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ESTIMATION OF COVARIANCE COMPONENTS IN A REPEATED REGRESSION EXPERIMENT

LUBOMÍR KUBÁČEK

Dedicated to Academician Štefan Schwarz on the occasion of his 70th birthday

Introduction

In the regression model $\mathbf{Y}=\mathbf{X}\boldsymbol{\beta}+\boldsymbol{\varepsilon}$ the covariance matrix of the error vector $\boldsymbol{\varepsilon}$ is considered in the form $\boldsymbol{\Sigma}=\sum_{i=1}^{m}\mathbf{J}_i\mathbf{C}\mathbf{J}_i'$ [3]; $(n\times s)$-matrices $\mathbf{J}_i$, $i=1, \ldots, m$ are known. The elements of the unknown matrix $\mathbf{C}$ are called covariance components. When $s=1$ and $\mathbf{J}_i\mathbf{J}_i'$ is denoted $\mathbf{V}_i$, $i=1, \ldots, m$, the situation studied in [2] occurs. This paper completes paper [2].

The aim is to determine the estimator of the covariance components on the basis of the matrix $\mathbf{S}$,

$$k\mathbf{S}=\sum_{j=1}^{k+1}(\mathbf{Y}_j-\bar{\mathbf{Y}})(\mathbf{Y}_j-\bar{\mathbf{Y}})' \left(\bar{\mathbf{Y}}=[1/(k+1)]\sum_{j=1}^{k+1}\mathbf{Y}_j\right),$$

which is generated from the $(k+1)$-tuple stochastically independent random vectors $\mathbf{Y}_1, \ldots, \mathbf{Y}_{k+1}$ with the same normal distribution $N_n(\mathbf{X}\boldsymbol{\beta}, \boldsymbol{\Sigma})$. Thus the matrix $k\mathbf{S}$ has the Wishart distribution $W_n(k, \boldsymbol{\Sigma})$ [1].

1. Assumptions and auxiliary statements

Let $(\mathscr{S}_n, \langle\cdot, \cdot\cdot\rangle)$ be a Hilbert space of symmetric $(n\times n)$-matrices, $\langle\cdot, \cdot\cdot\rangle$ denotes the inner product given by $\langle\mathbf{A}, \mathbf{B}\rangle=\mathrm{Tr}(\mathbf{AB})$, $\mathbf{A}$, $\mathbf{B}\in\mathscr{S}_n$ [4]; $\mathrm{Tr}(\mathbf{AB})$ denotes the trace of the matrix $\mathbf{AB}$.

Let $\mathbf{J}_i$, $i=1, \ldots, m$ be given $(n\times s)$-matrices and let the covariance matrix $\boldsymbol{\Sigma}$ of the random vector $\mathbf{Y}\sim N_n(\mathbf{X}\boldsymbol{\beta}, \boldsymbol{\Sigma})$ be an element of the set

$$\boldsymbol{\Sigma}_*=\left\{\boldsymbol{\Sigma}:\boldsymbol{\Sigma}=\sum_{i=1}^{m}\mathbf{J}_i\mathbf{C}\mathbf{J}_i', \mathbf{C}\in\mathscr{C}\right\},$$

where $\mathscr{C}$ $(\subset\mathscr{S}_s)$ is a set of symmetric $(s\times s)$-matrices which satisfies the following condition:

$$(*)\begin{cases} \text{If for } \mathbf{M}\in\mathscr{S}_s \text{ there exists } \mathbf{A}\in\mathscr{S}_n \text{ such that for each} \\ \text{matrix } \mathbf{\Sigma}\in\mathbf{\Sigma}_* \text{ it is } \mathrm{Tr}(\mathbf{MC})=\mathrm{Tr}(\mathbf{A\Sigma}) \left(=\mathrm{Tr}\left(\sum_{i=1}^{m}\mathbf{J}_i'\mathbf{A}\mathbf{J}_i\mathbf{C}\right)\right), \\ \text{then } \sum_{i=1}^{m}\mathbf{J}_i'\mathbf{A}\mathbf{J}_i=\mathbf{M}. \end{cases}$$

Further it is assumed that each element of $\mathbf{\Sigma}_*$ is a positive definite matrix.

$g(\cdot)$ denotes the function $g(\cdot)\colon \mathscr{C}\to\mathscr{R}$, $g(\mathbf{C})=\mathrm{Tr}(\mathbf{MC})$, which is to be unbiasedly estimated on the basis of the realization of the matrix $k\mathbf{S}\sim W_n(k,\mathbf{\Sigma})$. (Procedure for estimating the function $g(\cdot)$ based on the realization of the vector $\mathbf{Y}$ see in [3].) The estimator of the function $g(\cdot)$ is considered in the form $\mathrm{Tr}(\mathbf{AS})$, $\mathbf{A}\in\mathscr{S}_n$.

By the symbol $\mathscr{M}_{m,n}$ the set of $(m\times n)$-matrices is denoted.

Definition 1.1. *The mappings*

$$\mathrm{vec}(\cdot)\colon \mathscr{M}_{m,n}\to\mathscr{R}^{mn};$$
$$\mathrm{vech}(\cdot)\colon \mathscr{S}_n\to\mathscr{R}^{n(n+1)/2};$$
$$(\mathrm{cR})[\mathrm{vec}(\cdot)]\colon \mathscr{S}_n\to\mathscr{R}^{n(n+1)/2}$$

are given by

$$\mathrm{vec}(\mathbf{T})=(t_{1,1}, t_{2,1}, \ldots, t_{m,1}; t_{1,2}, t_{2,2}, \ldots, t_{m,2}; \ldots; t_{1,n}, t_{2,n}, \ldots, t_{m,n})';$$
$$\mathrm{vech}(\mathbf{T})=(t_{1,1}, t_{2,1}, \ldots, t_{n,1}; t_{2,2}, t_{3,2}, \ldots, t_{n,2}; \ldots; t_{n-1,n-1}, t_{n,n-1}; t_{n,n})';$$
$$(\mathrm{cR})[\mathrm{vec}(\mathbf{T})]=(t_{1,1}, 2t_{2,1}, \ldots, 2t_{n,1}; t_{2,2}, 2t_{3,2}, \ldots, 2t_{n,2}; \ldots; t_{n-1,n-1}, 2t_{n,n-1}; t_{n,n}).$$

Here $t_{i,j}=\{\mathbf{T}\}_{i,j}$ *is the* (i,j)*-th element of the matrix* $\mathbf{T}$.

