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ON PARTIALLY DIRECTED P-GRAPHS

PAVOL HÍC

1. Introduction

P-graphs (undirected, directed or mixed) have been studied in several papers [1, 2, 3, 7]. Bosák [3] has proved that every loopless undirected graph G is a P-graph if and only if G is a T-graph. In [7] there was found a partially directed P-graph which is neither a quasitree, nor a graph similar to a T-graph. In the present paper we shall study directed and, more generally, partially directed P-graphs. We show that any partially directed P-graph without loops is either a quasitree or a homogeneous block with a finite diameter.

2. Notation and definitions

Throughout this paper all notions and notations not defined here will be used as in [3].

The graphs considered in this paper are directed or partially directed.

For a given graph G, $V(G)$ and $E(G)$ denote its vertex set and edge set, respectively.

Let v be a vertex of a partially directed graph G. Denote by id v [od v] the number of edges of G incident with v that are either directed to [from, respectively] v, or undirected. G is said to be a *homogeneous* graph of valency d if id $v =$ od $v = d$ for every vertex of G (where d is a cardinal number).

By a *semitrail* from u to v in a graph G we mean a finite sequence

$$Q = [v_0, e_1, v_1, \dots, e_n, v_n],$$

where n is a non-negative integer (called the *length* of Q); $v_0 = u$, $v_1, v_2, \dots, v_n = v$ are vertices of G; $e_1, e_2, \dots, e_n$ are mutually different edges of G (if $n \geq 1$) and for every $i \in \{1, 2, \dots, n\}$ v_{i-1} and v_i are the end vertices of e_i. If, moreover, every e_i is either undirected, or directed from v_{i-1} to v_i, then Q is said to be a *trail.* A semitrail [trail] whose vertices are mutually different is called a *semipath* [*path*]. A semitrail [trail] from u to v is said to be a *semicycle* [*cycle*] if its vertices are mutually different with the exception of $u = v$. A segment of Q between the vertices $v_i = x$ and $v_j = y (i \leq j)$ will be denoted by $Q[x, y]$. If P is a path from u to v and $w \in P$, then we shall write

$$P - P[u, w] + P[w, v].$$

A distance between vertices u and v of G is denoted by $\varrho_G(u, v)$ and it is the smallest length of a path from u to v, if any; otherwise we put $\varrho_G(u, v) = \infty$. A graph G is said to be *connected* [*strongly connected*] if for every ordered pair $[u, v]$ of vertices of G there exists a semipath [path] from u to v. The *diameter* of G is defined as $\sup \varrho_G(u, v)$, where the supremum is taken through all the ordered pairs $[u, v]$ of vertices of G.

A vertex v of a graph G is said to be a *cutpoint* of G if there exist two different edges e and f of G such that every semitrail containing e and f contains v between e and f. A maximal connected subgraph H of G containing no cutpoint of H is called a *block* of G.

If H is a subgraph of G, then $|E(H)|$ will denote a number of edges of H. If H is a path or a cycle, then $|E(H)|$ will denote its length.

Let $v \in V(G)$. Denote by $\Gamma^n(v)[\Gamma^{-n}(v)]$ the set of vertices $u \in V(G)$ such that there exists a trail from v to u [from u to v] of the length $n (\Gamma^1(v) = \Gamma(v))$. Thus, if G has no multiple edges, then $|\Gamma(v)| = \text{od } v$, $|\Gamma^{-1}(v)| = \text{id } v$ and $\Gamma^0(v) = \{v\}$.

3. Partially directed P-graphs

A partially directed graph G is said to be *P-graph* if for each ordered pair $[u, v]$ of vertices of G there exists in G exactly one path from u to v of a length not greater than the diameter of G. A P-graph, which is a block, is said to be a *P-block*.

A graph G is said to be a *quasitree* if fore each ordered pair $[u, v]$ of vertices of G there exists exactly one path from u to v.

Lemma 1 (Bosák [3, Theorem 6]). *A graph G is a quasitree if and only if G is connected and every block of G is isomorphic to K_2, C_1 or a directed cycle.*

A partially directed graph G is said to be a *T-graph* if for each ordered pair $[u, v]$ of vertices of G there exists in G exactly one trail from u to v of a length not greater than the diameter of G.

