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α r-SPACES AND SOME OF THEIR PROPERTIES

IGOR ZUZČÁK

In [2] the author has introduced the notion of an r-space and has shown that topological spaces are special cases of r-spaces. It is well known that if X is a topological space, then

(1) the closure of the union of two subsets of X is the union of their closures.

In the present paper there are studied r-spaces satisfying properties analogous to (1) for some subsets of X and some conditions of a decomposition of such r-spaces into topological spaces are given.

We shall use the notation from [1] and 2^X will denote the class of all subsets of X. The notation $A \subseteq B$ means that A is a subset of B and if A is a proper subset of B we write $A \subset B$. Specific terms will be explained when used for the first time.

1. r-spaces

In [2] the notion of an r-space was introduced:

Let X be a non-empty set and ϱ be a relation on 2^X satisfying

R_1) for each subset A of X there is a subset B of X such that $A\varrho B$

R_2) $\emptyset\varrho\emptyset$

R_3) if $A\varrho B$, then $A \subseteq B$

R_4) if $A\varrho B$, then $B\varrho B$

R_5) if $A \subseteq B$ and $B\varrho B$, then there is a subset C of X such that $A\varrho C$ and $C \subseteq B$

R_6) if $A\varrho B$, then there is no subset C of X such that $C\varrho C$ and $A \subseteq C \subset B$.

Then ϱ is called a relation of closure on 2^X. The pair (X, ϱ) is called an r-space if X is a nonempty set and ϱ is a relation of closure on 2^X. If (X, ϱ) is an r-space and for subsets A, B of X we have $A\varrho B$, then we say that B is a closure of A. A set $A \subseteq X$ satisfying $A\varrho A$ is called a closed set.

To simplify the notation we often refer to the r-space X instead of to the more proper form (X, ϱ).

Remark 1. If (X, ϱ) is an r-space, then the class $\mathcal{T} = \{A \subseteq X\colon A\varrho A\}$ of all closed subsets of X has the following properties

Ω_1: $\emptyset$, $X \in \mathcal{T}$

Ω_2: for each $A \subseteq X$ and each $B \in \mathcal{T}$ such that $A \subseteq B$ there is a minimal element C of the class $\{M \in \mathcal{T}\colon A \subseteq M \subseteq B\}$.

Now let X be a nonempty set and $\mathcal{T}$ be a class of subsets of X satisfying Ω_1 and Ω_2. Then, as shown in [2], the relation ϱ defined on 2^X by

(4) $A\varrho B$ iff B is a minimal element of $\{M \in \mathcal{T}\colon A \subseteq M \subseteq B\}$

has the properties R_1—R_6, (X, ϱ) is an r-space and $\mathcal{T}$ is precisely the class of all closed subsets of (X, ϱ).

Remark 2. From Example 1 of [2] it follows that a subset A of an r-space X may have more than one closure.

At the beginning of this paper we considered the condition (1) in connection with topological spaces. Our aim now is to generalize this condition to r-spaces.

Suppose first that (X, ϱ) is an r-space. By definition of the r-space we know that if A, B are subsets of X, then $A\varrho B$ means that B is a closure of A. Therefore the condition (1) can be described in the following way:

(2) if for the subsets A, B, C and D of X we have $A\varrho B$ and $c\varrho D$, then $(A \cup C)\varrho(B \cup D)$.

Now we shall study r-spaces satisfying (2) for some subsets of X.

In b) of Theorem 9 of [2] it was shown that if (X, ϱ) is an r-space, A, B and C are subsets of X such that $A \subseteq B \subseteq C$ and $A\varrho C$ holds, then also $B\varrho C$ holds. From this we have the following statement

Corollary 1. *If (X, ϱ) is an r-space and for the subsets A, B, C and D of X the relations $A\varrho B$, $C\varrho D$ and $B \subseteq D$ hold, then $(A \cup C)\varrho(B \cup D)$.*

From this corollary it is clear that if (X, ϱ) is an arbitrary r-space, then for some subsets A, B, C and D of X the condition (2) is satisfied. Now we shall show that in some r-spaces (X, ϱ) the condition (2) cannot be satisfied for all subsets A, B, C and D of X.