Lemma 1.1. *For arbitrary matrices* $\mathbf{A}\in\mathscr{M}_{m,n}$, $\mathbf{X}\in\mathscr{M}_{n,p}$, $\mathbf{B}\in\mathscr{M}_{p,s}$, $\mathbf{C}\in\mathscr{M}_{m,s}$ *it is true that* $\mathbf{AXB}=\mathbf{C}\Leftrightarrow(\mathbf{B}'\otimes\mathbf{A})\mathrm{vec}(\mathbf{X})=\mathrm{vec}(\mathbf{C})$ ($\otimes$ *denotes the tensor product*).

Proof is obvious.

Definition 1.2. *The mappings*

$$(\mathrm{cC})(\cdot)\colon \{\mathbf{B}'\otimes\mathbf{A}\colon \mathbf{A}, \mathbf{B}'\in\mathscr{M}_{p,r}\}\to\mathscr{M}_{p^2, r(r+1)/2};$$
$$(\mathrm{cR})(\cdot)\colon \{\mathbf{B}'\otimes\mathbf{A}\colon \mathbf{A}, \mathbf{B}'\in\mathscr{M}_{p,r}\}\to\mathscr{M}_{p(p+1)/2, r^2}$$

are given by

$$\{(\mathrm{cC})[\mathbf{B}'\otimes\mathbf{A}]\}_{\cdot[ir+(1+i)(2-i)/2]}=\{\mathbf{B}'\otimes\mathbf{A}\}_{\cdot(ir+i+1)},\quad i=0, 1, \ldots, r-1;$$

$$\{(\mathrm{cC})[\mathbf{B}'\otimes\mathbf{A}]\}_{\cdot[ir+i+j-i(i+1)/2]}=\{\mathbf{B}'\otimes\mathbf{A}\}_{\cdot(ir+i+j)}+\{\mathbf{B}'\otimes\mathbf{A}\}_{\cdot(i+j-1)r+i+1},$$
$$i=0, 1, \ldots, r-2;\ j=2, 3, \ldots, r-i$$

and

$$\{(\mathrm{cR})[\mathbf{B}'\otimes\mathbf{A}]\}_{[ip+(1+i)(2-i)/2]\cdot}=\{\mathbf{B}'\otimes\mathbf{A}\}_{(ip+i+1)\cdot},\quad i=0, 1, \ldots, p-1,$$

$$\{(\mathrm{cR})[\mathbf{B}'\otimes\mathbf{A}]\}_{[ip+i+j-i(i+1)/2]\cdot}=\{\mathbf{B}'\otimes\mathbf{A}\}_{(ip+i+j)\cdot}+\{\mathbf{B}'\otimes\mathbf{A}\}_{[(i+j-1)p+i+1]\cdot},$$
$$i=0, 1, \ldots, p-2;\ j=2, 3, \ldots, p-i.$$

Here $\{\mathbf{M}\}_{\cdot j}$ *and* $\{\mathbf{M}\}_{i\cdot}$ *denote the* j*-th column and the* i*-th row of the matrix* $\mathbf{M}$.

Corollary 1.1. *For arbitrary matrices* $\mathbf{A}, \mathbf{B}' \in \mathcal{M}_{p,t}$, $\mathbf{X} \in \mathcal{S}_t$, $\mathbf{C} \in \mathcal{S}_p$ *it is true that* $\mathbf{AXB} = \mathbf{C} \Leftrightarrow (\mathbf{B}' \otimes \mathbf{A})\mathrm{vec}(\mathbf{X}) = \mathrm{vec}(\mathbf{C}) \Leftrightarrow (\mathrm{cR})[\mathrm{vec}(\mathbf{C})] = (\mathrm{cR})(\mathrm{cC})[\mathbf{B}' \otimes \mathbf{A}]\mathrm{vech}(\mathbf{X})$.

Lemma 1.2. *The estimator* $\mathrm{Tr}(\mathbf{AS})$ *of the function* $g(\mathbf{C}) = \mathrm{Tr}(\mathbf{MC})$, $\mathbf{C} \in \mathscr{C}$ *is unbiased iff* $\sum_{i=1}^{m} \mathbf{J}_i'\mathbf{A}\mathbf{J}_i = \mathbf{M}$.

Proof. It is a consequence of the relation

$$E_{\mathbf{C}}[\mathrm{Tr}(\mathbf{AS})] = \mathrm{Tr}(\mathbf{A\Sigma}) = \mathrm{Tr}\left(\sum_{i=1}^{m} \mathbf{J}_i'\mathbf{A}\mathbf{J}_i\mathbf{C}\right),$$

$\mathbf{C} \in \mathscr{C}$ and of the assumption (*).

Lemma 1.3. *The function* $g(\mathbf{C}) = \mathrm{Tr}(\mathbf{MC})$, $\mathbf{C} \in \mathscr{C}$ *is unbiasedly estimable iff*

$$(\mathrm{cR})[\mathrm{vec}(\mathbf{M})] \in \mathcal{M}\left\{(\mathrm{cR})(\mathrm{cC})\left|\sum_{i=1}^{m} \mathbf{J}_i' \otimes \mathbf{J}_i\right|\right\}$$

($\mathcal{M}(\mathbf{D})$ *denotes the column space of the matrix* $\mathbf{D}$).

Proof. It is a consequence of Lemma 1.2, Lemma 1.1 and Corollary 1.1.

