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DECOMPOSITION OF COMPLETE BIPARTITE GRAPHS INTO FACTORS WITH GIVEN DIAMETERS AND RADII

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Introduction

Ľ. Niepel [3] studies the existence of a decomposition of the complete graph into factors with given diameters and radii. In the present paper we study the analogous problem for the complete q-partite graphs. Most of the results are concerned with the case $q=2$ of bipartite graphs.

All graphs in the present paper are undirected, without loops or multiple edges. Let an integer $q \geq 2$ be given. A graph G with the vertex set V is called q-partite if V can be partitioned into q mutually disjoint, nonempty subsets $V_1, V_2, \ldots, V_q$, which are called parts of G such that every edge of G joins vertices of two different parts of G. If every two vertices of different parts of G are joined by an edge, then G is said to be a complete q-partite graphs and we write $G=K_{m_1, m_2, \ldots, m_q}$, where the cardinality $|V_j| = m_j$ for $j=1, 2, \ldots, q$ (2-partite graphs are called bipartite).

By a factor of a graph G we mean a subgraph of G containing all the vertices of G. By a decomposition of a graph G into factors we mean a system $\mathcal{S}$ of factors of G such that every edge of G is contained in exactly one factor of $\mathcal{S}$. The eccentricity $e(v)$ of a vertex v is $\sup \varrho_G(u, v)$, for all $u \in V_G$, where V_G is the vertex set of G and $\varrho_G(u, v)$ denotes the distance between two vertices $u, v \in V_G$ in G. The radius $r(G)$ of a graph G is defined as $r(G) = \min e(v)$ and the diameter $d(G)$ of G as $d(G) = \max e(v)$. The diameter $d(G)$ and the radius $r(G)$ can be also equal to ∞ if G is a disconnected graph or if G is connected but $e(v)$ is infinite for all v. The remaining terms are used in the usual sense [1, 2, 3, 4, 5].

Let integers $p, q \geq 2$ and nonnegative integers (or symbols ∞) d_i, r_i for $1 \leq i \leq p$ and non-zero cardinal numbers m_j for $1 \leq j \leq q$ be given. Our aim is to determine the conditions for the existence of a decomposition of the graph $K_{m_1, m_2, \ldots, m_q}$ into p factors $F_1, F_2, \ldots, F_p$ with given diameters $d_1, d_2, \ldots, d_p$ and radii $r_1, r_2, \ldots, r_p$.

1. The general case

Let $q \geq 2$ and $p \geq 1$ be integers, m_i $(i = 1, 2, \ldots, q-1)$ — cardinal numbers ≥ 1, d_i, r_i $(i = 1, 2, \ldots, p)$ — positive integers or symbols ∞. For the diameter d_i and the radius r_i of the factor F_i of $K_{m_1, m_2, \ldots, m_q}$ the following inequalities hold:

$$r_i \leq d_i \leq 2r_i \quad (i = 1, 2, \ldots, p).$$

Denote by $D_{m_1, m_2, \ldots, m_{q-1}}$ $(d_1, d_2, \ldots, d_p, r_1, r_2, \ldots, r_p)$ the smallest cardinal number m_q such that the graph $K_{m_1, \ldots, m_q}$ can be decomposed into p factors $F_1, F_2, \ldots, F_p$ with $d(F_i) = d_i$ and $r(F_i)r_i$ $(i = 1, 2, \ldots, p)$. If such a number does not exist, we shall write

$$D_{m_1, m_2, \ldots, m_{q-1}}(d_1, d_2, \ldots, d_p, r_1, r_2, \ldots, r_p) = \infty.$$

The importance of the function $D_{m_1, m_2, \ldots, m_{q-1}}$ can be seen from the next theorem.

Theorem 1. *If the graph $K_{m_1, m_2, \ldots, m_q}$ is decomposable into p factors with given diameters $d_1, d_2, \ldots, d_p$ and radii $r_1, r_2, \ldots, r_p$, where $d_i > 1$ $(i = 1, 2, \ldots, p)$, then the graph $K_{M_1, M_2, \ldots, M_q}$ (where $M_1 \geq m_1$, $M_2 \geq m_2$, $\ldots$, $M_q \geq m_q$) is also decomposable into p factors with the same diameters $d_1, d_2, \ldots, d_p$ and radii $r_1, r_2, \ldots, r_p$.*

The proof of this theorem is analogous to that of Theorem 1 of [4] or [5].