Lemma 1. *Let (X, ϱ) be an r-space. Let A, B, C and D be subsets of X satisfying $A\varrho B$ and $C\varrho D$. If $(A \cup C)\varrho(B \cup D)$ holds, then either $A \neq C$ or $B = D$.*

Proof. Suppose that the lemma is false, i.e., $A\varrho B$, $C\varrho D$, $(A \cup C)\varrho(B \cup D)$, $A = C$ and $B \neq D$ hold. This means that $A\varrho(B \cup D)$, since we have $(A \cup C)\varrho(B \cup D)$. By a) of Theorem 9 of [2] we have $D \nsubseteq B$, which means that $B \subset (B \cup D)$. But $A\varrho B$, $A\varrho(B \cup D)$ and $B \subset (B \cup D)$ contradicts the condition R_6.

From this there follows immediately

Lemma 2. *Let (X, ϱ) be an r-space. If for the subsets A, B, C and D we have $A\varrho B$, $C\varrho D$, $A = C$ and $B \neq D$, then $(A \cup C)\varrho(B \cup D)$ does not hold.*

By Lemma 1 we may consider if we want to study r-spaces, condition (2) only if either $A \neq C$ or $B = D$. But by Corollary 1 the condition (2) for $B = D$ always holds. So it seems to be natural to study r-spaces (X, ϱ) satisfying the following condition

(2a) if $A\varrho B$, $C\varrho D$ and $A \neq C$, then $(A \cup C)\varrho(B \cup D)$.

It is easy to see that if an r-space is a topological space, then also the condition (2a) is satisfied. Our next theorem will show that the converse of this statement is also true.

Theorem 1. *Let (X, ϱ) be an r-space and let the relation of closure ϱ of this r-space satisfy the condition* (2a). *Then (X, ϱ) is a topological space.*

Proof. To prove that an r-space (X, ϱ) is a topological space it suffices to show that ϱ is a closure operator on X (see [1]). Thus we must show that each subset of X has only one closure and that ϱ satisfies the Kuratowski closure axioms.

First we shall show that for each subset M of X there is only one subset N of X such that $M\varrho N$. Suppose that this is not true, i.e., there is $A \subseteq X$ such that $A\varrho B$, $A\varrho D$ and $B \neq D$, which implies that $A\varrho(B \cup D)$ does not hold. Since A has two closures B and D such that $B \neq D$, then A cannot be closed. Therefore $A \neq B$. Since $A\varrho B$, then by R_4 there is $B\varrho B$ and by R_3 we have $A \subseteq B$, hence we have $A\varrho B$, $B\varrho B$ and $A \neq B$. Then by (2a) it follows that $(A \cup B)\varrho(B \cup D)$ which means that $B\varrho(B \cup D)$. From $B\varrho B$ and $B\varrho(B \cup D)$ it follows that $B = B \cup D$ by R_6. Since $A\varrho B$ holds, $B = B \cup D$ implies $A\varrho(B \cup D)$, which contradicts the fact that $A\varrho(B \cup D)$ does not hold.

Now let $A\varrho B$ and $C\varrho D$ hold.

If $A \neq C$, then $(A \cup C)\varrho(B \cup D)$ follows from (2a).

If $A = C$, then $B = D$, since each subset of X has only one closure. By Corollary 1 we have again $(A \cup C)\varrho(B \cup D)$.

This means that the condition (1) is satisfied. The remaining Kuratowski closure axioms follow immediately from R_1, R_2, R_3 and R_4.

Now let X be an r-space, $\mathcal{T}$ the class of all closed subsets of X and ϱ is the relation of closure of this r-space. The property (2) can be interpreted also in the following way: for each B, $D \in \mathcal{T}$ there holds that

(2b) if A, C are subsets of X satisfying $A\varrho B$ and $C\varrho D$, then $(A \cup C)\varrho(B \cup D)$.

But from Lemma 1 it follows that the condition (2b) can be considered for B, D, where $B \neq D$ only if B and D are not closures of the same set.