2. Natural estimation and γ-estimation

Let the error vector $\boldsymbol{\varepsilon}$ be of the form $\boldsymbol{\varepsilon} = \mathbf{J}_1\xi_1 + \ldots + \mathbf{J}_m\xi_m$, $\xi_j \sim N_s(\mathbf{0}, \mathbf{C})$, $j = 1, \ldots, m$, where $\mathbf{C}$ is a positive definite matrix and vectors ξ_j, $j = 1, \ldots, m$ are stochastically independent. As $k\mathbf{S} \sim W_n(k, \Sigma)$, then $k\mathbf{S} = \sum_{\alpha=1}^{k} \mathbf{Z}_\alpha \mathbf{Z}_\alpha'$, $\mathbf{Z}_\alpha \sim N(\mathbf{0}, \Sigma)$, $\alpha = 1, \ldots, k$ and $\mathbf{Z}_\alpha$, $\alpha = 1, \ldots, k$ are stochastically independent [1]. Similarly as in [2] the vector $\mathbf{Z}_\alpha$ can be expressed in the form $\mathbf{Z}_\alpha = \mathbf{J}_1\xi_{\alpha,1} + \ldots + \mathbf{J}_m\xi_{\alpha,m}$, $\alpha = 1, \ldots, k$, $\xi_{\alpha,j} \sim N_s(\mathbf{0}, \mathbf{C})$ and $\xi_{\alpha,j}$, $\alpha = 1, \ldots, k$; $j = 1, \ldots, m$ are stochastically independent.

The natural estimator $\hat{\mathbf{C}}$ of the matrix $\mathbf{C}$ based on the realization of the vectors $\xi_{\alpha,j}$, $\alpha = 1, \ldots, k$, $j = 1, \ldots, m$ (see also the corollary 3.1) is

$$\hat{\mathbf{C}} = [1/(mk)] \sum_{\alpha=1}^{k} \sum_{i=1}^{m} \xi_{\alpha,i}\xi_{\alpha,i}'$$

and the estimator of the function $g(\cdot)$ is then $\mathrm{Tr}(\mathbf{M}\hat{\mathbf{C}})$. The difference between the unbiased estimator $\tau_g(\mathbf{S}) = \mathrm{Tr}(\mathbf{AS})$ and the natural estimator $\mathrm{Tr}(\mathbf{M}\hat{\mathbf{C}})$ is

$$\mathrm{Tr}(\mathbf{AS}) - \mathrm{Tr}(\mathbf{M}\hat{\mathbf{C}}) = (1/k)\,\mathrm{Tr}\left\{[(1/m)(\mathbf{I} \otimes \mathbf{M}) - \mathbf{J}'\mathbf{A}\mathbf{J}] \sum_{\alpha=1}^{k} \xi_\alpha \xi_\alpha'\right\},$$

where $\mathbf{J} = (\mathbf{J}_1, \mathbf{J}_2, \ldots, \mathbf{J}_m)$ and $\xi_\alpha' = (\xi_{\alpha,1}', \ldots, \xi_{\alpha,m}')$.

Definition 2.1. *The estimator* $\mathrm{Tr}(\mathbf{AS})$ *of the function* $g(\mathbf{C}) = \mathrm{Tr}(\mathbf{MC})$, $\mathbf{C} \in \mathscr{C}$ *is the MINUE if*

$$\mathbf{J}_1'\mathbf{A}\mathbf{J}_1 + \ldots + \mathbf{J}_m'\mathbf{A}\mathbf{J}_m = \mathbf{M} \quad \textit{and} \quad \mathrm{Tr}\{[(1/m)(\mathbf{I}\otimes\mathbf{M}) - \mathbf{J}'\mathbf{A}\mathbf{J}]^2\} = \min.$$

Theorem 2.1. *The MINUE of the function* $g(\mathbf{C}) = \mathrm{Tr}(\mathbf{MC})$, $\mathbf{C}\in\mathscr{C}$ *is*

$$\tau_g(\mathbf{S}) = \mathrm{Tr}\left(\sum_{i=1}^{m}\mathbf{V}^{-1}\mathbf{J}_i\Lambda\mathbf{J}_i'\mathbf{V}^{-1}\mathbf{S}\right),$$

where $\mathbf{V} = \mathbf{J}_1\mathbf{J}_1' + \ldots + \mathbf{J}_m\mathbf{J}_m'$ *and* $\Lambda\in\mathscr{S}_s$ *is a matrix of Lagrange multipliers which satisfies the equation*

$$(\mathrm{cR})[\mathrm{vec}(\mathbf{M})] = (\mathrm{cR})(\mathrm{cC})\left[\sum_{i=1}^{m}\sum_{j=1}^{m}(\mathbf{J}_i'\mathbf{V}^{-1}\mathbf{J}_j)\otimes(\mathbf{J}_i'\mathbf{V}^{-1}\mathbf{J}_j)\right]\mathrm{vech}(\Lambda).$$

Proof. As $-2\mathrm{Tr}\{(1/m)(\mathbf{I}\otimes\mathbf{M})\mathbf{J}'\mathbf{A}\mathbf{J}\} = -2(1/m)\,\mathrm{Tr}\left(\mathbf{M}\sum_{i=1}^{m}\mathbf{J}_i'\mathbf{A}\mathbf{J}_i\right) = -(2/m)\,\mathrm{Tr}(\mathbf{M}^2)$, then $\mathrm{Tr}\{[(1/m)(\mathbf{I}\otimes\mathbf{M}) - \mathbf{J}'\mathbf{A}\mathbf{J}]^2\} = \mathrm{Tr}(\mathbf{AVAV}) - (1/m)\,\mathrm{Tr}(\mathbf{M}^2)$. Thus it is sufficient to minimize $\mathrm{Tr}(\mathbf{AVAV})$ under the side condition $\mathbf{J}_1'\mathbf{A}\mathbf{J}_1 + \ldots + \mathbf{J}_m'\mathbf{A}\mathbf{J}_m = \mathbf{M}$. The method of Lagrange multipliers is used. The auxiliary function is $\phi(\mathbf{A}) = \mathrm{Tr}(\mathbf{AVAV}) - 2\mathrm{Tr}[\varkappa'(\mathbf{J}_1'\mathbf{A}\mathbf{J}_1 + \ldots + \mathbf{J}_m'\mathbf{A}\mathbf{J}_m - \mathbf{M})]$, where $\varkappa'$ is a matrix of Lagrange multipliers.

