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# ON VECTOR LATTICE-VALUED MEASURES-I

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**Introduction.** The present paper prepares the background for [14], the background being the study of vector lattice-valued outer measures and the study of the Carathéodory extension of vector lattice-valued measures. In [14] we introduce the theory of $V$-valued contents and study the $\hat{V}$-valued outer measures induced by such contents. This study enables us to prove in [14] that every $V\cup\{\text{strict.}\,\infty\}$ — valued Baire measure $\mu_0$ on a locally compact Hausdorff space $T$ admits uniquely regular Borel and weakly Borel extensions when $V$ is weakly $(\sigma, \infty)$-distributive. This result obviously includes the known result of Wright ((i) $\Rightarrow$ (ii) of Theorem 3.3. of [20]) as a particular case. At this point we would like to remark that there is an error in the proof of Lemma 2.1. of [20], as it is tacitly assumed at the end of p. 280 of [20] that the sequence $\{U_k\}_k$ is increasing. This need not happen though $\{B_n\}_n$ is an increasing sequence.

Also, the study made here is interesting in itself as it unifies the Carathéodory estension procedure in the known special cases of numerical measures and spectral measures in Banach spaces [12].

If $V = C(S)$ is a Stone algebra and if each $f_n$ is a constant function on $S$, then $\bigvee_{n=1}^{\infty} f_n = \infty$ implies $\bigvee_{n=1}^{\infty} (f_n\chi_k) = \infty$ for each non-null clopen subset $K$ of $S$. But this is not necessarily true if $f_n$ are non-constant functions in $C(S)$ with $\bigvee_{n=1}^{\infty} f_n = \infty$. This odd behaviour of infinity in the present setup makes us impose certain restrictions on $V\cup\{\infty\}$-valued measures and we call such measures $V\cup\{\text{strict.}\,\infty\}$-valued measures (see definitions 4.7 and 4.9). It turns out that an extended real valued measure is always an $\mathbf{R}\cup\{\text{strict.}\,\infty\}$-valued measure.

Here we assume that $V$ is weakly $(\sigma, \infty)$-distributive and prove that a bounded $V$-valued or a $V\cup\{\text{strict.}\,\infty\}$-valued (See definitions 4.7 and 4.9) measure $\mu$ on a ring $\mathcal{R}$ of sets admits the Carathéodory extension.

However, in this connection we may recall here the work of Fremlin [4], Matthes [10] and Wright [18, 19] in the extension problem of $V$-valued measures. They have proved that the weaker hypothesis of weak $\sigma$-distributivity of the vector lattice $V$ would itself ensure the solution of the extension problem of $V$-valued measures. But the extended measure in their work is not required to be defined and countably subadditive on $\mathscr{H}(\mathscr{R})$, the hereditary $\sigma$-ring generated by $\mathscr{R}$, as it is required in the Carathéodory extension procedure. Thus it is not known whether the Carathéodory extension is still possible when $V$ is just weakly $\sigma$-distributive and not weakly $(\sigma, \infty)$-distributive. Also see Riecan [22] and Volauf [23].

In § 1 we give the basic definitions and known results from [16, 17, 18, 20], which are needed in the sequel. In § 2 the notion of an onter measure is extended to vector lattice-valued set functions and some basic results of such outer measures are obtained.

As a preliminary to the Carathéodory extension procedure of vector lattice-valued measures, we develop in § 3 the theory of induced vector lattice-valued inner measures. In § 4 we introduce the notion of $V\cup\{\text{strict.}\quad\}$-valu d measures. Any bounded $V$-valued measure is $V\cup\{\text{strict.}\,\infty\}$-valued. An extended real valued measure is $V\cup\{\text{strict.}\,\infty\}$-valued when $V=\mathbf{R}$. We prove that when $V$ is weakly $(\sigma, \infty)$-distributive, every $V\cup\{\text{strict.}\,\infty\}$-valued measure on a ring $\mathscr{R}$ admits the Carathéodory extension. The classical Carathéodory extension of extended real valued measures follows as a particular ca e of this theorem. § 5 is devoted to the study of measurable covers and outer regularity of $V\cup\{\text{strict.}\,\infty\}$-valued measures to obtain the $\sigma$-ring of all $\mu^*$-measurable sets as the completion of $\mathscr{S}(\mathscr{R})$, the $\sigma$-ring generated by $\mathscr{R}$. The last section deals with applications to positive operator valued measures in Banach spaces and the Carathéodory extension theorem of [12] for spectral measures in Banach spaces is obtained as a particular case of the general situation studied in § 4.

## 1. Preliminaries

Throughout this paper $V$ will denote a boundedly $\sigma$-complete vector lattice with $\hat{V}$ its Dedekind completion. $V^+=\{x\in V\colon x\geqslant 0\}$. We adjoin an object $+\infty$ not in $V$ and extend the partial ordering and addition operation of $V$ to $V\cup\{\infty\}$ in the obvious way. The supremum of any unbounded collection of elements in $V^+$ or $\hat{V}^+$ is taken to be $\infty$.

**Definition 1.1.** *A $V\cup\{\infty\}$-valued measure is a map $\mu\colon \mathscr{R}\to V\cup\{\infty\}$, where $\mathscr{R}$ is a ring of subsets of a set $T$ such that*

(i) *$\mu(E)\geqslant 0$ for $E$ in $\mathscr{R}$;*

(ii) *$\mu(\emptyset)=0$;*

(iii) *$\mu\left(\bigcup_1^\infty E_n\right)=\bigvee_{n=1}^{\infty}\sum_{i=1}^{n}\mu(E_i)$, where $\{E_i\}$ is a sequence of pairwi e disjoint sets in $\mathscr{R}$ with $\bigcup_1^\infty E_i\in\mathscr{R}$.*

For each positive element $h$ in $V$ let

$$V[h]=\{b \in v: -rh \leqslant b \leqslant rh \text{ for some positive } r \in \mathbf{R}\},$$

where $\mathbf{R}$ denotes the real line.