In the remaining parts of this paper there are studied r-spaces called αr-spaces, satisfying for each B, $D \in T$, where B, D are not closure of the same set, the condition (2b).

For easy reference we introduce the following notation.

Definition 1. *Let T be the class of all closed subsets of an r-space X and $w \subseteq T \times T$ be a relation satisfying the following condition*

(3) *AwB iff $A \neq B$ and there is no subset C of X such that $C\varrho A$ and $C\varrho B$.*

Then the relation w is called an r-relation of the r-space X.

We are now ready to define the notion of an αr-space.

Definition 2. *Let (X, ϱ) be an r-space and w the r-relation of (X, ϱ). Suppose that the following condition is fulfilled*

R_7) *if $A\varrho B$, $C\varrho D$ and BwD, then $(A\cup C)\varrho(B\cup D)$.*

Then the r-space (X, ϱ) is called an αr-space.

Remark 3. If (X, ϱ) is an r-space and the condition (2a) is satisfied, then the condition R_7 is satisfied too. This follows from the fact that (X, ϱ) satisfying the condition (2a) is by Theorem 1 a topological space.

Now we give two examples illustrating relations between the above described types of r-spaces. First we give an example of an αr-space.

Example 1. Let R_2 be a real Euclidean 2-space (see [1]) and let N be the set of all positive integers. Define the following sets:

$$A_i = \{(x, y): x = i,\ y \in (0, \infty)\}, \quad i \in N$$
$$B_j = \{(x, y): x \in (0, \infty),\ y = j\}, \quad j \in N$$

$$A = \bigcup_1^\infty A_i, \quad B = \bigcup_1^\infty B_j \quad \text{and} \quad X = A \cup B.$$

Let us define a relation ϱ on 2^X as follows:

a) if $M \subseteq A$, $N_1 = \{i \in N: A_i \cap M \neq \emptyset\}$ and $A_m = \bigcup_{i \in N_1} A_i$ then $M\varrho A_M$

b) if $M \subseteq B$, $N_2 = \{j \in N: B_j \cap M \neq \emptyset\}$ and $B_M = \bigcup_{j \in N_2} B_j$ then $M\varrho B_M$

c) if $M \not\subseteq A$ and $M \not\subseteq B$, then $M\varrho X$

d) if $M = \emptyset$, then $M\varrho M$.

It is easy to see that (X, ϱ) is an αr-space. The class of closed subsets of X consists of:

1) $\emptyset$, X

2) all subsets of X of the form $\bigcup_{i \in N'} A_i$, where $N' \subseteq N$

3) all subsets of X of the form $\bigcup_{j \in N''} B_j$, where $N'' \subseteq N$.

r-spaces are not necesserily αr-spaces as shown by the following example.

Example 2. Let $X = \{a, b, c\}$ and let ϱ be the relation on 2^X defined by:

$$\{a\}\varrho\{a, b\}, \quad \{a, b\}\varrho\{a, b\}, \quad \{c\}\varrho\{c\}$$

$$\{a\}\varrho\{a, c\}, \quad \{a, c\}\varrho\{a, c\}, \qquad \emptyset\varrho\emptyset$$
$$\{b\}\varrho\{a, b\}, \quad \{b, c\}\varrho X, \qquad X\varrho X.$$

It is clear that (X, ϱ) is an r-space. The following sets are closed subsets of this r-space: $\emptyset$, X, $\{c\}$, $\{a, b\}$ and $\{a, c\}$. Next we see that

$$\{a\}\varrho\{a, b\}, \quad \{c\}\varrho\{c\} \quad \text{and} \quad \{a, b\}w\{c\}.$$

On the other hand

$$\{a, c\}\varrho\{a, b, c\} \quad \text{does not hold.}$$

This means that the condition R_7 is not satisfied.

In the following two sections of this paper we assume that there is given an αr-space X, the relation closure ϱ, the r-relation w and the class $\mathcal{T}$ of closed subsets of this αr-space.

2. Some properties of αr-spaces

First we give two statements proved in [2]:

K_1: Let A, B be closed subsets of X. Then A, B are closure of the same set iff A, B are closures of $A \cap B$.