$$\left(\frac{\partial\phi(\mathbf{A})}{\partial\mathbf{A}}\right) = 4\mathbf{VAV} - 4\sum_{i=1}^{m}\mathbf{J}_i(1/2)(\varkappa+\varkappa')\mathbf{J}_i'$$

$$-2\,\mathrm{diag}\cdot\left\{\mathbf{VAV} - \sum_{i=1}^{m}\mathbf{J}_i(1/2)(\varkappa+\varkappa')\mathbf{J}_i'\right\} = \mathbf{0} \Leftrightarrow \mathbf{VAV} = \sum_{i=1}^{m}\mathbf{J}_i\Lambda\mathbf{J}_i',$$

where $\Lambda = (1/2)(\varkappa+\varkappa')$. For each matrix $\mathbf{D}\in\mathscr{S}_n$ satisfying the condition $\mathbf{J}_1'\mathbf{D}\mathbf{J}_1 + \ldots + \mathbf{J}_m'\mathbf{D}\mathbf{J}_m = \mathbf{0}$ there holds

$$\mathrm{Tr}(\mathbf{DVAV}) = \mathrm{Tr}\left(\sum_{i=1}^{m}\mathbf{J}_i'\mathbf{D}\mathbf{J}_i\Lambda\right) = 0$$

and thus

$$\mathrm{Tr}[(\mathbf{A}+\mathbf{D})\mathbf{V}(\mathbf{A}+\mathbf{D})\mathbf{V}] = \mathrm{Tr}(\mathbf{AVAV}) + \mathrm{Tr}(\mathbf{DVDV}) \geq \mathrm{Tr}(\mathbf{AVAV})$$

because of $\mathrm{Tr}(\mathbf{DVDV}) = \mathrm{Tr}(\mathbf{J}'\mathbf{DJJ}'\mathbf{DJ}) \geq 0$. Therefore the matrix

$$\mathbf{A} = \sum_{j=1}^{m}\mathbf{V}^{-1}\mathbf{J}_j\Lambda\mathbf{J}_j'\mathbf{V}^{-1}$$

with Λ satisfying

$$\sum_{i=1}^{m}\sum_{j=1}^{m}\mathbf{J}_i'\mathbf{V}^{-1}\mathbf{J}_j\Lambda\mathbf{J}_j'\mathbf{V}^{-1}\mathbf{J}_i = \mathbf{M}$$

(unbiasedness) minimizes the quantity $\mathrm{Tr}(\mathbf{AVAV})$ under the side condition $\mathbf{J}_1'\mathbf{A}\mathbf{J}_1 + \ldots + \mathbf{J}_m'\mathbf{A}\mathbf{J}_m = \mathbf{M}$. The rest of the proof is a consequence of Corollary 1.1.

In the following the elements of the matrix $\mathbf{C}$ are assumed to be approximately known. The corresponding matrix of approximate values is denoted by γ and it is assumed to be positive definite; $\gamma=\gamma^{1/2}\gamma'^{1/2}$. The following denotation is used

$$\mathbf{J}_i^{(\gamma)}=\mathbf{J}_i\gamma^{1/2}, \quad i=1,\dots,m;$$
$$\xi_{\alpha,j}^{(\gamma)}=\gamma^{-1/2}\xi_{\alpha,j}, \quad j=1,\dots,m; \quad \alpha=1,\dots,k$$

(obviously $\xi_{\alpha,j}^{(\gamma)}\sim N_s(\boldsymbol{0},\gamma^{-1/2}\mathbf{C}\gamma'^{-1/2})$);

$$\xi^{(\gamma)'}=(\xi_{\alpha,1}^{(\gamma)'},\dots,\xi_{\alpha,m}^{(\gamma)'});$$
$$\mathbf{M}^{(\gamma)}=\gamma^{1/2}\mathbf{M}\gamma'^{1/2};$$
$$\mathbf{C}^{(\gamma)}=\gamma'^{-1/2}\mathbf{C}\gamma^{-1/2};$$
$$\mathbf{J}^{(\gamma)}=(\mathbf{J}_1^{(\gamma)},\dots,\mathbf{J}_m^{(\gamma)}).$$

The natural estimator of the matrix $\mathbf{C}^{(\gamma)}$ based on $\xi_{\alpha,j}^{(\gamma)}$, $j=1,\dots,m$; $\alpha=1,\dots,k$, is

$$\hat{\mathbf{C}}^{(\gamma)}=[1/(km)]\sum_{\alpha=1}^{k}\sum_{j=1}^{m}\xi_{\alpha,j}^{(\gamma)}\xi_{\alpha,j}^{(\gamma)'}$$

and the estimator of the function $g(\mathbf{C})=\operatorname{Tr}(\mathbf{MC})=\operatorname{Tr}(\mathbf{M}^{(\gamma)}\mathbf{C}^{(\gamma)})$ resulting from it is

$$\operatorname{Tr}(\mathbf{M}^{(\gamma)}\hat{\mathbf{C}}^{(\gamma)})=\operatorname{Tr}\left[(1/m)(\mathbf{I}\otimes\mathbf{M}^{(\gamma)})(1/k)\sum_{\alpha=1}^{k}\xi_{\alpha}^{(\gamma)}\xi_{\alpha}^{(\gamma)'}\right].$$

