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Mathematica Slovaca, Vol. 33 (1983), No. 4, 357--362

Persistent URL: http://dml.cz/dmlcz/136342

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ON REPRESENTATIONS OF LOGICS

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In [12] an embedding of a logic L into a lattice of all f-closed subspaces $L_f(V)$ of a vector space V with the Hermitian form f was found. In the presented paper it is shown that $L_f(V)$ has the Hilbertian property ($M+M^{\perp}=V$ for all $M\in L_f(V)$) if and only if the supremum $a\vee e$ exists in L for any $a\in L$ and any atom $e\in L$.

1. Basic concepts

Let L be an orthomodular σ-orthoposet, i.e. L is a partially ordered set with the first element 0 and the last element 1, with the orthocomplementation $\perp: L\to L$ such that

(*i*) $(a^{\perp})^{\perp}=a$, $a\in L$

(*ii*) $a\leqslant b \Rightarrow a^{\perp}\geqslant b^{\perp}$, $a, b\in L$

(*iii*) $a\vee a^{\perp}=1$, $a\in L$.

We say that a is orthogonal to b ($a\perp b$), $a, b\in L$ if $a\leqslant b^{\perp}$ and we suppose that

(*iv*) $\vee a_i\in L$ for any sequence $\{a_i\}$ of mutually orthogonal elements of L.

Finally, we suppose that L has the orthomodularity property, i.e.

(*v*) $a\leqslant b$ implies that there is $d\in L$, $d\perp a$ such that $b=a\vee d$.

A partially ordered set L with the properties (*i*)—(*v*) is called a logic.

A state on L is a map m: $L\to[0, 1]$ such that

(*i*) $m(1)=1$,

(*ii*) $m(\vee a_i)=\Sigma\, m(a_i)$ for any sequence $\{a_i\}$ of mutually orthogonal elements of L.

A state m on L is pure if it cannot be written as a convex combination of other states, i.e. if the equality $m(\cdot)=cm_1(\cdot)+(1-c)m_2(\cdot)$, $0<c<1$ implies $m=m_1=m_2$.

Let L be a logic and P a set of pure states on L. For $a\in L$ let us put $P_a=\{p\in P: p(a)=1\}$, and for $p\in P$ let us put $L_p=\{a\in L: p(a)=1\}$.

Definition 1 [2]. *We say that the pair* (L, P), *where* L *is a logic and* P *is a set of pure states on* L, *is a quantum logic if*

(*i*) $P_a \subset P_b \Rightarrow a \leq b$, a, $b \in L$,
and
(*ii*) $L_p \subset L_q \Rightarrow p = q$, p, $q \in P$.

Definition 2 [13]. *Let $M \subset P$. We say that a state m is a superposition of states of M if $M(a) = 1$ implies $m(a) = 1$, where $M(a) = 1$ means that $p(a) = 1$ for all $p \in M$.*

Let us put $\bar{M} = \{p \in P\colon\ M(a) = 1 \Rightarrow p(a) = 1\}$, i.e. $\bar{M}$ is the set of all pure superpositions of states in M.

Definition 3 [12]. *We say that $S \subset P$ is a subspace if $\{p, q\}^- \subset S$ for any $p, q \in S$. If S is a subset of P, we denote by $\Lambda(S)$ the smallest subspace of P containing S.*

Definition 4 [12]. *We say that $S \subset P$ is a closed subspace of P if $S = \bar{S}$.*

We denote by $L(P)$ the set of all subspaces of P and by $F(P)$ the set of all closed subspaces of P, i.e.

$$L(P) = \{S \subset P\colon\ S = \Lambda(S)\}$$

and

$$F(P) = \{S \subset P\colon\ S = \bar{S}\}.$$

It can be easily seen that $F(P) \subset L(P)$.

Definition 5 [3]. *We say that $p \in P$ is a minimal superposition of the set $S \subset P$ if $p \in \bar{S}$ and $p \notin \bar{Q}$ for any $Q \subset S$, $Q \neq S$.*

Definition 6 [3]. *We say that the minimal superposition postulate (MSP) holds in the quantum logic (L, P) if for any finite set $S = \{s_1, \ldots, s_n\} \subset P$ and any minimal superposition p of S there holds $\{p, S_1\}^- \cap \bar{S}_2 \neq \emptyset$ for any partition $\{S_1, S_2\}$ of S (i.e. such sets S_1 and S_2 that $S_1 \cup S_2 = S$ and $S_1 \cap S_2 = \emptyset$).*

Definition 7 [10]. *We say that the superposition principle holds in the quantum logic (L, P) if $\{p, q\}^- \neq \{p, q\}$ for any different states p, $q \in P$.* (Compare with [6].)

