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ON THE TWO-SIDED MULTIPARAMETER CONTROL

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One important task which statisticians have often to solve is to decide whether a statistical population is concentrated in prescribed boundaries. For example, if a producer of furniture receives a consignement of pressed planks, he wants to know whether they are sufficiently flat. This can be expressed by the requirement, that the thickness x_i of the plank in the i-th place of surface satisfies for $i=1, \ldots, m$ the relation

$$x_i \in \langle q, Q \rangle \tag{1}$$

where $q < Q$ are prescribed boundaries. Another similar problem is controlling several parameters $x_1, \ldots, x_m$ of the same product, which have to satisfy for $i=1, \ldots, m$ the relation

$$x_i \in \langle q_i, Q_i \rangle . \tag{2}$$

Here $q_i < Q_i$ and the interval $\langle q_i, Q_i \rangle$ is an admissible inaccuracy. If we assume that the vector $x=(x_1, \ldots, x_m)$ is normally distributed with diagonal covariance matrix, then both (1) and (2) can be expressed by the inequality

$$P\left(\prod_{i=1}^{m} \langle q_i, Q_i \rangle\right) \geqq 1-\Delta, \tag{3}$$

where $\Delta \in (0, 1)$ is a chosen number determining degree of quality of the statistical population. Unfortunately, parameters of the normal distribution satisfying (3) cannot be described in a simple analytical form, hence it is necessary to construct a statistical hypothesis implying (3).

Let us put

$$\Theta=\{(\mu, \sigma): \mu, \sigma \in R^m, \sigma_i>0 \text{ for } i=1, \ldots, m\} \tag{4}$$

and for $\theta=(\mu, \sigma) \in \Theta$ denote

$$f(x, \theta)=(2\pi)^{-m/2}\left(\prod_{i=1}^{m} \sigma_i\right)^{-1} \exp\left(-\frac{1}{2} \sum_{i=1}^{m} \frac{(x_i-\mu_i)^2}{\sigma_i^2}\right), \tag{5}$$

$$P_\theta(A)=\int_A f(x, \theta)\, \mathrm{d}x.$$

Let c be the $\frac{1+(1-\Delta)^{1/m}}{2}$ quantile of the $N(0, 1)$ distribution, i.e.

$$\Phi(c)=\frac{1+(1-\Delta)^{1/m}}{2}, \tag{6}$$

where Φ is distribution function of the standard one-dimensional normal distribution. If we denote

$$H_c=\{(\mu, \sigma)\in\Theta;\ \mu_i+c\sigma_i\leq Q_i,\ \mu_i-c_i\sigma_i\geq q_i\quad i=1, \ldots, m\} \tag{7}$$

then making use of well-known properties of normal distribution (5) we see that for each $\theta\in H_c$

$$P_\theta\left(\prod_{i=1}^{m}\langle q_i, Q_i\rangle\right)\geq P_\theta\left(\prod_{i=1}^{m}\langle\mu_i-c\sigma_i, \mu_i+c\sigma_i\rangle\right)=$$

$$=(\Phi(c)-\Phi(-c))^m=(2\Phi(c)-1)^m=1-\Delta.$$

We have shown that the hypothesis (7) implies (3), which means that it is reasonable to test its validity. We shall give an explicit formula for the maximum likelihood ration test statistic and describe its asymptotic distribution.

Let $x^{[1]}=(x_1^{[1]}, \ldots, x_m^{[1]}), \ldots, x^{[n]}=(x_1^{[n]}, \ldots, x_m^{[n]})$ be independet observations of the random vector x, $\bar{x}_i$ be the sample mean and s_i be the sample standard deviation of the i-th coordinate, i.e.

$$\bar{x}_i=\frac{1}{n}\sum_{j=1}^{n}x_i^{[j]},\quad s_i=\left(\frac{1}{n}\sum_{j=1}^{n}(x_i^{[j]}-\bar{x}_i)^2\right)^{1\,2}.$$