The difference between the unbiased estimator $\operatorname{Tr}(\mathbf{AS})$ and the natural γ-estimator $\operatorname{Tr}(\mathbf{M}^{(\gamma)}\hat{\mathbf{C}}^{(\gamma)})$ is

$$\operatorname{Tr}(\mathbf{AS})-\operatorname{Tr}(\mathbf{M}^{(\gamma)}\hat{\mathbf{C}}^{(\gamma)})=$$
$$=\operatorname{Tr}\left\{[\mathbf{J}^{(\gamma)'}\mathbf{A}\mathbf{J}^{(\gamma)}-(1/m)(\mathbf{I}\otimes\mathbf{M}^{(\gamma)})]\sum_{\alpha=1}^{k}\xi_{\alpha}^{(\gamma)}\xi_{\alpha}^{(\gamma)'}\right\}.$$

Definition 2.2. *The estimator* $\tau_g(\mathbf{S})=\operatorname{Tr}(\mathbf{AS})$ *of the function* $g(\mathbf{C})=\operatorname{Tr}(\mathbf{MC})$, $\mathbf{C}\in\mathscr{C}$ *is the MINUγE if*

$$\mathbf{J}_1'\mathbf{A}\mathbf{J}_1+\dots+\mathbf{J}_m'\mathbf{A}\mathbf{J}_m=\mathbf{M} \quad \textit{and} \quad \operatorname{Tr}\{[\mathbf{J}^{(\gamma)'}\mathbf{A}\mathbf{J}^{(\gamma)}-(1/m)(\mathbf{I}\otimes\mathbf{M}^{(\gamma)}]^2\}=\min.$$

Theorem 2.2. *The MINUγE of the function* $g(\mathbf{C})=\operatorname{Tr}(\mathbf{MC})$, $\mathbf{C}\in\mathscr{C}$, *is*

$$\tau_g(\mathbf{S})=\operatorname{Tr}\left(\sum_{i=1}^{m}\mathbf{V}^{(\gamma)^{-1}}\mathbf{J}_i\mathbf{\Lambda}^{(\gamma)}\mathbf{J}_i'\mathbf{V}^{(\gamma)^{-1}}\mathbf{S}\right),$$

where $\mathbf{V}^{(\gamma)}=\sum_{i=1}^{m}\mathbf{J}_i\gamma\mathbf{J}_i'$. *The matrix* $\mathbf{\Lambda}^{(\gamma)}\in\mathscr{S}_s$ *is a solution of the matrix equation*

$$\mathbf{M}=\sum_{i=1}^{m}\sum_{j=1}^{m}\mathbf{J}_i'\mathbf{V}^{(\gamma)^{-1}}\mathbf{J}_j\mathbf{\Lambda}^{(\gamma)}\mathbf{J}_j'\mathbf{V}^{(\gamma)^{-1}}\mathbf{J}_i.$$

This equation can be expressed in the form

$$(**)\quad (\mathrm{cR})[\mathrm{vec}(\mathbf{M})]=(\mathrm{cR})(\mathrm{cC})\left[\sum_{i=1}^{m}\sum_{j=1}^{m}(\mathbf{J}_i'\mathbf{V}^{(\gamma)-1}\mathbf{J}_j)\otimes(\mathbf{J}_i'\mathbf{V}^{(\gamma)-1}\mathbf{J}_j)\right]\mathrm{vech}(\mathbf{\Lambda}^{(\gamma)}).$$

The proof is analogous to the proof of Theorem 2.1.

3. Properties of the MINUγE

Theorem 3.1. *The MINUγE of the function $g(\mathbf{C})=\mathrm{Tr}(\mathbf{MC})$, $\mathbf{C}\in\mathscr{C}$ is the locally best estimator in $\gamma\in\mathscr{C}$.*

Proof. With respect to Lemma 1.4 [2] (the denotation $\mathbf{V}^{(\gamma)}=\mathbf{\Sigma}$ is used) we have

$$\mathrm{cov}\left\{\mathrm{Tr}\left(\sum_{i=1}^{m}\mathbf{\Sigma}^{-1}\mathbf{J}_i\mathbf{\Lambda}^{(\gamma)}\mathbf{J}_i'\mathbf{\Sigma}^{-1}\mathbf{S}\right),\ \mathrm{Tr}(\mathbf{A}_0\mathbf{S})\right\}=(2/k)\ \mathrm{Tr}\left(\sum_{i=1}^{m}\mathbf{J}_i'\mathbf{A}_0\mathbf{J}_i\mathbf{\Lambda}^{(\gamma)}\right).$$

The last expression is zero for each matrix $\mathbf{A}_0\in\mathscr{S}_n$ satisfying the condition

$$\forall\{\gamma\in\mathscr{C}\}\,E_\gamma[\mathrm{Tr}(\mathbf{A}_0\mathbf{S})]=0\left(\Leftrightarrow\forall\{\gamma\in\mathscr{C}\}\ \mathrm{Tr}\left(\sum_{i=1}^{m}\mathbf{J}_i'\mathbf{A}_0\mathbf{J}_i\gamma\right)=0\right).$$

With respect to the assumption $(*)$ this condition is equivalent with $\sum_{i=1}^{m}\mathbf{J}_i'\mathbf{A}_0\mathbf{J}_i=\mathbf{0}$. On the basis of the Lehmann—Scheffé theorem (see also Lemma 1.5 [2]) the statement is immediately proved.

Remark 3.1. The matrix $\mathbf{A}$ from the MINUγE minimizes the quantity $\mathrm{Tr}(\mathbf{AV}^{(\gamma)}\mathbf{AV}^{(\gamma)})$ under the side condition of the unbiasedness $\sum_{i=1}^{m}\mathbf{J}_i'\mathbf{AJ}_i=\mathbf{M}$. If $\mathbf{C}=\gamma$, then $\mathbf{V}^{(\gamma)}$ is the covariance matrix of the vector $\boldsymbol{Y}$ and regarding Lemma 1.4 [2] $(2/k)\mathrm{Tr}(\mathbf{AV}^{(\gamma)}\mathbf{AV}^{(\gamma)}$ is dispersion of the statistic $\mathrm{Tr}(\mathbf{AS})$.

