then

(i) $a \subseteqq a$,
(ii) $a \subseteqq b$ and $b \subseteqq c$ imply that $a \subseteqq c$.

Proof. (i) If $l(a)=0$, the $a \in L_r$ for a unique $r \in R$. Since $a = a_{(r)} = a^{(r)}$, the containment $a \subseteqq a$ holds by (1). Let $a = \bigwedge S$. Since $s \subseteqq s$ holds for all $s \in S$ by induction on the rank, it follows by (2) that $\bigwedge S \subseteqq s$ for all $s \in S$. Hence, applying (4), $a = \bigwedge S \subseteqq \bigwedge S = a$. Let $a = \bigvee S$. Since $s \subseteqq s$ for all $s \in S$, by induction it follows by (3) that $\bigvee S \subseteqq s$ for all $s \in S$. Hence, applying (5), $a = \bigvee S \subseteqq \bigvee S = a$.

(ii) Proof is by induction on $l(a) + l(b) + l(c)$.

If $a \subseteqq b$ holds by (2), then $a \bigwedge S$ and $s \subseteqq b$ for some $s \in S$. Hence, $s \subseteqq c$ and $a \subseteqq c$ holds by (2).

If $a \subseteqq b$ holds by (3), then $a = \bigvee S$ and $s \subseteqq b$ for all s. Hence, $s \subseteqq c$ for all s and $a \subseteqq c$ holds by (3).

If $a \subseteqq b$ holds by (5), then $b = \bigvee T$ and $a \subseteqq t$ for some $t \in T$. From $t \subseteqq b$, $b \subseteqq c$ it follows $t \subseteqq c$, hence, $a \subseteqq c$ by transitivity.

If $b \subseteqq c$ holds by (2), then $b = \bigwedge S$ and $s \subseteqq c$ for some $s \in S$. From $a \subseteqq b$, $b \subseteqq s$ it follows $a \subseteqq s$, hence $a \subseteqq c$ by transitivity.

If $b \subseteqq c$ holds by (4), then $c = \bigwedge T$ and $b \subseteqq t$. From $a \subseteqq b$, $b \subseteqq t$ it follows $a \subseteqq t$, hence $a \subseteqq c$ by (4).

If $b \subseteqq c$ holds by (5), then $c = \bigvee T$ and $b \subseteqq t$ for some $t \in T$. From $a \subseteqq b$, $b \subseteqq t$ it follows $a \subseteqq t$, hence $a \subseteqq c$ by (5).

If $a \subseteqq b$ holds by (1), then there are p, $r \in R$ such that $a^{(p)}$, $b_{(r)}$ are proper and $a^{(p)} \leqq b_{(r)}$. Therefore $a \subseteqq b_{(r)}$, $b_{(r)} \subseteqq c$, hence $a \subseteqq c$ by transitivity.

It $b \subseteqq c$ holds by (1), then there are p, $r \in R$ such that $b^{(p)}$, $c_{(r)}$ are proper and $b^{(p)} \leqq c_{(r)}$. Therefore $a \subseteqq b^{(p)}$, $b^{(p)} \subseteqq c$, hence $a \subseteqq c$ by transitivity.

Now there remains the case when $a \subseteqq b$ holds by (4) and $b \subseteqq c$ holds by (3). That means, $b = \bigwedge T$ and $a \subseteqq t$ for all $t \in T$ and $b = \bigvee S$ and $s \subseteqq c$ for all $s \in S$. But $b = \bigwedge T = \bigvee S$ is possible only if $b \in Q$. Therefore there is an $r \in R$ such that $b \in L_r$, $s \in L_r$ for all $s \in S$, $t \in L_r$ for all $t \in T$. Hence, the sets $A = \{x \mid x \in L_r, x \supseteqq a\}$, $C = \{x \mid x \in L_r, x \subseteqq c\}$ are nonempty, because $t \in A$ for all $t \in T$ and $s \in C$ for all $s \in S$. Since L_r is an m-complete lattice, $a^{(r)}$ and $c_{(r)}$ both exist and $a^{(r)} \subseteqq b \subseteqq c_{(r)}$. Hence, $a \subseteqq c$ by (1).

Lemma 2 is proved. By lemma 2, $\subseteqq$ is a quasi-ordering. Therefore, the relation $\equiv$ defined by

$$a \equiv b \text{ if and only if } a \subseteqq b \text{ and } a \supseteqq b$$

is an equivalence relation. Further, $C(a)=\{b \mid a \equiv b\}$ is the equivalence class containing a. $C(Q)=\{C(a) \mid a \in W_m(Q)\}$ is a poset with $C(a) \leqq C(b)$ if and only if $a \subseteqq b$.