If $\bar{x}_i\in(q_i, Q_i)$, we denote

$$M_i=\bar{x}_i,\ D_i=\begin{cases} s_i & \bar{x}_i+cs_i\leq Q_i, \quad \bar{x}_i-cs_i\geq q_i \\ \dfrac{\bar{x}_i-q_i}{c} & \bar{x}_i\in\left(q_i, \dfrac{q_i+Q_i}{2}\right\rangle, \quad \bar{x}_i-cs_i<q_i \\ \dfrac{Q_i-\bar{x}_i}{c} & \bar{x}_i\in\left\langle\dfrac{q_i+Q_i}{2}, Q_i\right), \quad \bar{x}_i+cs_i>Q_i. \end{cases} \tag{8}$$

If $\bar{x}_i\geq Q_i$, we put

$$D_i=\min\left\{\frac{Q_i-q_i}{2c}, \frac{c(\bar{x}_i-Q_i)}{2}+\left[s_i^2+(\bar{x}_i-Q_i)^2\left(1+\frac{c^2}{4}\right)\right]^{1\,2}\right\} \tag{9}$$

$$M_i=Q_i-cD_i \tag{10}$$

and finally for $\bar{x}_i\leq q_i$ let

$$D_i=\min\left\{\frac{Q_i-q_i}{2c}, \frac{c(q_i-\bar{x}_i)}{2}+\left[s_i^2+(\bar{x}_i-q_i)^2\left(1+\frac{c^2}{4}\right)\right]^{1/2}\right\} \tag{11}$$

$$M_i=q_i+cD_i.$$

It is shown in [2] that the mapping (M_i, D_i) defined by the formulas (8)—(12) is the maximum likelihood estimator of (μ_i, σ_i) under the constraints $\mu_i + c\sigma_i \leqslant Q_i$, $\mu_i - c\sigma_i \geqslant q_i$. Thus if we denote

$$\tilde{\theta} = (M_1, \ldots, M_m, D_1, \ldots, D_m) \tag{13}$$

then (cf. (5))

$$\prod_{j=1}^{n} f(x^{[j]}, \tilde{\theta}) = \sup \left\{ \prod_{j=1}^{n} f(x^{[j]}, \theta);\ \theta \in H_c \right\}$$

and $\tilde{\theta}$ is the maximum likelihood estimator under the hypothesis H_c.

Theorem. *If* $\hat{\theta} = (\bar{x}_1, \ldots, \bar{x}_m, s_1, \ldots, s_m)$, *then for each parameter* $\theta \in H_c$ *and for every positive number* t

$$\lim_{n\to\infty} P_\theta \left[-2 \sum_{j=1}^{n} \log \frac{f(x^{[j]}, \tilde{\theta})}{f(x^{[j]}, \hat{\theta})} > t \right] \leqslant 1 - F(t), \tag{14}$$

where log *is logarithm to the base* e *and*

$$F(t) = 2^{-m} \sum_{j=0}^{m} \binom{m}{j} \left(1 - \frac{2}{\pi} \operatorname{arc\,tg} \frac{\sqrt{2}}{c}\right)^{m-j} \sum_{r=0}^{j} \binom{j}{r}$$
$$\left(1 - \frac{2}{\pi} \operatorname{arc\,tg} \frac{c}{\sqrt{2}}\right)^{j-r} F_{2j-r}(t). \tag{15}$$

Here F_j is the distribution function of the chi-square distribution on j degrees of freedom, $F_0(t) = 1$ for $t > 0$ and the function arc tg takes its values in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. If

$$\theta = \left(\frac{q_1 + Q_1}{2}, \ldots, \frac{q_m + Q_m}{2}, \frac{Q_1 - q_1}{2c}, \ldots, \frac{Q_m - q_m}{2c}\right) \tag{16}$$

then (14) holds with equality sign.

Before proving the theorem we recall that a set H is said to be approximable at a point $\theta \in H$ by a cone K, if for every sequence $\{a_n\}$ of positive numbers tending to zero

$$\sup \{\varrho(x, K + \theta);\ x \in H,\ \|x - \theta\| \leqslant a_n\} = o(a_n),$$
$$\sup \{\varrho(y + \theta, H);\ y \in K,\ \|y\| \leqslant a_n\} = o(a_n).$$

Here we use the notation

$$\varrho(x, D) = \inf \{\|x - z\|;\ z \in D\}$$

and by the cone K we mean any closed convex set such that $\alpha x \in K$ whenever $x \in K$ and $\alpha \geqslant 0$.