Lemma 3.1. *The Fisher information matrix of the distribution of the matrix*

$$k\mathbf{S}\sim W_n\left(k,\ \mathbf{\Sigma}=\sum_{i=1}^{m}\mathbf{J}_i\mathbf{C}\mathbf{J}_i'\right)$$

with respect to the parameter $\mathrm{vech}(\mathbf{C})$ *is*

$$\mathbf{F}(\mathbf{C})=(k/2)(\mathrm{cR})(\mathrm{cC})\left[\sum_{i=1}^{m}\sum_{j=1}^{m}(\mathbf{J}_i'\mathbf{\Sigma}^{-1}\mathbf{J}_j)\otimes(\mathbf{J}_i'\mathbf{\Sigma}^{-1}\mathbf{J}_j)\right].$$

Proof. The probability density function of the matrix $\mathbf{S}$ is

$$f(\mathbf{S},\mathbf{C})=(k/2)^{kn/2}\pi^{-n(n-1)/2}\left\{\prod_{j=1}^{k}[(1/2)(k+1-j)]\right\}^{-1}\det(\mathbf{S})\cdot$$

$$\cdot\exp\{-(k/2)\ \mathrm{Tr}(\mathbf{\Sigma}^{-1}\mathbf{S})\}[\det(\mathbf{\Sigma})]^{-k/2},$$

where $\boldsymbol{\Sigma}=\sum_{i=1}^{m}\mathbf{J}_i\mathbf{C}\mathbf{J}_i'$. If in the following $\{\mathbf{C}\}_{i,j}=c_{i,j}$ and the relations

$$\partial\boldsymbol{\Sigma}^{-1}/\partial c_{i,j}=-\boldsymbol{\Sigma}^{-1}(\partial\boldsymbol{\Sigma}/\partial c_{i,j})\boldsymbol{\Sigma}^{-1},$$
$$\partial\ln\det(\boldsymbol{\Sigma})/\partial c_{i,j}=\mathrm{Tr}(\boldsymbol{\Sigma}^{-1}\partial\boldsymbol{\Sigma}/\partial c_{i,j})$$

and

$$\partial\boldsymbol{\Sigma}/\partial c_{i,j}=\begin{cases}\sum_{r=1}^{m}\{\mathbf{J}_r\}_{\cdot i}\{\mathbf{J}_r\}_{\cdot i}' & \text{for } i=j,\\ \sum_{r=1}^{m}[\{\mathbf{J}_r\}_{\cdot i}\{\mathbf{J}_r\}_{\cdot j}'+\{\mathbf{J}_r\}_{\cdot j}\{\mathbf{J}_r\}_{\cdot i}'] & \text{for } i\neq j\end{cases}$$

are used, then

$$\partial\ln f(\mathbf{S},\mathbf{C})/\partial c_{i,j}=$$

$$=\begin{cases}(k/2)\sum_{r=1}^{m}\{\mathbf{J}_r\}_{\cdot i}'\boldsymbol{\Sigma}^{-1}\mathbf{S}\boldsymbol{\Sigma}^{-1}\{\mathbf{J}_r\}_{\cdot i}-(k/2)\sum_{r=1}^{m}\{\mathbf{J}_r\}_{\cdot i}'\boldsymbol{\Sigma}^{-1}\{\mathbf{J}_r\}_{\cdot i}, & i=j,\\ 2\left\langle(k/2)\sum_{r=1}^{m}\{\mathbf{J}_r\}_{\cdot i}'\boldsymbol{\Sigma}^{-1}\mathbf{S}\boldsymbol{\Sigma}^{-1}\{\mathbf{J}_r\}_{\cdot j}-(k/2)\sum_{r=1}^{m}\{\mathbf{J}_r\}_{\cdot i}'\boldsymbol{\Sigma}^{-1}\{\mathbf{J}_r\}_{\cdot j}\right\rangle, & i\neq j.\end{cases}$$

If in the same way the second derivatives are determined and $E(\mathbf{S})=\boldsymbol{\Sigma}$ is used, then

$$E(-\partial^2\ln f(\mathbf{S},\mathbf{C})/\partial c_{i,i}\partial c_{r,r})=(k/2)\sum_{p=1}^{m}\sum_{t=1}^{m}\langle\{\mathbf{J}_t\}_{\cdot r}'\boldsymbol{\Sigma}^{-1}\{\mathbf{J}_p\}_{\cdot i}\rangle^2;$$

$$E(-\partial^2\ln f(\mathbf{S},\mathbf{C})/\partial c_{i,i}\partial c_{r,q})=$$

$$=2\left\langle(k/2)\sum_{p=1}^{m}\sum_{t=1}^{m}\{\mathbf{J}_p\}_{\cdot i}'\boldsymbol{\Sigma}^{-1}\{\mathbf{J}_t\}_{\cdot q}\{\mathbf{J}_p\}_{\cdot i}'\boldsymbol{\Sigma}^{-1}\{\mathbf{J}_t\}_{\cdot r}\right\rangle,\quad r\neq q;$$

$$E(-\partial^2\ln f(\mathbf{A},\mathbf{C})/\partial c_{i,j}\partial c_{r,q})=2\left\langle(k/2)\sum_{p=1}^{m}\sum_{t=1}^{m}[\{\mathbf{J}_t\}_{\cdot r}'\boldsymbol{\Sigma}^{-1}\{\mathbf{J}_p\}_{\cdot i}\cdot\right.$$
$$\left.\cdot\{\mathbf{J}_p\}_{\cdot i}'\boldsymbol{\Sigma}^{-1}\{\mathbf{J}_t\}_{\cdot q}+\{\mathbf{J}_t\}_{\cdot r}'\boldsymbol{\Sigma}^{-1}\{\mathbf{J}_p\}_{\cdot i}\{\mathbf{J}_p\}_{\cdot j}'\boldsymbol{\Sigma}^{-1}\{\mathbf{J}_t\}_{\cdot q}]\right\rangle,\quad i\neq j,\ r\neq s.$$

The last three relations imply the statement.