Corollary. *The graph $K_{m_1, m_2, \ldots, m_q}$ can be decomposed into p factors with diameters $d_1, d_2, \ldots, d_p$ and radii $r_1, r_2, \ldots, r_p$ (where $d_i \geq 2$, $r_i \geq 2$, $i = 1, 2, \ldots, p$) if and only if*

$$m_q \geq D_{m_1, m_2, \ldots, m_{q-1}}(d_1, d_2, \ldots, d_p, r_1, r_2, \ldots, r_p).$$

2. Decomposition of $K_{m,n}$ into p factors

In the graph $K_{m,n}$ (where m, n are integers such that $2 \leq m \leq n$) there evidently exists a factor with an arbitrary diameter d such that $2 \leq d \leq m$ with the exception of $m = n$, $d = 2m$ and a factor with an arbitrary radius r such that $2 \leq r \leq m$. Moreover, a factor with another finite diameter or radius in $K_{m,n}$ does not exist. If $m = 1$, $n \geq 2$, then in the graph $K_{m,n}$ there exists a factor with the diameter 2 or ∞ only and with the radius 1 or ∞ only.

Lemma 1 *[3]. If a finite connected graph has order p, radius r and diameter d, then the following inequalities hold:*

(a) $d \leq 2r \leq 2d$.

(b) $p \geq \begin{cases} d+1, & \text{if } d \leq 2r \leq d+1, \\ d+r, & \text{if } d+2 \leq 2r \leq 2d. \end{cases}$

Theorem 2. *Let m, n and r be positive integers. Then in the complete bipartite graph $K_{m,n}$ there exists a factor with diameter d and radius r if and only if one of the following six cases occurs:*

(1) $m=n=d=r=1$.

(2) $m=1<n$, $d=2$, $r=1$.

(3) $m=n$, $3\leqslant d\leqslant 2r\leqslant d+1\leqslant 2m$.

(4) $m<n$, $3\leqslant d\leqslant 2r\leqslant d+1\leqslant 2m+1$.

(5) $m\leqslant n\leqslant m+r$, $d+2\leqslant 2r\leqslant 2d$, $d+r\leqslant m+n$.

(6) $n>m+r$, $d+2\leqslant 2r\leqslant 2d$, $d\leqslant 2m$.

Proof. I. Let $K_{m,n}$ have a factor with diameter d and radius r. If $m\leqslant n$, then we evidently have:

(7) $d\leqslant 2m-1$, if $m=n$,

(8) $d\leqslant 2m$, if $m<n$.

If $d=1$ or $r=1$, then (1) or (2) evidently holds. Therefore let $d>1$, $r>1$. If $d=2$, $r>1$, then from (a) we have $r=2$ and (5) or (6) holds. Therefore let $d\geqslant 3$. According to (a) we have either

(9) $3\leqslant d\leqslant 2r\leqslant d+1$

or

(10) $d+2\leqslant 2r\leqslant 2d$.

In the case (9) according to (7) and (8) either (3) or (4) holds. Therefore let (10) hold. Put $p=m+n$. If $m\leqslant n\leqslant m+r$, then according to (b) $d+r\leqslant m+n$ and (5) holds. If $n>m+r$, then according to (8) $d\leqslant 2m$ and (6) holds. Thus some of the conditions (1)—(6) always holds.

II. Let some of the conditions (1)—(6) hold. We shall construct a factor F of $K_{m,n}$ with diameter d and radius r. Denote by A and B the parts of $K_{m,n}$, where $|A|=m$, $|B|=n$ and $m\leqslant n$.

If (1) or (2) holds, then it is sufficient to set $F=K_{m,n}$. When (3) or (4) holds, then the factor F contains the edges of the path $v_1v_2\ldots v_{d+1}$ where $v_1, v_3, v_5, \ldots\in B$, v_2, $v_4, v_6, \ldots\in A$ and all edges (if they exist) v_2x and v_3y, where x [or y] is the vertex from the part B [or A, respectively] not belonging to the path $v_1v_2v_3\ldots v_{d+1}$. It is clear that F has diameter d and radius

$$r=\left[\frac{d+1}{2}\right].$$

If (5) or (6) holds, then the factor F is defined by the edges of the path $v_1v_2\ldots v_{d+r}$, where $v_1, v_3, v_5, \ldots\in B$, $v_2, v_4, \ldots\in A$ and by the edge v_1v_{2r} and all the edges (if they exist) v_2x and v_3y where x [or y] is the vertex of the part B [or A, respectively] do not belonging to the path $v_1v_2v_3\ldots v_{d+r}$.