Let G be a partially directed graph. Denote by G^0 the loopless graph obtained from G by deleting all the loops of G. Denote by G^* the directed graph obtained from G by replacing each undirected odge by two oppositely directed edges.

Lemma 2 (Bosák [1, Lemma 1]).

(i) *A graph G is a P-graph if and only if the loopless graph G^0 is a P-graph.*

(ii) *A graph G is a P-graph if and only if the directed graph G^* is a P-graph.*

Lemma 2 enables us to restrict ourselves to directed and loopless P-graphs.

We obviously have (see Bosák [1, 3]) the following assertions (k is the diameter of G):

Proposition 1. *Let G be a P-graph. Then G has no two edges with the same initial and final vertices.*

Proposition 2. *Every quasitree is a P-graph.*

Proposition 3. *Every directed edge of a P-graph G lies in a directed cycle with $\leq k+1$ edges.*

Proposition 4. *No directed edge of a P-graph G can be contained in two cycles of length $\leq k+1$.*

Proposition 5. *Every block B of a directed P-graph G contains a cycle of length $\leq k+1$.*

Lemma 3. *Let G be a P-graph. Then for each $v \in V(G)$*

$$\mathrm{id}\, v = \mathrm{od}\, v\,.$$

Proof. By Propositions 3 and 4 every edge directed from v lies in exactly one directed cycle with $\leq k+1$ edges. In each of the cycles there is an edge directed to v so that $\mathrm{od}\, v \leq \mathrm{id}\, v$. Analogously, the inequality $\mathrm{id}\, v \leq \mathrm{od}\, v$ can be proved. Hence $\mathrm{id}\, v = \mathrm{od}\, v$.

Q.E.D.

Lemma 4. *Every loopless directed P-graph is either a quasitree or a block with a finite diameter.*

Proof. Let G be a loopless directed P-graph of diameter k. Distinguish three cases:

I. The diameter $k=1$. Then every P-graph is a complete graph and the assertion holds.

II. The diameter k of G is finite, $k \geq 2$. Let G have at least two blocks. Let B be a block and B' be a block which meets B at a cutpoint v. We shall prove that B is a directed cycle. Let w be a vertex of B' and $w \notin B$ (see Fig. 1). By Propositions 3 and 4 every directed edge in B lies in exactly one directed cycle C with $\leq k+1$ edges. We assert that $|E(C)| \leq k$ for every $C \subseteq B$. Let $|E(C)| = k+1$; then there exists a vertex $x \in C$ and a path P from v to x, $|E(P)| = k$, $P = P[v, y] + P[y, x]$ (see Fig. 1). Further, there exists a path P' from w to x, $P' = P'[w, v] + P'[v, x]$, $|E(P'[v, x])| < k$. Evidently, $P \neq P'[v, x]$ and this is a contradiction to the definition of a P-graph (there are two distinct paths from v to x of length $\leq k$). Hence $|E(C)| \leq k$.

Evidently, there is a cycle of B which contains v; let C be such a cycle. Suppose that there exists a vertex u of B that does not lie in C. As B is a block, there exists (see e.g. [6, Theorem 3.3]) a semipath

$$[u_0, e_1, u_1, \ldots, u_r = u, \ldots, e_s, u_s],$$

where $s \geq 2$, $1 \leq r \leq s-1$, u_0 and u_s are in C, $u_0 \neq u_s$ but $u_1, u_2, \ldots, u_{s-1}$ are not in

C. By the preceding, each e_i, $i \in \{1, 2, \ldots, s\}$ lies in a cycle C_i of B, where $|E(C_i)| \leq k$ (see Fig. 2).

Put $C_0 = C$. We obtained a sequence $S = \{C_0, C_1, \ldots, C_s\}$ of cycles. Now we shall prove that S has the following properties:

(i) *For every pair of adjacent cycles* C_i, $C_{i+1} \in S$ *either* $V(V_i) \cap V(C_{i+1}) = \{u_i\}$ *for* $i \in \{0, 1, \ldots, s-1\}$ *and* $V(C_s) \cap V(C_0) = \{u_s\}$, *or* $C_i = C_{i+1}$ *for* $i \in \{1, 2, \ldots, s-1\}$.