Since $\hat{\theta}$, $\tilde{\theta}$ are consistent maximum likelihood estimators, we can use the Chernoff theorem (cf. [1] or [2]). If the set H_c is approximable at the true value θ of the parameter by a cone K, the making use of this theorem it is easy to see that the left-hand side of (14) is of the form

$$P_\theta = P_\theta\ [\inf_{\theta^* \in K_\theta}(z-\theta^*)'J(z-\theta^*) > t \mid N(O, J^{-1})]$$

where J is a diagonal matrix with diagonal $\sigma_1^{-2}, \dots, \sigma_m^{-2}, 2\sigma_1^{-2}, \dots, 2\sigma_m^{-2}$. If θ is an interior point of H_c, then $K_\theta = R^{2m}$ and $P_\theta = 0$. If $\theta = (\mu, \sigma)$ belongs to the boundary of H_c, then it satisfies the relations

$$\begin{array}{lll}
\mu_{i_j} + c\sigma_{i_j} = Q_{i_j}, & \mu_{i_j} - c\sigma_{i_j} > q_{i_j} & j = 1, \dots, v \\
\mu_{i_j} + c\sigma_{i_j} = Q_{i_j}, & \mu_{i_j} - c\sigma_{i_j} = q_{i_j} & j = v+1, \dots, s \\
\mu_{i_j} + c\sigma_{i_j} < Q_{i_j}, & \mu_{i_j} - c\sigma_{i_j} = q_{i_j} & j = s+1, \dots, r \\
\mu_{i_j} + c\sigma_{i_j} < Q_{i_j}, & \mu_{i_j} - c\sigma_{i_j} > q_{i_j} & j = r+1, \dots, m.
\end{array}$$

The set H_c is in this case approximable at θ by the cone K_θ consisting of all vectors $(x, y) \in R^{2m}$, satisfying the relations

$$\begin{array}{lll}
x_{i_j} + cy_{i_j} \leq 0 & & j = 1, \dots, v \\
x_{i_j} + cy_{i_j} \leq 0, & x_{i_j} - cy_{i_j} \geq 0 & j = v+1, \dots, s \\
x_{i_j} - cy_{i_j} \geq 0 & & j = s+1, \dots, r.
\end{array} \tag{17}$$

Since the set $J^{1/2}K_\theta$ is determined by the inequalities (17) where c is replaced by $\frac{c}{\sqrt{2}}$, this set is smallest if θ is determined by (16). This means, that for $P = \sup_{\theta \in H_c} P_\theta$

$$P = 1 - P[\varrho^2(z, \tilde{K}) \leq t \mid N(O, I_{2m})], \tag{18}$$

$$\tilde{K} = \{(x, y);\ x, y \in R^m,\ x_i + \gamma y_i \leq 0,\ x_i - \gamma y_i \geq 0,\ i = 1, \dots, m\} \tag{19}$$

$$\gamma = \frac{c}{\sqrt{2}}, \tag{20}$$

where I_{2m} is the unit matrix of the type $2m \times 2m$. Denoting

$$K = \{(z_1, z_2);\ z_1 + \gamma z_2 \leq 0,\ z_1 - \gamma z_2 \geq 0\} \tag{21}$$

we see that

$$\varrho^2((x, y), \tilde{K}) = \sum_{i=1}^{m} \varrho^2((x_i, y_i), K) \tag{22}$$

and putting $x^+ = (|x_1|, \dots, |x_m|)$ we obtain

$$\varrho^2((x, y), \tilde{K}) = \varrho^2((x^+, y), \tilde{K}).$$

Hence if A_j is the set of all vectors (x, y) such that $x, y \in R^m$ and

$$x_1 > 0, \ldots, x_m > 0, \quad (x_i, y_i) \notin K \quad i = 1, \ldots, j, \quad (x_i, y_i) \in K \quad i = j+1, \ldots, m,$$

then aditivity of probability, symmetry of $N(0, I)$, (18), (19) and (22) imply

$$1 - P = 2^m \sum_{j=0}^{m} \binom{m}{j} P[A_j \cap \{\varrho^2((x, y), \tilde{K}) \leq t\} | N(0, I_{2m})] = 2^m \sum_{j=0}^{m} \binom{m}{j} (P_K)^{m-j} P(j). \tag{23}$$