Proposition 1 [11]. *If the MSP holds in the quantum logic (L, P), then*
(*i*) *$p \in \Lambda\{r, q\} \Rightarrow r \in \Lambda\{p, q\}$, $q \in \Lambda\{r, p\}$ for any mutually different states p, q, $r \in P$.*
(*ii*) *$\Lambda(S) = \bar{S}$ for any finite subset $S \subset P$.*

The states $p_1, \ldots, p_n$ are independent if $p_i \notin \Lambda\{p_j\colon j \neq i\}$, $i, j = 1, 2, \ldots, n$. The set $\{p_1, \ldots, p_n\}$ is a basis of an element $S \in L(P)$ if $p_1, \ldots, p_n$ are independent and $S = \Lambda\{p_1, \ldots, p_n\}$. If $S \in L(P)$ has a finite basis $\{p_1, \ldots, p_n\}$, then by (*ii*) of Proposition 1 $S \in F(P)$. In this case we say that S is finite-dimensional.

An element $a \in L$ is the support of a state $s \in P$ (in symbols: $a = \operatorname{supp} s$) if $s(b) = 0 \Leftrightarrow b \perp a$ $(b \in L)$. If $a = \operatorname{supp} s$, then $L_s = \{b \in L\colon\ b \geq a\}$. If a state s has a support, we say that s is supported.

Proposition 2 [4]. *Let (L, P) be a quantum logic such that all states in P are supported. Then*

(*i*) $\operatorname{supp} s$ *is an atom of* L *for any* $s \in P$ *and there is a one-to-one correspondence between states in* P *and atoms of* L.

(*ii*) $a = \vee\{\operatorname{supp} p \colon p \in P_a\}$ *for any* $a \in L$.

Let us define the following binary relation on P:

$p \perp q$ if there is $a \in L$ such that $p(a) = 1$ and $q(a) = 0$ (see [2]).

If $p \perp q$, we say that p is orthogonal to q. It can be easily seen that the relation $\perp$ is symmetric and antireflexive. If all states in P are supported, then $p \perp q$ iff $\operatorname{supp} p \perp \operatorname{supp} q$.

For $S \subset P$ let us write $S^{\perp} = \{s \in P \colon s \perp S\}$, where $s \perp S$ means that $s \perp p$ for any $p \in S$.

Proposition 3 [4]. *If (L, P) is a quantum logic such that all states are supported, then* $S^{\perp\perp} = \bar{S}$ *for any* $S \subset P$.

For $S_\alpha \subset P$, $\alpha \in A$ let us set

$$\bigvee_{\alpha \in A} S_\alpha = \overline{\left(\bigcup_{\alpha \in A} S_\alpha\right)}, \quad \sum_{\alpha \in A} S_\alpha = \Lambda\left(\bigcup_{\alpha \in A} S_\alpha\right).$$

Proposition 4 [2, 4, 12].

(*i*) *The set $F(P)$ is a complete lattice with the operations $\vee$ and $\wedge = \cap$ (set-theoretical intersection). If all states are supported, then $S \mapsto S^{\perp}$ is an orthocomplementation in $F(P)$.*

(*ii*) *The set $L(P)$ is a complete lattice with lattice operations Σ and $\wedge = \cap$. If the MSP holds, then $S_1 + S_2 = \{p \in P \colon p \in \Lambda\{r, q\}, r \in S_1, q \in S_2\}$. The singleton subsets $\{p\}$ of P are atoms in both $F(P)$ and $L(P)$.*

Proposition 5 [4]. *Let (L, P) be a quantum logic such that all states in P are supported. Then $P_a \in F(P)$ for any $a \in L$ and the map $a \mapsto P_a \colon L \to F(P)$ is an orthoinjection, i.e. preserves ordering and orthocomplementation.*

The following representation theorem was proved in [12].

Theorem 1. *Let (L, P) be a quantum logic such that the superposition principle (SP) and minimal superposition postulate (MSP) hold in it. Let there be at least four independent states in P. Then there is a division ring K and a vector space V over K such that $L(P)$ is isomorphic to the set $L(V)$ of all linear subspaces of V (i.e. there is a bijection between them that preserves the ordering).*

If, in addition, all states in P are supported, then there is an involutorial anti-automorphism $* \colon \lambda \mapsto \lambda^*$ in K and a Hermitian form $f \colon V \times V \to K$ such that the set $F(P)$ is isomorphic to the set $L_f(V)$ of all f-closed subspaces of V (i.e. there is a bijection between them that preserves ordering and orthocomplementation).