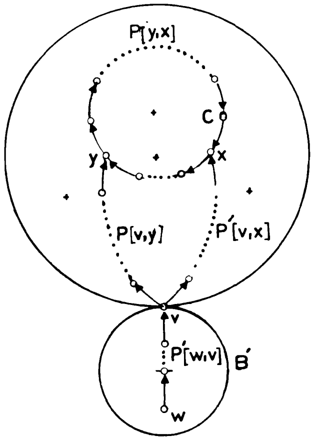

Fig 1

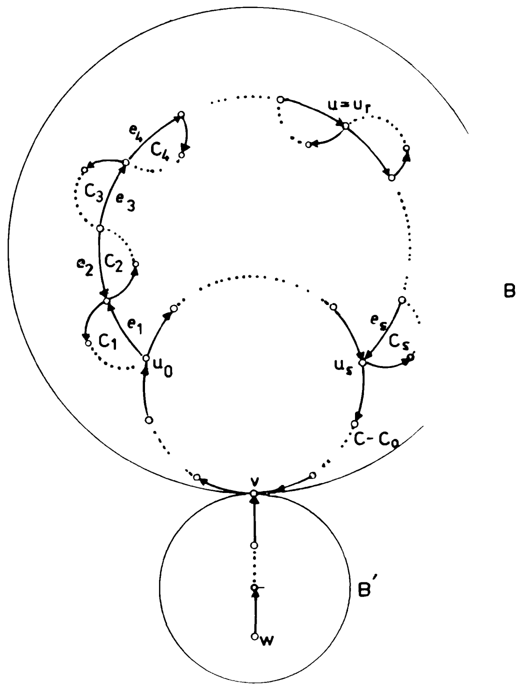

Fig 2

(ii) *For every pair of nonadjacent cycles* C_i, $C_j \in S$ *either* $V(C_i) \cap V(C_j) = \emptyset$ *for* $i \neq j$ *and* $i, j \in \{0, 1, \ldots, s\}$, *or* $C_i = C_j = C_k$ *for every* k *with* $i < k < j$.

Proof. (i) It is obvious that $V(C_0) \cap V(C_s) = \{u_s\}$ and $V(C_0) \cap V(C_1) = \{u_0\}$. Let $C_i \neq C_{i+1}$ and $V(C_i) \cap V(C_{i+1}) \supset \{u_i\}$. Let e_{i+1} be the first edge of the cycle C_{i+1} and x be the first vertex of C_{i+1}, which is in C_i too, and $x \neq u_i$. Then there exist distinct $u_i - x$ paths of length $\leq k$. One of them is contained in C_i and the other in C_{i+1}. This is a contradiction to the definition of a P-graph.

(ii) Obviously from $C_i - C_j$, $|i-j| > 1$ it follows that $C_i = C_j = C_k$ for every integer k with $i < k < j$. Otherwise, there is at least one edge, which is contained in two cycles of length $< k$. One cycle is $C_i - C_j$ and the other is

$$K = C_j[u_i, u_{j-1}] + C_{j-1}[u_{j-1}, u_{j-2}] + \ldots +$$
$$+ C_{i+1}[u_{i+1}, u_i].$$

This is a contradiction to the Proposition 4.

Now let $C_i \neq C_j$, $|i-j| = r > 1$, $V(C_i) \cap V(C_j) \neq \emptyset$ and r be the least number with this property. Let e_j be the first edge of the cycle C_j and y be the last vertex of C_j which is in C_i, too. Then there is at least one edge which is contained in two cycles of length $\leq k$. One cycle is C_j and the other is

$$K = C_j[y, u_{j-1}] + C_{j-1}[u_{j-1}, u_{j-2}] + \ldots + C_i[u_i, y].$$

This is a contradiction to the Proposition 4 and the assertions (i), (ii) are proved.