**Theorem 1.2.** (Stone—Krein—Kakutani—Yosida) *There exists a compact Hausdorff space $S$ such that $V[h]$ is vector lattice isomorphic to $C(S)$, the algebra of all real valued continuous functions on $S$. When $V$ is boundedly complete ($\sigma$-complete), then so is $V[h]$, $V[h]$ is a Banach space in the order unit norm, the isomorphism is also isometric and $C(S)$ is a Stone algebra ($\sigma$-Stone algebra) in the sense that $S$ is extremally disconnected ($S$ is totally disconnected with the property that the closure of every countable union of clopen subsets of $S$ is open).*

For details one may refer Kadison [6] and Vulikh [15].

We shall use the terms Stone algebra and $\sigma$-Stone algebra in the above sense.

From the results of Wright [20] one can define a weakly $(\sigma, \infty)$-distributive vector lattice as below.

**Definition 1.3.** *A $\sigma$-Stone algebra $C(S)$ is said to be weakly $(\sigma, \infty)$-distributive if and only if each meagre subset of $S$ is nowhere dense. Consequently, a boundedly $\sigma$-complete vector lattice $V$ is said to be weakly $(\sigma, \infty)$-distributive if for $h>0$ in $V$, $V[h]$ is weakly $(\sigma, \infty)$-distributive.*

**Proposition 1.4.** *A boundedly $\sigma$-complete vector lattice $V$ is weakly $(\sigma, \infty)$-distributive if and only if $\hat{V}$ is so.*

For related results confer also Matthes [21].

## 2. Vector lattice-valued outer measures

The notion of an outer measure is extended here to $V$-valued set functions and some basic results of such $V$-valued outer measures are obtained.

We refer to Halmos [5] for definitions of (i) ring of sets (ii) $\sigma$-ring of sets (iii) hereditary $\sigma$-ring of sets (iv) algebra or field of sets (v) $\mathscr{S}(\mathscr{R})$, the $\sigma$-ring generated by a ring $\mathscr{R}$ of sets and (vi) $\mathscr{H}(\mathscr{R})$, the hereditary $\sigma$-ring generated by a ring $\mathscr{R}$ of sets.

**Definition 2.1.** *A set function $\mu^*$ on a hereditary $\sigma$-ring $\mathscr{H}$ is called a $V\cup\{\infty\}$-valued outer measure if it satisfies the following conditions:*

(i) *its range is contained in $V^+\cup\{\infty\}$;*

(ii) *it is monotone (i.e. $\mu^*(E)\geqslant\mu^*(F)$ if $E\supseteq F$, $E$ and $F\in\mathscr{H}$);*

(iii) *it is countably subadditive (i.e. $\mu^*\left(\bigcup_1^\infty E_n\right)\leqslant \bigvee_{n=1}^\infty \sum_{i=1}^n \mu^*(E_i)$, $E_i\in\mathscr{H}$, $i=1, 2, \ldots$);*

(iv) $\mu^*(\emptyset)=0$.

**Definition 2.2.** *Let $\mu^*$ be a $V \cup \{\infty\}$-valued outer measure on a hereditary $\sigma$-ring $\mathscr{H}$, $M_{\mu^*}$ be the collection of all sets $E$ in $\mathscr{H}$ for which*

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \setminus E)$$

*holds for every $A$ in $\mathscr{H}$. The members of $M_{\mu^*}$ are called $\mu^*$-measurable sets.*

Remark. A set $E$ in $\mathscr{H}$ is in $M_{\mu^*}$ if and only if

$$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \setminus E)$$

for every $A$ in $\mathscr{H}$.

**Definition 2.3.** *A $V \cup \{\infty\}$-valued measure $\mu$ on a $\sigma$-ring $\mathscr{S}$ is said to be complete if whenever $E \in \mathscr{S}$ and $\mu(E) = 0$, then every subset $F$ of $E$ is in $\mathscr{S}$.*

**Lemma 2.4.** *Let $\mu^*$ be a $V \cup \{\infty\}$-valued outer measure on a hereditary $\sigma$-ring $\mathscr{H}$. Then $M_{\mu^*}$ is a ring and $\mu^*$ is finitely additive on $M_{\mu^*}$. Further, for $A \in \mathscr{H}$ and $E$, $F \in M_{\mu^*}$ with $E \cap F = \emptyset$ we have*

$$\mu^*(A \cap (E \cup F)) = \mu^*(A \cap E) + \mu^*(A \cap F). \tag{1}$$

Proof. The proof is similar to that of Theorem A, § 11 of Halmos [5].

**Lemma 2.5.** *Under the hypothesis of Lemma 2.4, $M_{\mu^*}$ is a $\sigma$-ring. If $A \in \mathscr{H}$ and if $\{E_n\}_n$ is a disjoint sequence of sets in $M_{\mu^*}$ with $\bigcup_1^\infty E_n = E$, then*

$$\mu^*(A \cap E) = \bigvee_{n=1}^{\infty} \sum_{i=1}^{n} \mu^*(A \cap E_i). \tag{2}$$

*Consequently, every set of outer measure zero belongs to $M_{\mu^*}$ and the set function $\hat{\mu}$ defined for $E$ in $M_{\mu^*}$ by $\hat{\mu}(E) = \mu^*(E)$ is a complete $V \cup \{\infty\}$-valued measure on $M_{\mu^*}$.*

Proof. To prove (2) observe that by equation (1) of Lemma 2.4, for each $n$, we have

$$\mu^*\left(A \cap \left(\bigcup_1^n E_i\right)\right) = \sum_{i=1}^{n} \mu^*(A \cap E_i)$$

for every $A$ in $\mathscr{H}$ and that $\bigcup_{i=1}^{n} E_i \in M_{\mu^*}$. Hence for each $n$,

$$\begin{aligned}\mu^*(A) &= \mu^*\left(A \cap \left(\bigcup_{i=1}^{n} E_i\right)\right) + \mu^*\left(A \setminus \bigcup_{i=1}^{n} E_i\right)\\ &\geq \sum_{i=1}^{n} \mu^*(A \cap E_i) + \mu^*\left(A \setminus \bigcup_{i=1}^{\infty} E_i\right).\end{aligned}$$

Now taking the supremum on both sides of the above inequality as $n$ varies from 1 to $\infty$, we obtain

$$\mu^*(A) \geq \bigvee_{n=1}^{\infty} \sum_{i=1}^{n} \mu^*(A \cap E_i) + \mu^* \left(A \setminus \bigcup_{i=1}^{\infty} E_i\right)$$

so that

$$\mu^*(A) \geq \mu^* \left(A \cup \left(\bigcup_{i}^{\infty} E_i\right)\right) + \mu^* \left(A \setminus \bigcup_{i=1}^{\infty} E_i\right)$$

as $\mu^*$ is countably subadditive. Replacing $A$ by $A \cap E$ in the above inequality we obtain (2). The rest of the lemma follows on similar lines as the numerical analogues in Theorems A, B and C, §11 of Halmos [5].