K_2: Each closed subset of X has only one closure.

Lemma 3. *Let A, B and C be subsets of X such that $A \subseteq B$, $A\varrho C$ and BwC. Then $C \subseteq B$.*

Proof. Since B is closed, then $B\varrho B$. Thus we have $B\varrho B$, $A\varrho C$ and BwC. Therefore $(A \cup B)\varrho(B \cup C)$ by R_7. But $A \subseteq B$ and so $B\varrho(B \cup C)$. The relations $B\varrho B$ and $B\varrho(B \cup C)$ imply $B = B \cup C$, hence $C \subseteq B$.

We know that A is a closed subset of x iff $A\varrho A$. Therefore if A and B are closed subsets of X we have $A\varrho A$ and $B\varrho B$. If we assume AwB, the by R_7 we have $(A \cup B)\varrho(A \cup B)$, which means that $A \cup B$ is closed. We thus get the following result

Theorem 2. *If A and B are closed subsets of X such that AwB, then $A \cup B$ is closed.*

Theorem 3. *Let A and B closed subsets of X and $A \neq B$. Then $A \cap B$ is closed iff AwB.*

Proof. First we show that if $A \cap B$ is closed, then AwB. Suppose $A \cap B$ is closed, $A \neq B$ and AwB is not true, i.e., A and B are closures of the same set. Then by K_1 the sets A and B are distinct closures of $A \cap B$, which is impossible, since $A \cap B$ is closed and has only one closure.

There remains to be shown that if $A \neq B$ and AwB, then $A \cap B$ is closed. If

AwB, then at least one of the sets A and B is not a closure of $A\cap B$. We may suppose that A is not a closure of $A\cap B$, i.e., $(A\cap B)\varrho A$ does not hold. Therefore by R_5 there is a closed set C such that $(A\cap B)\varrho C$ and $C\subseteq B$. From $(A\cap B)\varrho C$ it follows that $A\cap B\subseteq C$ by R_3. From $C\subseteq B$ we have $B\cap C=C$ and since $A\cap B\subseteq C$, then $(A\cap C)=A\cap(B\cap C)=(A\cap B)\cap C=A\cap B$.

We thus have $(A\cap C)\varrho C$, A is closed and AwC. By virtue of Lemma 3 we see that $C\subseteq A$ and therefore $A\cap C=C$, but this means that $A\cap B$ is closed, since $A\cap B=A\cap C=C$ and C is closed.

3. Complete r-systems

Definition 3. *A class $\mathscr{A}$ of closed subsets of X is called an r-system in X iff AwB for each A, $B\in\mathscr{A}$, $A\neq B$.*

If A is an r-system in X, then by Theorems 2 and 3 for every A, $B\in\mathscr{A}$ both the sets $A\cup B$ and $A\cap B$ are closed. We may also wonder whether or not for A, $B\in\mathscr{A}$ we have $A\cup B\in\mathscr{A}$ or $A\cap B\in\mathscr{A}$. This question is answered in the following section of the paper.

Let $\mathfrak{U}$ be the class of all r-systems in X. The class $\mathfrak{U}$ is partially ordered by the relation of inclusion $\subseteq$. According to Hausdorff's maximal principle there exists the family Φ of maximal chains in $\mathfrak{U}$ such that each chain in $\mathfrak{U}$ is contained in an element or Φ. From the definition of r-systems it is not hard to see that if $\mathfrak{U}_1$ is a chain in $\mathfrak{U}$, then the union of all elements of $\mathfrak{U}_1$ is itself an element of $\mathfrak{U}$. Therefore if $\mathfrak{U}_0$ is an element of Φ, then the union of all elements of $\mathfrak{U}_0$ is a maximal element of $\mathfrak{U}$. Each maximal r-system we call a complete r-system in X. It is evident that to every r-system $\mathscr{A}_1$ in X there is in X at least one complete r-system $\mathscr{A}_2$ such that $\mathscr{A}_1\subseteq\mathscr{A}_2$.