Here

$$P_k = P[(x_1, y_1) \in K, x_1 > 0 | N(0, I_2)], \tag{24}$$

$P(0) = 1$ and for $j = 1, \ldots, m$

$$P(j) = P\left[\sum_{i=1}^{j} \varrho^2((x_i, y_i), K) \leq t, (x_i, y_i) \notin K, x_i > 0 \quad i = 1, \ldots, j | N(0, I_{2j})\right]. \tag{25}$$

Obviously,

$$P(j) = \sum_{r=0}^{j} \binom{j}{r} P_{jr}, \tag{26}$$

where

$$P_{jr} = P\left[B_{jr} \cap \left\{\sum_{i=1}^{j} \varrho^2((x_i, y_i), K) \leq t\right\} | N(0, I_{2j})\right] \tag{27}$$

and B_{jr} is the set of all vectors $(x, y) \in R^{2j}$ such that $x, y \in R^j$ and

$$x_i > 0, \quad (x_i, y_i) \notin K \quad i = 1, \ldots, j,$$
$$y_i < \gamma x_i \quad i = 1, \ldots, r \quad y_i \geq \gamma x_i \quad i = r+1, \ldots, j.$$

If $x_i > 0$ and π is the projection on the cone (21), then (cf. Fig. 1)

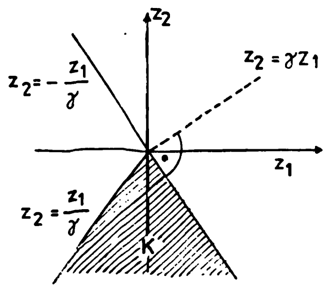

Fig. 1

$$\pi(x_i, y_i) = \begin{cases} (x_i, y_i) & y_i \leq -x_i/\gamma, \\ -\left(\dfrac{\gamma^2 x_i - \gamma y_i}{1+\gamma^2}, \dfrac{y_i - \gamma x_i}{1+\gamma^2}\right) & y_i \in (-x_i/\gamma, \gamma x_i). \\ (0, 0) & y_i \geq \gamma x_i. \end{cases}$$

When the transformation

$$z_i(1+\gamma^2)^{-1/2}(x_i+\gamma y_i), \quad u_i=(1+\gamma^2)^{-1/2}(-\gamma x_i+y_i)$$

is applied to (27) for $i=1, 2, \ldots, r$, we see that

$$P_{jr}=2^{-r}P\left[D_{jr}\cap\left\{\sum_{i=1}^{r} z_i^2+\sum_{i=r+1}^{j}(x_i^2+y_i^2)\leqslant t\right\}\,|N(0, I_{2j-r})\right] \tag{28}$$

where D_{jr} is the set of all vectors $(z_1, \ldots, z_r, x_{r+1}, \ldots, x_j, y_{r+1}, \ldots, y_j)$ such that

$$z_i>0 \quad i=1, \ldots, r, \quad x_i>0, \quad y_i\geqslant\gamma x_i \quad i=r+1, \ldots, j.$$

To calculate the probability (28) we need

Lemma 1. *If $\mathscr{L}(X)$ $N(0, I_p)$ and $c_1, \ldots, c_k$ are vectors belonging to R^p, then in notation of the theorem*

$$P[c_j'x\leqslant 0\ j=1, \ldots, k, x_1^2+\ldots+x_p^2\leqslant v]=F_p(v)P[c_j'x\leqslant 0\ j=1, \ldots, k] \tag{29}$$