2. Hilbertian property of $L_f(V)$

A question may arise if the lattice $L_f(V)$ from Theorem 1 has the Hilbertian property, i.e. if $M+M^{\perp}=V$ for any $M \in L_f(V)$ (M_1+M_2 denotes the least linear subspace of V containing both M_1 and M_2).

It is known [8, (33.4) and (29.13)] that $M+M^{\perp}=V$ holds iff $L_f(V)$ is orthomodular. By Theorem 1, $L_f(V)$ is orthomodular iff $F(P)$ is orthomodular. As there is a one-to-one correspondence between atoms of L and elements of P, $p \perp q$ iff $\operatorname{supp} p \perp \operatorname{supp} q$, and the set of all atoms is join-dense in L, the set $F(P)$ is isomorphic to the completion by cuts $\tilde{L}$ of the logic L ([7, Th. 2.4 and 2.5]. See also the remarks at the end of [5]). The orthomodularity of L under somewhat different assumptions was studied in [1]. The proof of the following theorem requires a refinement of the technique of [1]. Before stating the theorem, we shall need some lemmas. In the sequel we suppose that (L, P) is a quantum logic such that all states are supported and MSP holds.

Lemma 1. *For any $S \in F(P)$ and any finite-dimensional $Q \in F(P)$ we have $S \vee Q = S+Q$.*

Proof. (The technique of the proof is similar to [9, p. 55].) It is enough to show that $S \vee \{p\} = S+\{p\}$ for any $p \in P$, $p \notin S$. By Theorem 1 in [12], the set $L(P)$ has the covering property, i.e. $S+\{p\}$ covers S. But then $S^{\perp}$ covers $(S+\{p\})^{\perp}$, and there exists $q \in P$ such that $(S+\{p\})^{\perp}+\{q\}=S^{\perp}$. Similarly we have that $(S+\{p\})^{\perp\perp}$ covers $[(S+\{p\})^{\perp}+\{q\}]^{\perp}=S^{\perp\perp}=S$ in $L(P)$. From this it follows that $S+\{p\}=(S+\{p\})^{\perp\perp}=S \vee \{p\}$.

Lemma 2. *Let L have the following property:*

$$\text{for any } a \in L \text{ and any atom } e \in L,\ a \vee e \in L. \qquad (*)$$

Then the following statements are equivalent

(i) $a \leqslant x \leqslant a \vee e$ implies $x=a$ or $x=a \vee e$ for any $a \in L$ and any atom $e \in L$ (covering property),

(ii) if e, f are atoms in L and $a \in L$, $a \wedge e=0$, then $e \leqslant a \vee f$ implies that $f \leqslant a \vee e$ (atomic exchange property).

Proof. $(i) \Rightarrow (ii)$: If $a \wedge e=0$ and $e \leqslant a \vee f$, then $a \wedge f=0$, because if not, then $f \leqslant a$, which implies $e \leqslant a$, a contradiction. Since $a \leqslant a \vee e \leqslant a \vee f$, by (i) $a \vee e = a \vee f \geqslant f$.

$(ii) \Rightarrow (i)$: Let $a \wedge e=0$ and $a \leqslant x \leqslant a \vee e$, $a \neq x$. As L is atomistic, there is an atom $f \leqslant x$, $f \nleqslant a$. From $f \wedge a=0$ we get by (ii) that $a \vee e=a \vee f$. Since $a \vee f \leqslant x \leqslant a \vee e$, we get $x=a \vee e$.

Lemma 3. *$F(P)$ has the covering property.*

Proof. We show that $F(P)$ has the atomic exchange property. It can be shown as in the proof of Lemma 2 that this is equivalent to the covering property. Let $S \in F(P)$, p, $q \in P$, $p \notin S$, $p \in S \vee \{q\}$. By Lemma 1, $S \vee \{q\} = S + \{q\}$, i.e. there is $s \in S$ such that $p \in \{s\} + \{q\}$ (Proposition 4 (*ii*)). By Proposition 1 (*i*), $q \in \{p\} + \{s\}$, which means that $q \in S \vee \{p\}$.