### 3. The inner measure $\mu_*$ induced by a $V \cup \{\infty\}$-valued measure $\mu$

In this section as a preliminary to the Carathéodory extension procedure of vector lattice-valued measures, we develop the theory of vector lattice-valued inner measures induced by vector lattice-valued measures.

We fix the following notations in the sequel. $\mathcal{R}$ is a ring of subsets of a set $X$, $\mu$ is a $V \cup \{\infty\}$-valued measure on $\mathcal{R}$ where $V$ is a boundedly $\sigma$-complete vector lattice and $\mathcal{R}_\sigma = \left\{E \subseteq X \colon E = \bigcup_{1}^{\infty} E_n,\ E_n \in \mathcal{R}\right\}$. We say that $\mu(E) < \infty$ or $\mu(E)$ is finite if $\mu(E) \in V$.

**Lemma 3.1.** *Let $\mu$ be a $V \cup \{\infty\}$-valued measure on $\mathcal{R}$. If $\{E_n\}_n$ is an increasing (decreasing) sequence of sets in $\mathcal{R}$ with $\bigcup_{1}^{\infty} E_n \in \mathcal{R}$ $\left(\bigcap_{1}^{\infty} E_n \in \mathcal{R}\right.$ and $\mu\ (E_n) < \infty$ for some $\left.n\right)$, then*

$$\mu\left(\bigcup_{1}^{\infty} E_n\right) = \bigvee_{1}^{\infty} \mu(E_n) \left(\mu\left(\bigcap_{1}^{\infty} E_n\right) = \bigwedge_{1}^{\infty} \mu(E_n)\right).$$

Proof. The statement for an increasing sequence is an easy consequence of the countable additivity of $\mu$. In the decreasing case the result follows from Lemma 3.1 of Wright [16] and Theorem III.2.2 of Vulikh [15].

**Lemma 3.2.** *Let $A$ be in $\mathcal{R}_\sigma$ with $A = \bigcup_{1}^{\infty} E_m = \bigcup_{1}^{\infty} F_n$, where $\{E_n\}_n$ and $\{F_n\}_n$ are increasing sequences of members of $\mathcal{R}$. Then*

$$\bigvee_{1}^{\infty} \mu(E_n) = \bigvee_{1}^{\infty} \mu(F_n)$$

*if $\mu$ is a $V \cup \{\infty\}$-valued measure on $\mathcal{R}$.*

Proof. Let $A_{n,k}=E_n\cap F_k$. Then $A_{n,k}\nearrow E_n\cap A=E_n$ as $k\to\infty$. Hence by Lemma 3.1

$$\mu(E_n)=\bigvee_{k=1}^{\infty}\mu(A_{n,k})\leqslant\bigvee_{k=1}^{\infty}\mu(F_k) \tag{3}$$

and thus

$$\bigvee_{n=1}^{\infty}\mu(E_n)\leqslant\bigvee_{k=1}^{\infty}\mu(F_k). \tag{4}$$

The reverse inequality can be proved similarly.

**Definition 3.3.** *Let $A$ be in $\mathcal{R}_\sigma$ and $\mu$ be a $V\cup\{\infty\}$-valued measure on the ring $\mathcal{R}$. Then the inner measure $\mu*$ induced by $\mu$ is defined on $\mathcal{R}_\sigma$ by*

$$\mu*(A)=\bigvee_{1}^{\infty}\mu(E_n)$$

*where $\{E_n\}_n$ is an increasing sequence of members of $\mathcal{R}$ with $\bigcup_{1}^{\infty}E_n=A$.*

Note that if $A$ is in $\mathcal{R}_\sigma$, by definition of $\mathcal{R}_\sigma$, $A=\bigcup_{1}^{\infty}F_n$, $F_n\in R$. Taking $E_n=\bigcup_{i=1}^{n}F_i$, we see that $A=\bigcup_{1}^{\infty}E_n$ and $\{E_n\}_n$ is an increasing sequence of members of $\mathcal{R}$. Thus $\mu*$ has $\mathcal{R}_\sigma$ as its domain.

**Lemma 3.4.** *$\mu*|\mathcal{R}=\mu$. Further $\mu*$ is finitely additive, monotone and $V^+\cup\{\infty\}$-valued on $\mathcal{R}_\sigma$.*

Proof. The first statement follows from the definition of $\mu*$ and Lemma 3.2. The monotoneity and the non-negativeness of the range of $\mu*$ are evident. We shall now prove the finite additivity of $\mu*$.

Let $A$, $B$ be in $\mathcal{R}_\sigma$ with $A\cap B=\emptyset$. If $A=\bigcup_{1}^{\infty}E_n$ and $B=\bigcup_{1}^{\infty}F_n$, where $\{E_n\}_n$ and $\{F_n\}_n$ are increasing sequences of members of $\mathcal{R}$, then obviously by Definition 3.3

$$\mu*(A\cup B)=\bigvee_{n=1}^{\infty}\mu(E_n\cup F_n).$$

If either $\mu*(A)=\infty$ or $\mu*(B)=\infty$, then by the monotoneity of $\mu*$, $\mu*(A\cup B)=\infty=\mu*(A)+\mu*(B)$. Let $\mu*(A)$ and $\mu*(B)$ be finite. Let $\mu*(A)+\mu*(B)=h\in V$.