Now according to (i) and (ii) there are two paths from u_0 to u_s of the length $< k$ (see Fig. 2). The first path is inside C and the other is

$$P = C_1[u_0, u_1] + C_2[u_1, u_2] + \ldots + C_s[u_{s-1}, u_s].$$

The length of P must be $< k$ since otherwise a cycle

$$C^* = C[v, u_0] + P + C[u_s, v]$$

would have the length $> k$ and this is a contradiction, as for any cycle C in B, $|E(C)| \leq k$. But the existence of two such $u_0 - u_s$ paths is a contradiction to the definition of a P-graph. Hence B must be a cycle.

III. The diameter k is infinite. Let B be a block of G and C be a cycle contained in B. Suppose that there exists a vertex v of B that does not lie in C. As B is a block, there exists a semipath $[u_0, e_1, u_1, \ldots, u_r = v, \ldots, e_s, u_s]$, where $s \geq 2$, $1 \leq r \leq s-1$, u_0 and u_s are in C, but $u_1, u_2, \ldots, u_{s-1}$ are not in C. As in the case II we can construct two distinct paths from u_0 to u_s of length $\leq k$. However, this is a contradiction to the definition of a P-graph.

Q.E.D.

Lemma 5. *Let G be a directed P-block of diameter k. Then for every $u, v \in V(G)$ such that $\varrho_G(u, v) = k$ we have:*

$$\text{od } u = \text{id } v.$$

Proof. Let $[u, e_1, v_1, \ldots, e_k, v]$ be a path of length k from u to v. Let $y_1, y_2, \ldots, y_s$ be all the vertices of G different from v_1 such that there exists an edge directed from u to each of them. For $i \in \{1, 2, \ldots, s\}$ we have:

$$\varrho_G(y_i, v) = k$$

and no two of the corresponding paths of length k have a vertex different from v in common. Therefore od $u \leq$ id v. Analogously, considering the edges directed to v, the inequality od $u \geq$ id v can be obtained.

Q.E.D.

Now, we describe the structure of a P-graph. A similar structure for Moore graphs is decribed in [9].

Let G be a P-graph of a diameter k, and $w \in V(G)$. We define (see Fig. 3):

$$M_i(w) = \{v \mid \varrho_G(w, v) = i, v \in V(G)\} \text{ for } i \in \{0, 1, \ldots k\}.$$

Thus for any $w \in V(G)$ we have:

$$\sum_{i}^{k} |M_i(w)| = |V(G)|.$$

Denote $M_1(w) = \{a_1, a_2, \ldots, a_d\}$ and then.

$$A_i = \{v \mid \varrho_G(a_i, v) = k-1 \wedge w \notin P\} \text{ for } i = 1, 2, \ldots d.$$

(P is the shortest path from a_i to v.)

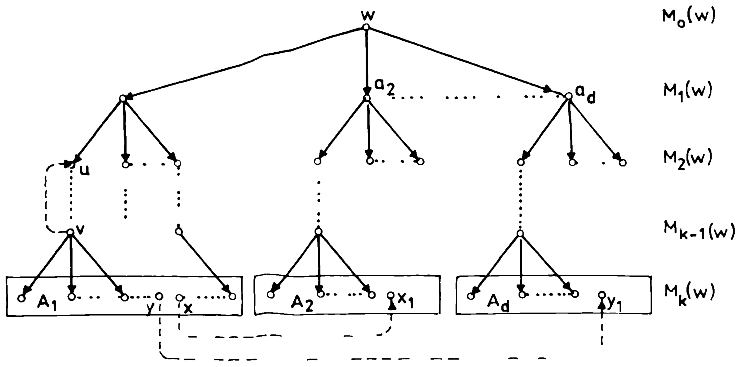

Fig. 3

We see that for every $i \in \{1, 2, \ldots, k-1\}$ and an arbitrary vertex $v \in M_i(w)$ there exists no edge directed from v to u, $u \in M_j(w)$, $j \in \{1, 2, \ldots, k\}$ with a possible exception $j < i$ and $u \in \Gamma^{i-j}(v)$ (see Fig. 3). In the other case we would have a contradiction to the definition of a P-graph. The example of a P-grahp G of diameter 3 and its structure in such a form are given in Fig 4.