Remark 4. If A is a closed subset of X, then by Corollary 1 it is clear that the class $\{\emptyset, A, X\}$ is an r-system. From this it follows that for each closed subset A of X there is a complete r-system $\mathscr{A}$ in X such that $A\in\mathscr{A}$.

Theorem 4. *Let $\mathscr{A}$ be a complete r-system in X. If $\{A_s\}_{s\in S}\subseteq\mathscr{A}$ and S is a finite set, then $\bigcup_S A_s\in\mathscr{A}$ and also $\bigcap_S A_s\in\mathscr{A}$.*

Proof. It suffices to prove that for each A, $B\in\mathscr{A}$ there is $A\cap B\in\mathscr{A}$ and $A\cup B\in\mathscr{A}$.

First we show $A\cap B\in\mathscr{A}$. Suppose this is not true, i.e., $A\cap B\notin\mathscr{A}$. Since $\mathscr{A}$ is a complete r-system there is $C\in\mathscr{A}$ such that $(A\cap B)wC$ is not true. Hence $A\cap B\neq C$ and $A\cap B$ and C are closures of the same set. This implies that $A\cap B$ and C are closures of $(A\cap B)\cap C$. Next it is clear that $(A\cap B)\cap C\subseteq A\cap C\subseteq C$.

Since $A, C \in \mathscr{A}$ it follows from Theorem 3 that $A \cap C$ is closed. This means that $A \cap C = C$ and therefore $C \subseteq A$. Analogously it is clear that $C \subseteq B$ and hence $C \subseteq (A \cap B)$. From the fact that $A \cap B$ and C are closed sets, $C \subseteq A \cap B$, $(A \cap B \cap C)\varrho C$ and $(A \cap B \cap C)\varrho(A \cap B)$ it follows that $C = (A \cap B)$ this is a contradiction, since $C \neq A \cap B$.

There remains to be proved that if $A, B \in \mathscr{A}$, then $A \cup B \in \mathscr{A}$. Let C be an arbitrary element of $\mathscr{A}$. If $A \cup B = C$, the proof is completed, since $A \cup B \in \mathscr{A}$.

Suppose now that $A \cup B \neq C$. We shall show that $Cw(A \cup B)$. To prove this by Theorem 3 it suffices to show that $(A \cup B) \cap C$ is closed. However, A, B and C are elements of $\mathscr{A}$ and hence $A \cap C$ and $B \cap C$ are also elements of $\mathscr{A}$. This follows from the first part of this theorem. Using the fact that $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$, we see that $(A \cup B) \cap C$ is a closed set.

Hence $Cw(A \cup B)$ for each $C \in \mathscr{A} - \{A \cup B\}$ and $A \cup B$ must belong to $\mathscr{A}$ by the maximality of $\mathscr{A}$.

Theorem 5. *Let X be an αr-space satisfying the following condition:*

(4) *for each r-system $\{A_s\}_{s \in S}$ in X the set $\bigcap_S A_s$ is closed.*

Then if $\mathscr{A}$ is a complete r-system in X, the intersection of any number of elements of $\mathscr{A}$ belongs to $\mathscr{A}$.

Proof. Let $\mathscr{A}$ be a complete r-system in X. Let $\{A_s\}_{s \in S} \subseteq \mathscr{A}$ and let $A = \bigcap_S A_s$. Analogously to the proof of the previous theorem it suffices to show that if $C \in \mathscr{A}$ and $A \neq C$, then $A \cap C$ is closed. But $A \cap C = \bigcap_S A_s \cap C$ and $\{A_s\}_{s \in S} \cup \{C\}$ is a r-system in $\mathscr{A}$. Therefore by (4) $A \cap C$ is closed.

Remark 5. Let X be αr-space given in Example 1. Let

$$\mathscr{T}_1 = \left\{A = \bigcup_{i \in N'} A_i \colon N' \subseteq N\right\} \cup \{0, X\}$$

and

$$\mathscr{T}_2 = \left\{B = \bigcup_{j \in N'} B_j \colon N' \subseteq N\right\} \cup \{\emptyset, X\}$$

be classes of closed subsets of X.