Then $V[h]$ is boundedly $\sigma$-complete and $V[h]\simeq C(S)$, a $\sigma$-Stone algebra by Theorem 1.2. Then as $\mu$ is additive on $\mathcal{R}$ and as $E_n\cap F_n=\emptyset$ for $n=1,2,\ldots$,

$$\mu*(A\cup B)=\bigvee_{1}^{\infty}\mu(E_n\cup F_n)=\bigvee_{1}^{\infty}(\mu(E_n)+\mu(F_n))\leqslant h.$$

Thus $\mu*(A\cup B)\in V[h]\simeq C(S)$. Let us identify $V[h]$ with $C(S)$. By the dual result of Lemma $K$ of Wright [18] and by the fact that the finite union of $\sigma$-meagre sets is $\sigma$-meagre, there exists a $\sigma$-meagre subset $M$ of $S$ such that for $s \in S \backslash M$

$$\mu*(A \cup B)(s) = \sup_n \{\mu(E_n) + \mu(F_n)\}(s) = \lim_n \{\mu(E_n) + \mu(F_n)\}(s)$$
$$= \lim_n \mu(E_n)(s) + \lim_n \mu(F_n)(s) = \sup_n \mu(E_n)(s) + \sup_n \mu(F_n)(s)$$
$$= \mu*(A)(s) + \mu*(B)(s).$$

Since $\mu*(A \cup B)$, $\mu*(A) + \mu*(B)$ are in $C(S)$ and differ on a meagre subset of $S$, $\mu*(A \cup B) = \mu*(A) + \mu*(B)$ by Theorem 34 Chapter 6 of Kelley [7]. Thus $\mu*$ is additive on $\mathcal{R}_\sigma$ and hence finitely additive on $\mathcal{R}_\sigma$ by finite induction.

**Lemma 3.5.** *If $\{A_n\}_n$ is an increasing sequence of members in $\mathcal{R}_\sigma$ with $\bigcup_1^\infty A_n = A$, then $A \in \mathcal{R}_\sigma$ and*

$$\mu*(A) = \bigvee_1^\infty \mu*(A_n).$$

Proof. For each $n$, let $A_n = \bigcup_{j=1}^\infty E_{n,j}$, $\{E_{n,j}\}_{j=1}^\infty$ being an increasing sequence of members of $\mathcal{R}$. If $B_n = \bigcup_{i,j=1}^n E_{i,j}$, then $B_n \subseteq A_n$ and $\{B_n\}_n$ is an increasing sequence of members of $\mathcal{R}$ with $A = \bigcup_1^\infty B_n$. Hence $A$ is in $\mathcal{R}_\sigma$. Now, by Definition 3.3 and Lemma 3.4, we have

$$\mu*(A) = \bigvee_1^\infty \mu(B_n) \leqslant \bigvee_1^\infty \mu*(A_n) \leqslant \mu*(A).$$

**Lemma 3.6** *$\mu*$ is countably subadditive on $\mathcal{R}_\sigma$.*

Proof. Let $\{A_n\}_n$ be a sequence of members of $\mathcal{R}_\sigma$ with their union $A$. Evidently by Lemma 3.4. $\mu*$ is finitely subadditive. Hence by Lemma 3.5

$$\mu*(A) = \bigvee_{n=1}^\infty \mu* \left(\bigcup_{i=1}^n A_i\right) \leqslant \bigvee_{n=1}^\infty \sum_{i=1}^n \mu*(A_i).$$

## 4. Carathéodory extension of vector lattice-valued measures

In this section we prove mainly that the Carathéodory extension procedure is valid for bounded $V$-valued and suitably restricted $V \cup \{\infty\}$-valued measures on a ring $\mathcal{R}$ of subsets of $X$ when $V$ is a weakly $(\sigma, \infty)$-distributive vector lattice. The classical Carathéodory extension of extended real valued measures follows as a particular case of this result.

$\mathscr{R}$ will denote a ring of sets, and $\mathscr{S}(\mathscr{R})$ ($\mathscr{H}(\mathscr{R})$) will he the $\sigma$-ring (hereditary $\sigma$-ring) generated by $\mathscr{R}$ in the sequel.

**Definition 4.1.** *Let $\mu$ be a $V \cup \{\infty\}$-valued measure on $\mathscr{R}$ and $\mu_*$ on $\mathscr{R}_\sigma$ be the inner measure induced by $\mu$. The set function $\mu^*$ on $\mathscr{H}(\mathscr{R})$ induced by $\mu$ is defined by*

$$\mu^*(A) = \bigwedge_{\hat{V}} \{\mu_*(F) : A \subseteq F \in \mathscr{R}_\sigma\}$$

*for $A \in \mathscr{H}(\mathscr{R})$, where $\hat{V}$ is the Dedekind completion of $V$.*

**Lemma 4.2.** *If $\mu$ is a $V \cup \{\infty\}$-valued measure on $\mathscr{R}$, then $\mu^*$ is a $\hat{V}^+ \cup \{\infty\}$-valued set function on $\mathscr{H}(\mathscr{R})$. $\mu^* | \mathscr{R}_\sigma = \mu_*$ and $\mu^*$ is monotone.*

Proof. The first statement follows from Lemma 3.4 and Definition 4.1. The restriction of $\mu^*$ to $\mathscr{R}_\sigma$ coincides with $\mu_*$ by the monotoneity of $\mu_*$. The monotoneity of $\mu^*$ is obvious from Definition 4.1.

**Definition 4.3.** *A $V$-valued measure $\mu$ on $\mathscr{R}$ is said to be bounded if there exists an $h \in V^+$ such that $\mu(E) \leqslant h$ for every $E$ in $\mathscr{R}$. Then we say $\mu$ is bounded by $h$.*

Note that a $V$-valued measure $\mu$ on an algebra $\mathscr{R}$ of subsets of a set $X$ is necessarily bounded by $\mu(X)$.

**Lemma 4.4.** *Let $\mu$ be a bounded $V$-valued measure on $\mathscr{R}$, with $\mu(E) \leqslant h$ for all $E \in \mathscr{R}$. Then $\mu_*(F) \leqslant h$ for all $F \in \mathscr{R}_\sigma$. Consequently, $\mu^*(A) \leqslant h$ for all $A \in H(\mathscr{R})$, where $\mu^*$ is the set function on $H(\mathscr{R})$ induced by $\mu$.*

Proof. If $F \in \mathscr{R}_\sigma$, then $F = \bigcup_1^\infty E_n$, $\{E_n\}_n$ an increasing sequence of members of $\mathscr{R}$. Thus $\mu_*(F) = \bigvee_1^\infty \mu(E_n) \leqslant h$. The last part follows from the first part and Definition 4.1.