Lemma 3. $C(Q)$ is an m-lattice with $\bigwedge\{C(s) \mid s \in S\} = C(\bigwedge S)$ and $\bigvee\{C(s) \mid s \in S\} = C(\bigvee S)$ whenever $S \subseteqq W_m(Q)$ and $0<|S|<m$. Furthermore, Q is embedded in $C(Q)$.

Proof. $\bigwedge S \subseteqq s$ for all $s \in S$, therefore $C(\bigwedge S) \leqq C(s)$ for all $s \in S$, hence $C(\bigwedge S) \leqq C(s)$. On the other hand, if $t \subseteqq s$ for all $s \in S$, then $t \subseteqq \bigwedge S$ by (4). Therefore, if $C(t) \leqq C(s)$ for all $s \in S$, then $C(t) \leqq C(\bigwedge S)$. Hence, $\bigwedge C(s) \leqq C(\bigwedge S)$. The first equality is proved and the second follows by duality.

Let $x = \inf Y$ in L_r with $x \in L_r$, $Y \subseteqq L_r$ and $0<|Y|<m$. Then $x \subseteqq y$ for all $y \in Y$, and therefore $x \subseteqq \bigwedge Y$. Since $(\bigwedge Y)^{(r)} = x$, $\bigwedge Y \subseteqq x$ holds by (1). Hence $x \equiv \bigwedge Y$. Then means, $C(x) = C(\bigwedge Y)$. Therefore each L_r, $r \in R$ is an m-sublattice of $C(Q)$. From the definition of the relation $\equiv$ and of the class $C(a)$ it follows that for $x, y \in Q$ from $x \leqq y$ it follows that $C(x) \leqq C(y)$ and from $x \neq y$ there follows $C(x) \neq C(y)$. Lemma 3 is proved.

To complete the proof of the theorem, it remains to show that $C(Q)$ is the m-poproduct of $(L_r, r \in R)$. Each L_r is an m-sublattice of $C(Q)$ by lemma 3 and $C(Q)$ is clearly m-generated by Q. Let K be an m-lattice and let the m-homomorphisms $f_r \colon L_r \to K$ be given for $r \in R$. We define a mapping $g \colon W_m(Q) \to K$ inductively as follows:

if $x \in L_r$, then $g(x) = f_r(x)$,

if $a = \bigwedge S$ and $g(s)$ is already given for each $s \in S$, then $g(a) = \bigwedge(g(s) \mid s \in S)$,

if $a = \bigvee S$, then $g(a)$ is defined dually.

We require the following

Lemma 4. Let $a, b \in W_m(Q)$ and $r \in R$.

(i) If $a_{(r)}$ is proper, then $g(a_{(r)}) \leqq g(a)$.

(ii) If $a^{(r)}$ is proper, then $g(a) \leqq g(a^{(r)})$.

(iii) $a \subseteqq b$ implies that $g(a) \leqq g(b)$.

Proof. (i) If $a \in Q$, then $a = a_{(r)}$, hence $g(a_{(r)}) \leqq g(a)$. If $a = \bigwedge S$, then $g(a_{(r)}) = g(\bigwedge(s_{(r)}) \mid s \in S) = \bigwedge(g(s_{(r)}) \mid s \in S) \leqq \bigwedge(g(s) \mid s \in S) = g(a)$.

(By induction, $g(s_{(r)}) \leqq g(s)$ for all $s \in S$.) For $a = \bigvee S$ dually.

(ii) This is dual to (i).

(iii) If $a \subseteqq b$ follows by (1), the $a^{(p)} \leqq b_{(r)}$ for some $p, r \in R$, $p \leqq r$. Applying (i) and (ii), $g(a) \leqq g(a^{(p)}) \leqq g(b_{(r)}) \leqq g(b)$. If $a \subseteqq b$ holds by (2) with $a = \bigwedge S$, then $s \leqq b$ for some $s \in S$. Hence, $g(a) \leqq g(s) \leqq g(b)$. The remaining cases are analogous.

Thus, g induces a map $f \colon C(Q) \to K$ that extends each f_r. If $S \subseteqq W_m(Q)$ with $0<|S|<m$, then

$$f(\bigwedge(C(S)/s\in S))=f(C(\bigwedge S))=g(\bigwedge S)=(g(s)\mid s\in S)=\bigwedge(f(C(s))\mid s\in S)).$$

We conclude that f is an m-homomorphism, completing the proof of the theorem.

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Matematický ústav SAV
Obrancov mieru 49
814 73 Bratislava

m-ПОПРОДУКТ СТРУКТУР

Zuzana Ladzianska

Резюме

В работе изучаются свойства m-попродукта. m-попродукт является обобщением свободного m-произведения структур.