Theorem 2. *Let (L, P) be a quantum logic such that MSP holds and all states are supported. Then the lattice $F(P)$ is orthomodular if and only if L has the property $(*)$ of Lemma 2.*

Proof. I. Let L have the property $(*)$. By Lemma 3, $F(P)$ has the covering property. As $a \mapsto P_a$ is an orthoinjection from L into $F(P)$, L has the covering property as well. Indeed, if $a \wedge b$ exists in L, then $P_{a \wedge b} = P_a \cap P_b = P_a \wedge P_b$. From this it follows that if $a \vee b$ exists in L, then $P_{a \vee b} = P_{(a^\perp \wedge b^\perp)^\perp} = (P_a^\perp \wedge P_b^\perp)^\perp = P_a \vee P_b$. If $a \leqslant x \leqslant a \vee e$, then $P_a \leqslant P_x \leqslant P_{a \vee e} = P_a \vee \{p\}$, where $p = \text{supp}^{-1} e$. The last inequality implies that $P_x = P_a$ or $P_x = P_a \vee \{p\} = P_{a \vee e}$. It follows that $x = a$ or $x = a \vee e$. Since L is orthomodular, it has the Varadarajan property: if $a \in L$ with $0 < a < 1$ and if e is an atom of L, then there exist two atoms x and y such that $e \leqslant x \vee y$, $x \leqslant a$, $y \leqslant a^\perp$ (see [8, (30.7)]).

For $M \in F(P)$, let B_M denote the maximal set of orthogonal states in M. Such a set exists by Zorn's lemma. We show that $M = \bar{B}_M$. Clearly, $\bar{B}_M \subset M$. Let $s \in M$, $s \notin \bar{B}_M$. It can be shown that for any $p \in B_M$ $s(\text{supp}\, p) \neq 0$ only for at most a countable subset $\{p_1, p_2, \ldots\}$ of B_M. Hence $s \perp p$ for $p \notin \{p_1, p_2, \ldots\}$. Put $a = \bigvee_{i=1}^{\infty} \text{supp}\, p_i$. Using the Varadarajan property we show that there is an atom $e \in L$, $e \leqslant a^\perp$ such that $a \vee \text{supp}\, s = a \vee e$. Let $q = \text{supp}^{-1} e$. Then $q \in \left(\bigvee_{i=1}^{\infty} \{p_i\}\right)^\perp \cap M$, i.e. $q \perp p_i$, $i = 1, 2, \ldots$ Let $p \in B_M$, $p \notin \{p_1, p_2, \ldots\}$. Then $e \leqslant a \vee \text{supp}\, s \leqslant (\text{supp}\, p)^\perp$, hence $q \perp p$ for all $p \in B_M$, which contradicts the maximality of B_M. Hence there is no atom in $M \backslash \bar{B}_M$ and since $F(P)$ is atomistic, this implies that $M = \bar{B}_M$. Now let $M_1 \subseteq M_2$, M_1, $M_2 \in F(P)$. Let B_1 be the maximal orthogonal set of states in M_1. It can be extended to the maximal orthogonal set B_2 in M_2. Let $B_3 = B_2 \backslash B_1$ and $\bar{B}_3 = M_3$. Then $B_3 \subseteq B_1^\perp = \bar{B}_1^\perp = M_1^\perp$ and thus $M_3 = \bar{B}_3 \subseteq M_1^\perp$. In addition, $M_1 \vee M_3 = (M_1 \cup M_3)^- = (\bar{B}_1 \cup \bar{B}_3)^- = (B_1 \cup B_3)^- = \bar{B}_2 = M_2$. This proves the orthomodularity of $F(P)$.

II. Let $F(P)$ be orthomodular. Then $F(P)$ has the Varadarajan property and hence L has it, too. Let $a \in L$, $a \neq 0,1$ and e be an atom of L. Then there exist atoms $x \leqslant a$, $y \leqslant a^\perp$ such that $e \leqslant x \vee y \leqslant a \vee y$. It can be easily seen that $\text{supp}^{-1} x = (P_a^\perp \vee \{\text{supp}^{-1} e\}) \wedge P_a$, $\text{supp}^{-1} y = (P_a \vee \{\text{supp}^{-1} e\}) \wedge P_a^\perp$ in $F(P)$. Now let $c \geqslant a$, e. Then $\text{supp}^{-1} y = (P_a \vee \{\text{supp}^{-1} e\}) \wedge P_a^\perp \leqslant P_c \wedge P_a^\perp \leqslant P_c$, i.e. $y \leqslant c$ and thus $c \geqslant a \vee y$. We have shown that $a \vee e = a \vee y$ and this completes the proof.

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Received September 17, 1981

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О ПРЕДСТАВЛЕНИЯХ ЛОГИК

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Резюме

В [12] показано вложение логики L в решетку всех f-замкнутых подпространств $L_f(V)$ линейного пространства V с эрмитовой формой f. В предлагаемой статье показано, что $L_f(V)$ имеет качество Гильберта $M+M^{\perp}=V$ для всех $M \in L_f(V)$ тогда и только тогда, когда $a \vee e$ существует в L для всех $a \in L$ и всех атомов $e \in L$.