**Lemma 4.5.** (Countable subadditivity lemma) *If $V$ is a weakly $(\sigma, \infty)$-distributive vector lattice and if $\mu^*$ is the set function induced by a $V \cup \{\infty\}$-valued measure $\mu$ on the ring $\mathscr{R}$ of sets, then*

$$\mu^*\left(\bigcup_{i=1}^\infty A_i\right) \leqslant \bigvee_{n=1}^\infty \sum_{i=1}^n \mu^*(A_i) \tag{5}$$

*when $\mu^*\left(\bigcup_{i=1}^\infty A_i\right) \in V$, where $A_i \in \mathscr{H}(\mathscr{R})$, $i = 1, 2, \ldots$.*

Proof. If the right-hand side of (5) is infinity, trivially inequality (5) holds. Hence let $\bigvee_{n=1}^\infty \sum_{i=1}^n \mu^*(A_i) = h_1 \in \hat{V}$.

By hypothesis that $\mu^*\left(\bigcup_{i=1}^\infty A_i\right) \in V$ and by Definition 4.1 there exists an $F_0 \in \mathscr{R}_\sigma$

such that $\mu_*(F_0)\in\hat{V}$ and $\bigcup_{i=1}^{\infty} A_i\subseteq F_0$. Let $\mu_*(F_0)=h_2$. Let $h=h_1\vee h_2$ in $V$. Then as $V$ is weakly $(\sigma,\infty)$-distributive, $\hat{V}$ and $\hat{V}[h]$ are weakly $(\sigma,\infty)$-distributive by Proposition 1.4 and Definition 1.3. Further, by Theorem 1.2, $\hat{V}[h]\simeq C(S)$, a weakly $(\sigma,\infty)$-distributive Stone algebra. In the proof we shall hereafter identify $\hat{V}[h]$ with $C(S)$.

From Definition 4.1.,

$$\mu^*(A_i)=\bigwedge_{\hat{V}}\{\mu_*(F):\ A_i\subseteq F\in\mathcal{R}_\sigma\}. \tag{6}$$

For $A_i\subseteq F\in\mathcal{R}_\sigma$, $F\cap F_0\in\mathcal{R}_\sigma$ and $\mu_*(F\cap F_0)\leqslant\mu_*(F)$. Hence

$$\mu^*(A_i)\leqslant\bigwedge_{\hat{V}}\{\mu_*(F\cap F_0):A_i\subseteq F\in\mathcal{R}_\sigma\}\leqslant\bigwedge_{\hat{V}}\{\mu_*(F):\ A_i\subseteq F\in\mathcal{R}_\sigma\}=\mu^*(A_i)$$

by (6). Thus for each $i$

$$\mu^*(A_i)=\bigwedge_{\hat{V}}\{\mu_*(F\cap F_0):\ A_i\subseteq F\in\mathcal{R}_\sigma\}=\bigwedge_{\hat{V}}\{\mu_*(F):A_i\subseteq F\in\mathcal{R}_\sigma,\ F\subseteq F_0\} \tag{6'}$$

so that $\mu^*(A_i)$ is realized as the infimum of a decreasing net of elements in $\hat{V}[h]\simeq C(S)$ for $i=1,2,\ldots$. Hence by Lemma 1.1 of Wright [16] there exists a meagre set $M_i\subseteq S$ such that

$$\mu^*(A_i)(s)=\inf\{\mu_*(F)(s):\ A_i\subseteq F\in\mathcal{R}_\sigma,\ F\subseteq F_0\}$$

for $s\in S\backslash M_i$. This holds for $i=1,2,\ldots$. Since a countable union of meagre sets is meagre, $M=\bigcup_{i=1}^{\infty} M_i$ is meagre and for $S\in S\backslash M$ and for $i=1,2,\ldots$,

$$\mu^*(A_i)(s)=\inf\{\mu_*(F)(s):\ A_i\subseteq F\in\mathcal{R}_\sigma,\ F\subseteq F_0\}.$$

Since $\hat{V}[h]\simeq C(S)$ is weakly $(\sigma,\infty)$-distributive, by Definition 1.3 the meagre set $M$ is nowhere dense in $S$, so that $S\backslash\bar{M}$ is open and dense in $S$. Let $s_0\in S\backslash\bar{M}$. Then there exists a clopen neighbourhood $K$ of $s_0$ such that $K\subseteq S\backslash\bar{M}$. Then the decreasing net $\{\mu_*(F)\chi_K:\ A_i\subseteq F\in\mathcal{R}_\sigma,\ F\subseteq F_0\}$ descends pointwise on the compact set $S$ to $\mu_*(A_i)\chi_K$, where $\chi_K$ is the characteristic function of $K$ and hence by Dini's Theorem the convergence is uniform. Hence given $\varepsilon>0$, for each positive integer $i$, there exists an $F_i\in\mathcal{R}_\sigma$, $F_0\supseteq F_i\supseteq A_i$ so that $\mu_*(F_i)\in C(S)$ such that

$$\mu^*(A_i)\chi_K+\varepsilon/2^i\geqslant\mu_*(F_i)\chi_K.$$

Hence

$$\sum_{i=1}^{n}\mu^*(A_i)\chi_K+\sum_{1}^{n}\varepsilon/2^i\geqslant\sum_{i=1}^{n}\mu_*(F_i)\chi_K$$

so that

$$\bigvee_{n=1}^{\infty} \sum_{i-1}^{n} \mu^*(A_i)\chi_K + \varepsilon \geqslant \bigvee_{n=1}^{\infty} \left(\sum_{i=1}^{n} \mu^*(A_i)\chi_K + \sum_{i=1}^{n} \varepsilon/2^i\right)$$

$$\geqslant \bigvee_{n=1}^{\infty} \sum_{i=1}^{n} \mu_*(F_i)\chi_K. \tag{7}$$

By hypothesis $\bigvee_{n=1}^{\infty} \sum_{i-1}^{n} \mu^*(A_i) \in C(S)$ and it is clear that $\mu^*\left(\bigcup_{1}^{\infty} F_i\right) \in C(S)$. Hence by the dual result of corollary on p. 109 of Wright [16] and by the fact that $\mu_*$ is finitely subadditive, we have from the inequality (7) that