From definition of the sets A_i, B_j, $i, j = 1, 2, 3, \ldots$ it follows that $\mathscr{T}_1$ and $\mathscr{T}_2$ are complete r-systems in X and each r-system in X is either a part of $\mathscr{T}_1$ or a part of $\mathscr{T}_2$. On the other hand it is clear that both pairs $(X, \mathscr{T}_1)$ and $(X, \mathscr{T}_2)$ are topological spaces, where $\mathscr{T}_1$ and $\mathscr{T}_2$ are their classes of closed sets. This means that the αr-space X satisfies the condition (4).

Now consider an r-space X and let $\mathscr{T}$ be the class of closed subsets of X. If for each closed subset A of X we denote

$$\mathcal{T}_A = \{\emptyset, A, X\},$$

then it is clear that $(X, \mathcal{T}_A)$ is a topological space.

Definition 4. *The topological space $(X, \mathcal{T}_A)$ is called a trivial topological space for the r-space X. A topological space $(X, \mathcal{T}')$ such that $\mathcal{T}' \subseteq \mathcal{T}$ is said to be a topological space for the r-space X and $\mathcal{T}'$ a topology for the r-space X.*

From Remark 5 it follows that there is an r-space having the following property:

There is a class $\{\mathcal{T}_s\}_{s \in S}$ of topologies for the r-space X satisfying

a) for each $s \in S$ there is no topology $\mathcal{T}'$ for X such that $\mathcal{T}_s \subseteq \mathcal{T}'$

b) if $\mathcal{T}'$ is a topology for X, then there is $s \in S$ such that $\mathcal{T}' \subseteq \mathcal{T}_s$.

Since for each $A \in \mathcal{T}$ there is a trivial topology for X containing A, then $\bigcup_S \mathcal{T}_s = \mathcal{T}$.

It is also clear that for each $s \in S, (X, \mathcal{T}_s)$ is a topological space.

Definition 5. *Let X be an r-space and $\mathcal{T}$ be the class of all closed subsets of X. If there is a class $\{\mathcal{T}_s\}_{s \in S}$ of topologies for X satisfying a) and b), then X is said to be a topological r-space.*

As an immediate consequence of Theorems 4 and 5 and Definition 5 we have the following

Corollary 2. *Each αr-space satisfying (4) is a topological r-space.*

Finally we give an example to show that there is a topological αr-space which does not satisfy the condition (4).

Example 3. Let $X = \{0, 1, 2, 3, 4, 5, \dots\}$. Consider the following subsets of X:

$$\begin{array}{llll} \text{for} & i = 1, 3, 5, 7, \dots & \text{let} & A_i = \{0, i, i+1, i+3, i+5, \dots\}, \\ \text{for} & i = 2, 4, 6, 8, \dots & \text{let} & A_i = \{0, i, i+1, i+2, i+3, \dots\} \end{array}$$

and $A_0 = X$.

In the following we shall show that there is an αr-space (X, ϱ) such that

a) $\mathcal{T} = \{\emptyset, A_0, A_1, A_2, A_3, \dots\}$ is the class of all closed subsets of (X, ϱ)

b) (X, ϱ) is a topological αr-space, but does not satisfy (4).

Let $M \subseteq X$ and N be the set of all positive integers. Define a relation ϱ on 2^X as follows

a) if there are at least two elements $i, j \in \{1, 3, 5, 7, \dots\}$ such that $i \neq j$, $i, j \in M$ and $k = \inf\{p \in N\colon\ p \in M\}$, we put

$$\begin{array}{lll} M\varrho A_{k-1} & \text{if} & k \text{ is odd} \\ M\varrho A_k & \text{if} & k \text{ is even} \end{array}$$

b) if there is only one odd positive integer i such that $i \in M$, then if $k = \inf\{p \in N\colon p \in M\}$, we put

$$M\varrho A_k$$

c) if $M \neq \{0\}$ and M contains only elements of $\{0, 2, 4, 6, \dots\}$, then if $k = \inf\{i \in N\colon i \in M\}$, we put

$$M \varrho A_k$$

and $$M \varrho A_p \quad \text{for} \quad p < k, \text{ where } p \text{ is odd}$$

d) if $M = \{0\}$, then

$$M \varrho A_k \quad \text{for} \quad k = 1, 3, 5, 7, \dots$$

e) $\emptyset \varrho \emptyset$ and $X \varrho X$.