It is easy to see that the maximum [or the minimum] of the eccentricity of a vertex is d [or r] and it is reached for the vertex v_{d+r} [or v_{2r}, respectively]. Hence the factor F has diameter d and radius r.

3. Decomposition of $K_{m,n}$ into two factors

Our results are complete in the case of bipartite graphs (i.e. $q=2$), two factors (i.e. $p=2$) and diameters equal to radii ($d_i=r_i$). There are found for every given diameters d_1, d_2 and radii r_1, r_2 such that $d_1=r_1$, $d_2=r_2$ all the couples (m, n) such that $m \leqslant n$, $D_m(d_1, d_2, r_1, r_2)=n$, and $D_M(d_1, d_2, r_1, r_2)=N$ does not hold for any $M \leqslant m$, $N \leqslant n$ and $(M, N) \neq (m, n)$. These couples are given in table I.

Table I

$d_2=r_2$ \ $d_1=r_1$	∞	1	2	3	4	5	6	7...
∞	(1, 2)	(1, 1)	(2,)	(3, 3)				
1	(1, 1)			In this area no decomposition exists for any $K_{m,n}$				
2	(2, 2)							
3	(3, 3)			(6, 6)	(5, 7) (6, 6)	(5, 5)	(6, 6)	(7, 7)
4				(5, 7) (6, 6)	(4, 4)			
5				(5, 5)				
6				(6, 6)	In this area no decomposition exists for any $K_{m,n}$			
7				(7, 7)				
				⋮				

Theorem 3. Let d_1, d_2, r_1, r_2 be positive integers or symbols ∞ and m, n be cardinal numbers such that $d_1 \leqslant d_2$, $d_1=r_1$, $d_2=r_2$ and $m \leqslant n$ hold. Then the complete bipartite graph $K_{m,n}$ is decomposable into two factors with diameters d_1 and d_2 and radii r_1 and r_2 if and only if one of the following cases occurs:

(1) $d_1=d_2=\infty$, $m \geqslant 1$, $n \geqslant 2$.
(2) $d_1=1$, $d_2=\infty$, $m=1$, $n=1$.
(3) $d_1=2$, $d_2=\infty$, $m \geqslant 2$.
(4) $d_1=3$, $d_2=\infty$, $m \geqslant 3$.
(5) $d_1=3$, $d_2=3$ or 4, $m \geqslant 6$.
(6) $d_1=3$, $d_2=4$, $m \geqslant 5$, $n \geqslant 7$.
(7) $d_1=3$, $5 \leqslant d_2 < \infty$, $m \geqslant d_2$.
(8) $d_1=4$, $d_2=4$, $m \geqslant 4$.

Proof of the Theorem 3 follows from Theorem 4 of [5] and Theorem 11 of [4]. It is sufficient to check that the graphs constructed there have $r_1 = d_1$ and $r_2 = d_2$.

The next Corollary shows for which diameters and radii it is possible to decompose a complete bipartite graph.

Corollary. Let positive integers $d_1 = r_1$, $d_2 = r_2$ $(d_1 \leq d_2)$ be given. A complete bipartite graph decomposable into two factors with diameters d_1 and d_2 and with radii r_1 and r_2 exists if and only if one of the following cases occurs:

(1) $d_1 = 3$.

(2) $d_1 = d_2 = 4$.

Proof. This is obvious from Theorem 3.

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РАЗЛОЖЕНИЕ ПОЛНЫХ ДВУДОЛЬНЫХ ГРАФОВ НА ФАКТОРЫ С ДАННЫМИ ДИАМЕТРАМИ И РАДИУСАМИ

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Резюме

Рассматривается проблема существования разложения полных двудольных графов на факторы с данными диаметрами и радиусами. Для случая факторов с равняющимися диаметрами и радиусами проблема решена полностью.