$$\left\{\bigvee_{n-1}^{\infty} \sum_{i=1}^{n} \mu^*(A_i)\right\} \chi_K + \varepsilon = \bigvee_{n=1}^{\infty} \left(\sum_{i=1}^{n} \mu^*(A_i)\chi_K\right) + \varepsilon$$

$$\geqslant \bigvee_{n=1}^{\infty} \left(\sum_{i=1}^{n} \mu_*(F_i)\chi_K\right) \geqslant \bigvee_{n=1}^{\infty} \left(\mu_* \left(\bigcup_{i=1}^{n} F_i \sigma\ \chi_K\right)\right.$$

$$\geqslant \mu_* \left(\bigcup_{i=1}^{\infty} F_i\right) \chi_K \geqslant \mu^* \left(\bigcup_{i=1}^{\infty} A_i\right) \chi_K. \tag{8}$$

Since $\varepsilon$ is arbitrary, giving special attention to the inequality (8) at $s_0$

$$\left(\bigvee_{n=1}^{\infty} \sum_{i=1}^{n} \mu^*(A_i)\right) (s_0) \geqslant \mu^* \left(\bigcup_{i=1}^{\infty} A_i\right) (s_0).$$

Since $s_0$ is arbitrary in the dense set $S \backslash \bar{M}$ and since $\bigvee_{n=1}^{\infty} \sum_{i=1}^{n} \mu^*(A_i)$ and $\mu^* \left(\bigcup_{i-1}^{\infty} A_i\right)$ are in $C(S)$, the above inequality implies that

$$\bigvee_{n=1}^{\infty} \sum_{i=1}^{n} \mu^*(A_i) \geqslant \mu^* \left(\bigcup_{i=1}^{\infty} A_i\right)$$

in $C(S)$ and hence in $\hat{V}$.

This completes the proof of the lemma.

**Definition 4.6.** *If $\{g_n\}_n$ is a sequence of functions in a Stone algebra $C(S)$, we say that $\bigvee_{n=1}^{\infty} g_n = \infty$ if there exists no $g \in C(S)$ such that $g \geqslant g_n$ for every $n$. We say that $\bigvee_{n=1}^{\infty} g_n$ is strictly infinity* (strict. $\infty$ *in notation*) *if for each non-null clopen subset $K$ of* $S \bigvee_{n=1}^{\infty} (g_n \chi_K) = \infty$, where $\chi_K$ denotes the characteristic function of $K$.

We observe that the supremum of any unbounded sequence of non-negative constant functions in $C(S)$ is strict $\infty$ in the above sense.

**Definition 4.7.** *Let $C(S)$ be a Stone algebra. A $C(S)\cup\{\infty\}$-valued measure $\mu$ on $\mathcal{R}$ of sets is said to be strictly infinity $C(S)$-valued ($C(S)\cup$(strict. $\infty$)-valued in notation) if for each increasing sequence $\{E_n\}_n$ of sets in $\mathcal{R}$ with $\mu(E_n)\in C(S)$ and $\{\mu(E_n)\}$ not bounded above, $\bigvee_1^\infty \mu(E_n)=\text{strict.}\,\infty$, in the sense of Definition 4.6.*

If $\mu$ is an extended real valued measure on $\mathcal{R}$, then observe that $\mu$ is strictly infinity valued.

**Lemma 4.8.** *If $\mu$ is a $C(S)\cup\{\text{strict.}\,\infty\}$-valued measure on a ring $\mathcal{R}$ of sets, where $C(S)$ is a weakly $(\sigma, \infty)$-distributive Stone algebra, the set function $\mu^*$ induced by $\mu$ is countably subadditive on $H(\mathcal{R})$.*

Proof. Let $\{A_i\}_i$ be a sequence of sets in $\mathcal{H}(\mathcal{R})$, with $A=\bigcup_1^\infty A_i$. If $\mu^*(A)$ is finite, then by Lemma 4.5,

$$\mu^*(A)\leq \bigvee_{n=1}^{\infty} \sum_{i=1}^{n} \mu^*(A_i).$$

Thus it suffices to prove that if $\mu^*(A)=\infty$, then $\bigvee_{n=1}^{\infty} \sum_{i=1}^{n} \mu^*(A_i)=\infty$. If possible, let $\bigvee_{n=1}^{\infty} \sum_{i=1}^{n} \mu^*(A_i)=h_1\in C(S)$ when $\mu^*(A)=\infty$. Since

$$\mu^*(A_i)=\bigwedge_{C(S)}\{\mu_*(F): A_i\subseteq F\in \mathcal{R}_\sigma\}$$

and since $\mu^*(A_i)$ is finite, there exists $G_i\in\mathcal{R}_\sigma$ such that $A_i\subseteq G_i$ and $\mu_*(G_i)\in C(S)$. As discussed at the beginning of the proof of Lemma 4.5, it can be shown that for $i=1, 2, \ldots$

$$\mu^*(A_i)=\bigwedge_{C(S)}\{\mu_*(F): A_i\subseteq F\in\mathcal{R}_\sigma,\ F\subseteq G_i\}$$

so that each member in the infimum collection is in $C(S)$. Hence by Lemma 1.1. of Wright [16] there exists a meagre set $M_i\subseteq S$ such that for $s\in S\backslash M_i$

$$\mu^*(A_i)(s)=\inf\{\mu_*(F)(s): A_i\subseteq F\in\mathcal{R}_\sigma,\ F\subseteq G_i\}.$$

Hence $M=\bigcup_{i=1}^{\infty} M_i$ is meagre and for $s\in S\backslash M$ and for $i=1, 2, \ldots$

$$\mu^*(A_i)(s)=\inf\{\mu_*(F)(s): A_i\subseteq F\in\mathcal{R}_\sigma,\ F\subseteq G_i\}.$$