Proof. Let us consider the transformation

$$\begin{aligned} x_1&=u\cos\varphi_1\\ x_2&=u\sin\varphi_1\cos\varphi_2\\ &\vdots\\ x_{p-1}&=u\sin\varphi_1\cdot\ldots\cdot\sin\varphi_{p-2}\cos\varphi_{p-1}\\ x_p&=u\sin\varphi_1\cdot\ldots\cdot\sin\varphi_{p-2}\sin\varphi_{p-1}, \end{aligned}$$

where $(u, \varphi_1, \ldots, \varphi_{p-1})\in(0, \infty)\times\varphi$, $\varphi=(0, 2\pi)\times(0, \pi)^{p-2}$. The absolute value of the Jacobian of this mapping is of the form

$$|J|=u^{p-1}Q(\varphi_1, \ldots, \varphi_{p-2}).$$

Making use of this transformation and Fubini's theorem we see that

$$F_p(v)=\int_{(0, v^{1/2})}\exp\left(-\frac{u^2}{2}\right)u^{p-1}\,du\int_{\varphi}(2\pi)^{-p/2}Q(\varphi_1, \ldots, \varphi_{p-2})\,d\varphi_1\ldots d\varphi_{p-1} \tag{30}$$

$$1=\int_{(0, \infty)}\exp\left(-\frac{u^2}{2}\right)u^{p-1}\,du\int_{\varphi}(2\pi)^{-p/2}Q(\varphi_1, \ldots, \varphi_{p-2})\,d\varphi_1\ldots d\varphi_{p-1}. \tag{31}$$

Hence if we denote by A the set of all vectors $(\varphi_1, \ldots, \varphi_{p-1})$ satisfying for $j=1, \ldots, k$ the inequality

$$c_j'(\cos\varphi_1, \sin\varphi_1\cos\varphi_2, \ldots, \sin\varphi_1\cdot\ldots\cdot\sin\varphi_{p-1})\leqslant 0,$$

the repeated use of the transformation, Fubini's theorem, (30) and (31) yield

$$P\left[c_j'x\leqslant 0 \quad j=1, \ldots, k \ \sum_{i=1}^{p} x_i^2 \leqslant v\right]=$$

$$=F_p(v)\int_{(0,\infty)} \exp\left(-\frac{u^2}{2}\right) u^{p-1}\,\mathrm{d}u \int_A (2\pi)^{-p/2} Q(\varphi_1, \ldots, \varphi_{p-2})\,\mathrm{d}\varphi_1 \ldots \mathrm{d}\varphi_{p-1},$$

which completes the proof of the lemma.

Now if we take into account both (28) and (29) we obtain

$$\begin{aligned} P_{jr} &= F_{2j-r}(t)4^{-r}P[x_1>0,\ y_1 \geqq \gamma x_1 | N(0, I_2)]^{j-r} = \\ &= F_{2j-r}(t)4^{-r}(4^{-1}-(2\pi)^{-1}\operatorname{arc\,tg}\gamma)^{j-r}, \end{aligned} \tag{32}$$

where the second equality can be easily shown by means of the transformation $x_1 = u\cos\varphi_1$, $y_1 = u\sin\varphi_1$. Making use the same transformation we can show that

$$P_K = 4^{-1}-(2\pi)^{-1}\operatorname{arc\,tg}\frac{1}{\gamma}. \tag{33}$$

Obviously, the relations (23), (33), (26), (32) and (20) imply (14).

Now we prove that the likelihood ratio test is consistent. Since this assertion can be proved in a general setting, we assume for a while that we are given a family of probabilities $\{P_\theta;\ \theta\in\Theta\}$, defined on a measurable space $(X, \mathscr{S})$ by densities

$$f(x,\theta)=\frac{\mathrm{d}P_\theta}{\mathrm{d}\mu}(x),$$

satisfying the following conditions.

(CI) Θ is an open subset of R^m and $P_{\theta_1}\neq P_{\theta_2}$ whenever $\theta_1\neq\theta_2$.

(CII) The functions $\{f(x,\cdot);\ x\in X\}$ are continuous functions of the variable θ.