Lemma 6. *Let G be a directed P block of diameter k; then for any $w \in V(G)$ we have:*

$$M_k(w) \neq \emptyset.$$

Proof. Let there be a vertex $w \in V(G)$ such that $M_k(w) = \emptyset$. Then $A_i = \emptyset$ for every $i \in \{1, 2, \ldots, d\}$. Let $k-r$ be the maximal index with $M_{k-r}(w) \neq \emptyset$. Denote $A'_i = \{v \mid \varrho_G(a_i, v) = k-r-1 \wedge w \notin P\}$ for $i \in \{1, 2, \ldots, d\}$. Obviously, $A'_i \subset M_{k-r}(w)$. Let us construct the structure of a graph G with $w \in M_k(w)$.

Then there exists no edge directed from a vertex $v \in A'_i$ to a vertex $u \in A'_j$ for $i \neq j$ (otherwise we have a contradiction to the definition of a P-graph), so there can only be an edge directed from $v \in A'_i$ to $u \in \Gamma^{-j}(v)$, $1 \leqslant j \leqslant k-r-1$. Then w is a cutpoint and this is a contradiction to the definition of a block.

Q.E.D.

Lemma 7. *Let G be a directed P-block of diameter k. Then G is either a directed cycle or for any vertex $w \in V(G)$ we have:*

$$\text{od } w = \text{id } w \geqslant 2 .$$

Proof. By Lemma 3 *id* $w = \text{od } w$. Let G be not a directed cycle. Suppose that there exist vertices $w, v \in V(G)$ with id $w = \text{od } w = 1$, id $v = \text{od } v \geqslant 2$ and there is

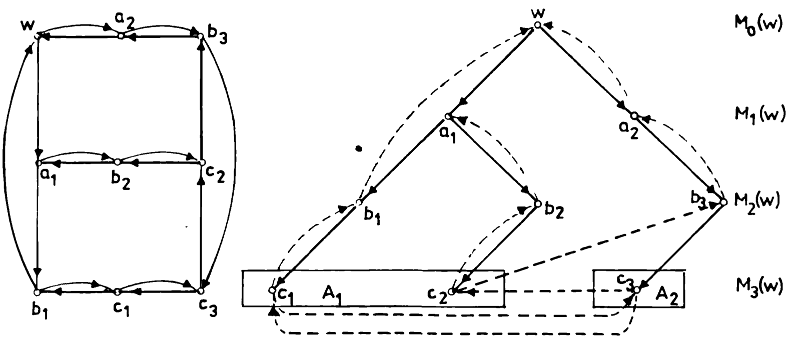

Fig. 4

an edge directed from w to v. Let us construct the structure of a graph G in the preceding form with $w \in M_0(w)$. Then there can only be edges directed from $u \in A_1$ to $x \in \Gamma\,'(u)$, $1 \leqslant r \leqslant k-1$, and this is a contradiction to the definition of a block (v is a cutpoint).

Q.E.D.

Lemma 8. *Let G be a directed P-block of diameter k. Let $w \in V(G)$ and* od $w = \text{id } w = d \geqslant 3$. *Then for every vertex $v \in V(G)$ we have:*

$$\text{id } v = \text{od } v = d .$$

Proof. We will proceed in two steps.

I. Let $v \in \Gamma^{-1}(w)$. By Lemmas 3 and 5 it is sufficient to prove that $M_k(v) \cap M_k(w) \neq \emptyset$. Let u be an element of $M_k(w)$ and let $[w, e_1, v_1, \ldots, e_k, u]$ be a path of length k from w to u. Let $v_1 = x_1, x_2, \ldots, x_d$ be all the vertices of G such that there is an edge directed from w to each of them. Evidently, for $\in \{2, 3, \ldots, d\}$ we have $\varrho_G(x_i, u) = k$ and no two of the corresponding paths of length k have a vertex different from u in common. Let $y_2, y_3, \ldots, y_d$ be the corresponding vertices of G with $y_i \in \Gamma^{-1}(u)$. Evidently, for $i \in \{2, 3, \ldots, d\}$, $y_i \in M_k(w)$. For the vertex v there must exist paths of length $\leqslant k$ from v to y_i $(i = 2, 3, \ldots, d)$ without vertices in common and at most one of them has length $< k$. If $d \geqslant 3$, then at least one of them has length k. Hence $M_k(v) \cap M_k(w) \neq \emptyset$.