Let $s_0\in S\backslash \bar{M}$. Then there is a clopen neighbourhood $K$ of $s_0$ such that $K\subseteq S\backslash M$. By an argument similar to the derivation of inequality (7), given $\varepsilon>0$, there exist sets $F_i\in\mathcal{R}_\sigma$ with $A_i\subseteq F_i\subseteq G_i$ such that

$$\mu^*(A_i)\chi_K+\frac{\varepsilon}{2^i}\geq \mu_*(F_i)\chi_K$$

and hence

$$\bigvee_{n=1}^{\infty}\left(\sum_{i=1}^{n}\mu^*(A_i)\chi_K\right)+\varepsilon\leq\bigvee_{n=1}^{\infty}\left[\left(\sum_{i=1}^{n}\mu_*(F_i)\right)\chi_K\right]\geq\bigvee_{n=1}^{\infty}\left(\mu_*\left(\bigcup_{i=1}^{n}F_i\right)\chi_K\right). \qquad (9)$$

By Lemma 3.5., $\mu_*\left(\bigcup_1^{\infty}F_n\right)=\bigvee_{n=1}^{\infty}\mu_*\left(\bigcup_{i=1}^{n}F_i\right)$. But $\bigcup_1^{\infty}F_n\supseteq A$ and since $\mu^*(A)=\infty$, by Definition 4.1, $\mu_*\left(\bigcup_1^{\infty}F_n\right)=\infty$. Let $\bigcup_{i-1}^{n}F_i=L_n$. Then $L_n\in\mathscr{R}_\sigma$ and hence let $L_n=\bigcup_{j=1}^{\infty}E_{n,j}$, where $\{E_{n,j}\}_{j=1}^{\infty}$ is an increasing sequence of members of $\mathscr{R}$. Then $B_n=\bigcup_{j-1}^{n}E_{n,j}$ is an increasing sequence of sets in $\mathscr{R}$, with $\bigcup_1^{\infty}B_n=\bigcup_1^{\infty}L_n=\bigcup_1^{\infty}F_n$. Thus

$$\infty=\mu_*\left(\bigcup_1^{\infty}F_n\right)=\bigvee_1^{\infty}\mu_*(B_i).$$

Since $\{B_i\}_i$ is an increasing sequence of sets in $\mathscr{R}$. Since $\mu_*|\mathscr{R}=\mu$, $\bigvee_1^{\infty}\mu(B_i)=\infty$.

$$\mu(B_n)=\mu\left(\bigcup_{j=1}^{n}E_{n,j}\right)\leq\mu_*\left(\bigcup_{i=1}^{n}L_i\right)$$

$$=\mu_*(L_n)=\mu_*\left(\bigcup_1^{n}F_i\right)\leq\sum_1^{n}\mu_*(F_i)\in C(S)$$

since each $\mu_*(F_i)\in C(S)$. Thus $\{P_n\}_n$ is an increasing sequence of sets in $\mathscr{R}$ with $\mu(B_n)\in C(S)$ and $\{\mu(B_n)\}_1^{\infty}$ is not bounded above. This implies by the hypothesis on $\mu$ that

$$\bigvee_{n=1}^{\infty}(\mu(B_n)\chi_K)=\infty.$$

But

$$\mu(B_n)\chi_K\leq\mu_*\left(\bigcup_1^{n}F_i\right)\chi_K$$

so that

$$\bigvee_{n=1}^{\infty}\left(\mu_*\left(\bigcup_{i=1}^{n}F_i\right)\chi_K\right)\geqq\bigvee_1^{\infty}(\mu(B_n)\chi_K)=\infty.$$

This contradicts inequality (9) and hence the lemma.

**Definition 4.9.** *Let $\mu$ be a $V\cup\{\infty\}$-valued measure on a ring $\mathscr{R}$ of sets. We say that $\mu$ is $V\cup\{$strict. $\infty\}$-valued on $\mathscr{R}$ if there exists an $h\in V^+$ such that $\mu$ is $\hat{V}[h]\cup\{$strict. $\infty\}$-valued and that for $E\in\mathscr{R}$ with $\mu(E)\in V$, $\mu(E)$ is in $V[h]$.*

Remark. Any bounded $V$-valued measure $\mu$ on $\mathscr{R}$ is vacously $V\cup\{$strict. $\infty\}$-valued on $\mathscr{R}$.

**Theorem 4.10.** (Outer measure theorem) *Let $V$ be a weakly $(\sigma, \infty)$-distributive vector lattice and let $\mu$ be a $V \cup \{\text{strict.}\,\infty\}$-valued measure on a ring $\mathcal{R}$ of subsets of a set $X$. Then the set function $\mu^*$ on $H(\mathcal{R})$ induced by $\mu$ is a $\hat{V} \cup \{\infty\}$-valued outer measure on $\mathcal{H}(\mathcal{R})$ and is an extension of $\mu$. If $\mu$ is bounded by $h$, then $\mu^*(A) \leq h$ for all $A \in \mathcal{H}(\mathcal{R})$.*

Proof. In view of Lemmas 4.2, 3.4 and 4.4, it suffices to show that $\mu^*$ is countably subadditive. But by hypothesis, there exists an $h_1 \in V^+$ such that $\mu$ is $\hat{V}[h_1] \cup \{\text{strict.}\,\infty\}$-valued and $\hat{V}[h_1]$ is a weakly $(\sigma, \infty)$-distributive Stone algebra. Then by Lemma 4.8 $\mu^*$ is countably subadditive on $\mathcal{H}(\mathcal{R})$. Thus if $\{A_i\}_i$ is a sequence of sets in $\mathcal{H}(\mathcal{R})$, then

$$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \underset{\hat{V}[h_1]}{\bigvee_{n=1}^{\infty}} \sum_{i=1}^{n} \mu^*(A_i) = \underset{\hat{V}}{\bigvee_{n=1}^{\infty}} \sum_{i=1}^{n} \mu^*(A_i)$$

so that $\mu^*$ is a $\hat{V} \cup \{\infty\}$-valued outer measure on $\mathcal{H}(\mathcal{R})$.