(CIII) If θ, θ^* belong to Θ, then there is a number $\delta>0$ such that the positive part of the function

$$G(x,\delta)=\sup\{\log f(x,\tilde\theta);\ \|\tilde\theta-\theta^*\|\leqslant\delta\}$$

is P_θ integrable, i.e.

$$\int \max\{0, G(x,\delta)\}\,\mathrm{d}P_\theta(x)<\infty.$$

(CIV) The function $\log f(\cdot,\theta)$ is P_θ integrable for each $\theta\in\Theta$. If we denote by $x^{(n)}=(x_1, \ldots, x_n)$ n independent observations and for $\tau\subset\Theta$ put

$$L(x^{(n)},\tau)=\sup\left\{\prod_{j=1}^{n} f(x_j,\theta^*);\ \theta^*\in\tau\right\}.$$

then we can state

Lemma 2. *Let $\theta \in \Theta - \tau$. If there are measurable mappings*

$$\hat{\theta}\colon X^n \to \Theta, \quad \tilde{\theta}_n\colon X^n \to \tau$$

and a compact set $K \subset \tau$ such that

$$P_\theta[L(x^{(n)}, \hat{\theta}_n) = L(x^{(n)}, O)] \to 1, \quad P_\theta[L(x^{(n)}, \tilde{\theta}_n) = L(x^{(n)}, K)] \to 1$$

then under the conditions (CI)—(CIV)

$$P_\theta \left[\log \frac{L(x^{(n)}, \hat{\theta}_n)}{L(x^{(n)}, \tilde{\theta}_n)} > M\right] \to 1 \tag{34}$$

for any real number M.

Proof. Since $K \cup \{\theta\}$ is a compact subset of R^m, the probabilities $\{P_{\theta^*};\ \theta^* \in K \cup \{\theta\}\}$ fulfill the conditions presented in [3]. But $\theta \notin K$, which according to Theorem 1 in [3] means that

$$\lim_{n\to\infty} P_\theta \left[\frac{L(x^{(n)}, K)}{L(x^{(n)}, \theta)} < e^{-M}\right] = 1.$$

Since

$$\frac{L(x^{(n)}, \hat{\theta}_n)}{L(x^{(n)}, \tilde{\theta}_n)} \geqslant \frac{L(x^{(n)}, \theta)}{L(x^{(n)}, K)}$$

with probability tending to 1 for $n \to \infty$, the lemma is proved.

Let us turn our attention again to testing the hypothesis (7). If we denote (cf. (13))

$$t_n = t_n(x^{[1]}, \ldots, x^{[n]}) = -2 \sum_{j=1}^{n} \log \frac{f(x^{[j]}, \tilde{\theta})}{f(x^{[j]}, \hat{\theta})},$$

then

$$t_n = \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{(x_i^{[j]} - M_i)^2}{D_i^2} + n \left[2 \sum_{i=1}^{m} \log \frac{D_i}{s_i} - m\right],$$

where M_i, D_i are expressions determined by the formulas (8)—(12).

Hence if F is the function (15) and

$$F(t(\Delta, \alpha)) = 1 - \alpha,$$

then the tests

$$\psi(x^{[1]}, \ldots, x^{[n]}) = \begin{cases} \text{reject } H_c \text{ if } t_n > t(\Delta, \alpha) \\ \text{accept } H_c \text{ if } t_n \leqslant t(\Delta, \alpha) \end{cases} \tag{35}$$

according to the theorem have asymptotic size α. Moreover, it is easy to see that the assumptions of Lemma 2 are fulfilled. This means, that if $\theta \in O - H_c$, then the tests (35) will reject the hypothesis H_c with probability tending to 1 for n tending to infinity.

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О ДВУХСТРОННОМ КОНТРОЛЕ МНОГОМЕРНОГО ПАРАМЕТРА

František Rublík

Резюме

Пусть m-мерный вектор нормально распределен и его координаты являются независимыми случайными величинами. В статье находится явная формула для статистики отношения правдоподобия для проверки гипотезы $\mu_i + c\sigma_i \leq Q_i$, $\mu_i - c\sigma_i \geq q_i$, $i = 1, \ldots, m$, где μ_i — средное значение, σ_i — стандартное отклонение i-той координаты и $q_i < Q_i$ произвольные фиксированные числа. Приведено тоже асимптотическое распределение этой статистики и показано, что проверка упомянутой гипотезы при помощи отношения правдоподобия имеет свойство состоятельности.