Theorem 3.2. *The dispersion of the MINUγE of the function $g(\mathbf{C})=\mathrm{Tr}(\mathbf{MC})$, $\mathbf{C}\in\mathscr{C}$, attains in $\mathbf{C}=\boldsymbol{\gamma}$ the Rao—Cramér lower bound.*

Proof. With respect to Lemma 1.4 [2] the dispersion of the MINUγE is $\mathscr{D}_\gamma[\mathrm{Tr}(\mathbf{AS})]=$

$$=(2/k)\,\mathrm{Tr}\left(\sum_{i=1}^{m}\boldsymbol{\Sigma}^{-1}\mathbf{J}_i\boldsymbol{\Lambda}^{(\gamma)}\mathbf{J}_i'\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}\sum_{j=1}^{m}\boldsymbol{\Sigma}^{-1}\mathbf{J}_j\boldsymbol{\Lambda}^{(\gamma)}\mathbf{J}_j'\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}\right)=$$

$$=(2/k)\,\mathrm{Tr}\left(\sum_{i=1}^{m}\sum_{j=1}^{m}\mathbf{J}_j'\boldsymbol{\Sigma}^{-1}\mathbf{J}_i\boldsymbol{\Lambda}^{(\gamma)}\mathbf{J}_i'\boldsymbol{\Sigma}^{-1}\mathbf{J}_j\boldsymbol{\Lambda}^{(\gamma)}\right)=$$

$$= (2/k)\left[\operatorname{vec}\left\langle \sum_{i=1}^{m}\sum_{j=1}^{m} \mathbf{J}_i'\mathbf{\Sigma}^{-1}\mathbf{J}_j\mathbf{\Lambda}^{(\gamma)}\mathbf{J}_j'\mathbf{\Sigma}^{-1}\mathbf{J}_i \right\rangle\right]' \operatorname{vec}(\mathbf{\Lambda}^{(\gamma)}) =$$

$$= (2/k)\left\{(\mathrm{cR})(\mathrm{cC})\left[\sum_{i=1}^{m}\sum_{j=1}^{m}(\mathbf{J}_j'\mathbf{\Sigma}^{-1}\mathbf{J}_i)\otimes(\mathbf{J}_i'\mathbf{\Sigma}^{-1}\mathbf{J}_j)\right]\operatorname{vec}(\mathbf{\Lambda}^{(\gamma)})\right\}'\operatorname{vec}(\mathbf{\Lambda}^{(\gamma)}).$$

Regarding Lemma 3.1, the last expression can be rewritten as $(2/k)^2\{\operatorname{vech}(\mathbf{\Lambda}^{(\gamma)})\}'\mathbf{F}(\gamma)\operatorname{vech}(\mathbf{\Lambda}^{(\gamma)})$ and regarding $(**)$ in Theorem 2.2 it can again be rewritten as $\{(\mathrm{cR})[\operatorname{vec}(\mathbf{M})]\}'\mathbf{F}^{-1}(\gamma)(\mathrm{cR})\operatorname{vec}(\mathbf{M})$. This is the Rao—Cramér lower bound of dispersions of unbiased estimators of the function $g(\mathbf{C}) = \{(\mathrm{cR})[\operatorname{vec}(\mathbf{M})]\}'\operatorname{vech}(\mathbf{C}) = \operatorname{Tr}(\mathbf{MC})$, $\mathbf{C} \in \mathscr{C}$ for the value γ of the parameter $\mathbf{C}$.

Corollary 3.1. *If $m = 1$, $\mathbf{J}_1 = \mathbf{I}$ and $g(\mathbf{C}) = \operatorname{Tr}(\mathbf{MC}) = c_{i,j} = \{\mathbf{\Sigma}\}_{i,j}$, $i, j = 1, \ldots, n$, then regarding Theorem 2.2 the MINUγE is*

$$\tau_g(\mathbf{S}) = \operatorname{Tr}(\mathbf{V})^{(\gamma)-1}\mathbf{\Lambda}^{(\gamma)}\mathbf{V}^{(\gamma)-1}\mathbf{S} \quad \textit{and} \quad \mathbf{M} = \mathbf{V}^{(\gamma)-1}\mathbf{\Lambda}^{(\gamma)}\mathbf{V}^{(\gamma)-1}.$$

Thus $\tau_g(\mathbf{S}) = \{\mathbf{S}\}_{i,j}$. This estimator does not depend on γ and attains the Rao—Cramér lower bound in its dispersion. Thus it is uniformly best.

Corollary 3.2. *If $s = 1$, i.e. $\mathbf{\Sigma} = c\mathbf{V} = c\sum_{i=1}^{n}\mathbf{J}_i\mathbf{J}_i'$, where $(\mathbf{J}_1, \ldots, \mathbf{J}_n) = \mathbf{V}^{1/2}$ and $g(c) = c$, $c \in (0, \infty)$, then by Theorem 2.2 the MINUγE is $\tau_g(\mathbf{S}) = \operatorname{Tr}[(1/n)\mathbf{V}^{-1}\mathbf{S}]$ (the same result follows from Theorem 3.2 [2]). Its dispersion (see also Lemma 1.4 [2]) is $\mathscr{D}(\tau_g(\mathbf{S})) = [2/(kn)]c^2$ and by Theorem 3.2 this is identical with the lower Rao—Cramér bound. As $\tau_g(\mathbf{S})$ does not depend on γ, this estimator is the uniformly best one. It is necessary to remark that the distribution of the estimator $\operatorname{Tr}[(1/n)\mathbf{V}^{-1}\mathbf{S}]$ is identical with the distribution of the random variable $c\chi^2_{kn}/(kn)$, where χ^2_{kn} has the chi-square distribution with (kn) degrees of freedom.*

Remark 3.2. Similarly as in [2] the comparison of the estimator based on the realization of the vector $\boldsymbol{Y} \sim N_n(\mathbf{X}\boldsymbol{\beta}, \mathbf{\Sigma})$ with the estimator based on the realization of the matrix $k\mathbf{S} \sim W_n(k, \mathbf{\Sigma})$ (from a repeated regression experiment) can be done.