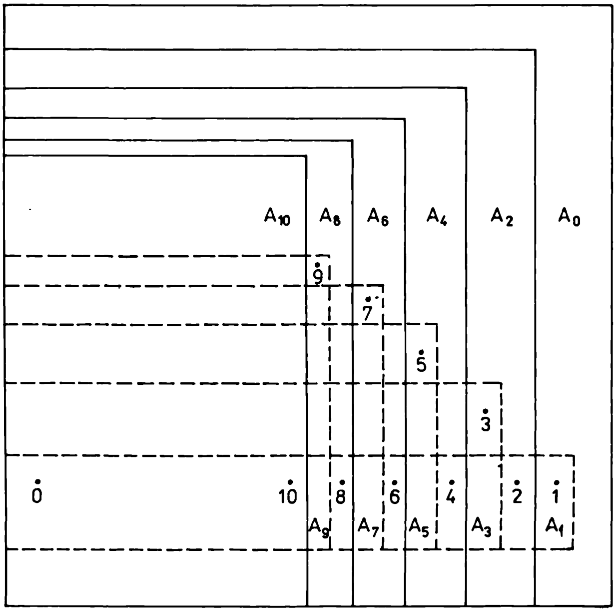

Fig. 1

From the definition of ϱ and from Figure 1 it is clear that:
1) the conditions R_1—R_6 are satisfied;
2) if $i, j \in \{1, 3, 5, \dots\}$, then A_i and A_j are closures of the set $\{0\}$ and therefore $A_i w A_j$ is not true;
3) if $i, j \in \{0, 2, 4, 6, \dots\}$ and $i < j$, then $A_j \subseteq A_i$;
4) if $i \in \{1, 3, 5, \dots\}$ and $j \in \{0, 2, 4, 6, \dots\}$, then either A_i and A_j are closures of the same set, i.e., $A_i w A_j$ is not true, or $A_i \subseteq A_j$.

From 2), 3) and 4) it is evident that for each closed subset A_i and A_j of X, where $i \neq j$, either $A_i w A_j$ is not true or $A_i \subseteq A_j$, resp., $A_j \subseteq A_i$ holds. From this and from

Corollary 1 it follows that the condition R_7 is satisfied too. Hence (X, ϱ) is an αr-space and $\mathcal{T} = \{\emptyset, A_0, A_1, A_2, A_3, \ldots\}$.

Now let for each $i \in \{1, 3, 5, 7, \ldots\}$

$$T_i = \{\emptyset, A_i, A_{i-1}, A_{i-3}, A_{i-5}, \ldots, A_0\}.$$

It is easy to verify that

— for each $i \in \{1, 3, 5, \ldots\}$ the class $\mathcal{T}_i$ is a topology for X

— the family $\{\mathcal{T}_i\}_{i \in \{1, 3, 5, \ldots\}}$ satisfies the conditions a) and b) of Definition 5.

This means that (X, ϱ) is a topological αr-space.

On the other hand the class $\{A_i\}_{i \in N_1}$, where $N_1 = 0, 2, 4, \ldots$ is an r-system in X, but $\bigcap_{N_1} A_i = \{0\}$ is not closed. Therefore (4) is not satisfied.

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αr-ПРОСТРАНСТВА И НЕКОТОРЫЕ ИХ СВОЙСТВА

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Резюме

r-пространство (X, ϱ) называется αr-пространством, если для всяких подмножеств A, B, C, D множества X таких, что $A\varrho B$, $C\varrho D$ и B, D не являются замиканиями того самого подмножества X, имеет место соотношение

$$(A \cup B)\varrho(C \cup D).$$

αr-пространства тоже являются обобщением топологических пространств.

В настоящей работе изучаются некоторые свойства αr-пространств и условия разложимости этих пространств на топологические пространства.