II. For any $v \in V(G)$ there exists a sequence of vertices $v = v_1, v_2, \ldots, v_n = w$, such that there is an edge directed from v_i to v_{i+1} $(i \in \{1, 2, \ldots, n-1\})$. The proof follows from step I.

Q.E.D.

Lemma 9. *Let G be a directed P-block of diameter k. Then G is a homogeneous graph.*

Proof. Distinguish two cases.

I. If for every vertex $v \in V(G)$ id $v =$ od $d < 3$, then a proof follows from Lemma 7.

II. If there exists a vertex $v \in V(G)$ such that id $v =$ od $v = d \geqslant 3$, then a proof follows from Lemma 8.

Q.E.D.

From Lemmas 4 and 9, the following theorem follows:

Theorem 1. *Every loopless directed P-graph is either a quasitree or a homogeneous block with a finite diameter.*

From Lemma 2 and Theorem 1 we immediately have:

Corollary 1. *Every partially directed P-graph without loops is either a quasitree or a homogeneous block with a finite diameter.*

Corollary 2. *Every partially directed P-graph G is either a quasitree or a graph of a finite diameter such that G^0 is a homogeneous block.*

A homogeneous P-graph of valency d and with a finite diameter k will be called a graph of the type $P(d, k)$.

From [4, Theorem 4] or [10, Theorem 2] and [3, Proposition 4] it follows:

Corollary 3. *For an arbitrary infinite cardinal number d and an arbitrary finite cardinal number k there exists an undirected [directed, mixed] P-graph of the type $P(d, k)$.*

Theorem 2. *Let G be a graph of diameter $\leqslant 2$ without loops and oppositely directed edges. Then G is a P-graph if and only if G is a T-graph.*

Proof. 1. If G is a T-graph, then from [1, Theorem 10] it follows that G is a P-graph.

2. If G is a P-graph of diameter ≤ 2 without loops and oppositely directed edges, then every trail S of a graph G of length ≤ 2 is a path and G is a T-graph.

Q.E.D.

From Theorem 2 and from [1, Theorem 8] it follows:

Corollary 4. *Let G be a finite graph of the type $P(d, 2)$. Then G is a totally homogeneous graph.*

A homogeneous graph G is said to be *totally homogeneous* with a directed valency z and an undirected valence r if for every vertex v of G exactly z directed edges going from [to] v and v is incident with exactly r undirected edges.

Problem. *For which positive integers d and k does there exist a graph of type $P(d, k)$?*

Remark. From [1, 3, 7, 8, 10] and this paper it follows that P(d, k)-graphs exist in the following cases:

(i) d arbitrary, $k=1$ (complete graphs undirected, directed or mixed).
(ii) $k=2$, $d=2, 3, 7$ (undirected Moore graphs [8]).
(iii) $k=2$, $d\geq 2$ (totally homogeneous prahs $(B(d, 1))^+$, with $d=z+r$, $r=1$ [1], [3]).
(iv) $k=2$, $d=4$ (a totally homogeneous graph M, with $d=z+r$, $r=3$, [1], [3]).
(v) $k\geq 3$, $d=2$ (graphs $Z_{3,d}$ [7]).
(vi) k arbitrary, $d=1$ (directed cycles [10]).
(vii) k arbitrary odd, $d=2$ (odd undirected cycles [3]).
(viii) $k\geq \aleph_0$, d an arbitrary finite number.

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О ЧАСТИЧНО ОРИЕНТИРОВАННЫХ Р-ГРАФАХ

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Резюме

Частично ориентированный граф G называется Р-графом, если для всякой упорядоченной пары $[u\ v]$ его вершин существует в G точно один $u-v$ путь длины, не превышающей диаметр графа G. Граф G называется однородным валентности d, если внешняя и внутренняя степень всякой вершины равны d. Граф G называется квазидеревом, если для всякой упорядоченной пары $[u, v]$ его вершин существует в G точно один $u-v$ путь. Показано, что всякий Р-граф является или квазидеревом, или однородным графом конечного диаметра.