Let $\mathbf{F}(\boldsymbol{\beta}, \mathbf{C})$ be the Fisher information matrix of the distribution of the vector $\boldsymbol{Y}$ related to the parameter $(\boldsymbol{\beta}', [\operatorname{vech}(\mathbf{C})]')'$. Analogously to Lemma 3.1 we obtain

$$(***) \qquad \mathbf{F}(\boldsymbol{\beta}, \mathbf{C}) = \begin{bmatrix} \mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X}, & \mathbf{0} \\ \mathbf{0}, & (1/k)\mathbf{F}(\mathbf{C}) \end{bmatrix},$$

where $\mathbf{F}(\mathbf{C})$ is the matrix from Lemma 3.1. The unbiased estimator based on the realization of the vector $\boldsymbol{Y}$ and respecting the approximate values of the elements of the matrix γ is $\boldsymbol{Y}'\mathbf{A}_*\boldsymbol{Y}$. The matrix $\mathbf{A}_*$ minimizes the quantity $\operatorname{Tr}(\mathbf{AV}^{(\gamma)}\mathbf{AV}^{(\gamma)})$ under the side condition $\mathbf{X}'\mathbf{AX} = \mathbf{0}$ and $\sum_{i=1}^{m}\mathbf{J}_i'\mathbf{AJ}_i = \mathbf{M}$, respectively (see [3]). There is

$$\mathbf{A}_* = \sum_{j=1}^{m} \mathbf{V}^{(\gamma)-1}\mathbf{J}_j \Lambda^{(\gamma)} \mathbf{J}_j'^{(\gamma)-1} - \sum_{j=1}^{m} \mathbf{V}^{(\gamma)-1}\mathbf{X}(\mathbf{X}'\mathbf{V}^{(\gamma)-1}\mathbf{X})^{-}\mathbf{X}'\mathbf{V}^{(\gamma)-1} \cdot$$

$$\cdot \mathbf{J}_j \Lambda^{(\gamma)} \mathbf{J}_j' \mathbf{V}^{(\gamma)-1}\mathbf{X}(\mathbf{X}'\mathbf{V}^{(\gamma)-1}\mathbf{X})^{-}\mathbf{X}'\mathbf{V}^{(\gamma)-1},$$

where the matrix of the Lagrange multipliers $\Lambda^{(\gamma)}$ satisfies the matrix equation $\mathbf{M} = \sum_{j=1}^{m} \mathbf{J}_j' \mathbf{A}_* \mathbf{J}_j$.

If the second term of the right-hand side, the expression (∗∗∗), Theorem 3.2 and the expression (∗∗) are taken into account then it can be seen that the dispersion of $\boldsymbol{Y}'\mathbf{A}_*\boldsymbol{Y}$ cannot in general attain its Rao—Cramér lower bound. Therefore, if there exists a possibility to obtain the realization of the matrix $\mathbf{S}$ from results of a repeated regression experiment, then the estimator should be based on the matrix $\mathbf{S}$ instead the vector $\boldsymbol{Y}$ (see also Part 4 of [2]).

Example. Let $\boldsymbol{Y} \sim N_n(\mathbf{X}\beta, c\mathbf{V})$ (see corollary 3.2). Then the MINQUE (see [3]) of the parameter c is $\hat{c} = \boldsymbol{Y}'[\mathbf{V}^{-1} - \mathbf{V}^{-1}\mathbf{X}(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-}\mathbf{X}'\mathbf{V}^{-1}]\boldsymbol{Y}/[n - R(\mathbf{X})]$ and its dispersion is $\mathscr{D}(\hat{c}) = 2c^2/[n - R(\mathbf{X})]$. Repeating this experiment $(k+1)$-times we get $\mathscr{D}\{[1/(k+1)](\hat{c}_1 + \ldots + \hat{c}_{k+1}) = 2c^2/\{(k+1)[n - R(\mathbf{X})]\}$, while $\mathscr{D}\{\mathrm{Tr}[(1/n)\mathbf{V}^{-1}\mathbf{S}]\} = 2c^2/(kn)$ and this value is substantially smaller; e.g. for $n = 5$, $R(\mathbf{X}) = 2$, $k + 1 = 7$ we have $\mathscr{D}\{[1/(k+1)](\hat{c}_1 + \ldots + \hat{c}_{k+1})\}/\mathscr{D}\{\mathrm{Tr}[(1/n)\mathbf{V}^{-1}\mathbf{S}]\} = 1{,}43$.

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ОЦЕНКА КОВАРИАЦИОННЫХ КОМПОНЕНТ В ПОВТОРЕННОМ РЕГРЕССИОННОМ ЭКСПЕРИМЕНТЕ

Lubomír Kubáček

Резюме

Предложена несмещенная оценка минимальной нормы (MINUE) элементов матрицы **C**, которые названы ковариационными компонентами случайного вектора

$$\mathbf{Y} \sim N_n \left(\mathbf{X}\boldsymbol{\beta}, \Sigma = \sum_{i=1}^{m} \mathbf{J}_i \mathbf{C} \mathbf{J}_i' \right),$$

основанная на реализации матрицы

$$\mathbf{S} = (1/k) \sum_{i=1}^{k+1} (\mathbf{Y}_i - \bar{\mathbf{Y}})(\mathbf{Y}_i - \bar{\mathbf{Y}})'.$$

Исследованы некоторые статистические свойства MINUE.