**Definition 4.11.** *When the set function $\mu^*$ induced by $\mu$ becomes a $\hat{V} \cup \{\infty\}$-valued outer measure on $\mathcal{H}(\mathcal{R})$, $\mu^*$ will be called the outer measure induced by $\mu$.*

**Lemma 4.12.** *Let $\mu$ be a $V \cup \{\text{strict.}\,\infty\}$-valued measure on a ring $\mathcal{R}$ of subsets of a set $X$ and $V$ be a weakly $(\sigma, \infty)$-distributive vector lattice. Then the set function $\mu^*$ induced by $\mu$ is a $\hat{V} \cup \{\infty\}$-valued outer measure on $\mathcal{H}(\mathcal{R})$ and $M_{\mu^*}$ is a $\sigma$-ring containing $\mathcal{S}(\mathcal{R})$, the $\sigma$-ring generated by $\mathcal{R}$.*

Proof. $\mu^*$ is a $\hat{V} \cup \{\infty\}$-valued outer measure on $\mathcal{H}(\mathcal{R})$ by Theorem 4.10 and $M_{\mu^*}$ is a $\sigma$-ring by Lemma 2.5. Thus the lemma follows if we prove that $\mathcal{R} \subseteq M_{\mu^*}$.

For this, let $E \in \mathcal{R}$ and $A \in H(\mathcal{R})$. Then

$$\mu^*(A) = \bigwedge_{\hat{V}} \{\mu_*(F) \colon A \subseteq F \in \mathcal{R}_\sigma\} = \bigwedge_{\hat{V}} \{\mu_*\{(F \cap E) \cup (F \backslash E)\},\ A \subseteq F \in \mathcal{R}_\sigma\}. \quad (10)$$

Since $E \in \mathcal{R}$ and $F \in \mathcal{R}_\sigma$, $F \cap E$ and $F \backslash E$ are in $\mathcal{R}_\sigma$ and hence by Lemma 3.4 (10) can be rewritten as

$$\mu^*(A) = \bigwedge_{\hat{V}} \{\mu_*(F \cap E) + \mu_*(F \backslash E) \colon A \subseteq F \in \mathcal{R}_\sigma\}.$$

Also,

$$\mu^*(F \cap E) + \mu^*(F \backslash E) \geq \mu^*(A \cap E) + \mu^*(A \backslash E).$$

Therefore,

$$\mu^*(A) = \bigwedge_{\hat{V}} \{\mu_*(F \cap E) + \mu_*(F \backslash E) \colon A \subseteq F \in \mathcal{R}_\sigma\} \geq \mu^*(A \cap E) + \mu^*(A \backslash E).$$

This completes the proof of the lemma.

Remark. The notion of a $\sigma$-finite $V \cup \{\infty\}$-valued measure can be introduced here and it can be shown that in Lemma 4.12, $\mu^*$ is $\sigma$-finite if $\mu$ is $\sigma$-finite.

**Lemma 4.13.** *Let $\mu$ be a $V \cup \{\text{strict.}\infty\}$-valued measure on $\mathcal{R}$ and let $V$ be a weakly $(\sigma, \infty)$-distributive vector lattice. Then the outer measure $\mu^*$ induced by $\mu$ is a complete $\hat{V} \cup \{\infty\}$-valued measure on $M_{\mu^*}$ extending $\mu$ to $M_{\mu^*}$ and $\mathcal{S}(\mathcal{R})$. If $\mu$ is bounded by $h$ in $V$, then $\mu - \mu^*|\mathcal{S}(\mathcal{R})$ is further $V$-valued and bounded by $h$. The extension $\bar{\mu} = \mu^*|\mathcal{S}(\mathcal{R})$ of $\mu$ to $\mathcal{S}(\mathcal{R})$ is unique when $\mu$ is $\sigma$-finite.*

Proof. By Theorem 4.10 and Lemma 2.5 $\mu^*$ is a complete $\hat{V} \cup \{\infty\}$-valued measure on $M_{\mu^*}$. The uniqueness of the extension $\bar{\mu}$ of $\mu$ to $\mathcal{S}(\mathcal{R})$ when $\mu$ is bounded or when $\mu$ is $\sigma$-finite follows from an argument analogous to the numerical case (proof of Theorem A, § 13 of Halmos [5]) due to the availability of Lemma 3.1. When $\mu$ is bounded by $h$ in $V$, $\mu^*$ and $\bar{\mu}$ are bounded by $h$, by Theorem 4.10.

Finally, we have to prove that the range of $\mu^*$ on $\mathcal{S}(\mathcal{R})$ is contained in $V$ if $\mu$ is bounded by $h$. Let $\mathcal{I}$ be the collection of all sets $A$ in $M_{\mu^*}$, for which $\mu^*(A) \in V$. $\mathcal{R} \subseteq \mathcal{I}$. In view of Theorem B, §6 of Halmo [5], it suffices to show that $\mathcal{I}$ is a monotone class. Since $\mu$ is bounded by $h$, $\mu^*(A) \leqslant h$ for every $A \in M_{\mu^*}$ by Theorem 4.10. Let $\{E_n\}_n$ be a monotone sequence of sets in $\mathcal{I}$.

Then as $\mu^*$ is a $\hat{V}$-valued mea ure on $M_{\mu^*}$, by Lemma 3.1.

$$\mu^*\left(\bigcup_1^\infty E_n\right) = \bigvee_1^\infty \mu^*(E_n) \quad (\text{if } \{E_n\}_n \text{ is increasing})$$

and

$$\mu^*\left(\bigcap_1^\infty E_n\right) = \bigwedge_1^\infty \mu^*(E_n) \quad (\text{if } \{E_n\}_n \text{ is decreasing}).$$

Consequently, as $V$ is boundedly $\sigma$-complete and $0 \leqslant \mu^*(E_n) \leqslant h \in V$ for all $n$, we obtain that $\mu^*\left(\bigcup_1^\infty E_n\right) \in V$ and $\mu^*\left(\bigcap_1^\infty E_n\right) \in V$. Thus $\mathcal{I}$ is a monotone class and hence $\mu^*$ is $V$-valued on $\mathcal{S}(\mathcal{R})$.

Thus in the foregoing lemmas of this section we have proved the following theorem.

**Theorem 4.14.** (Carath´odory extension theorem) *Let $\mu$ be a $V \cup \{\text{strict.}\infty\}$-valued measure on a ring $\mathcal{R}$ of subsets of a set $X$ and let $V$ be a weakly $(\sigma, \infty)$-distributive vector lattice, with $\hat{V}$ its Dedekind completion. Then $\mu^*$, the set function induced by $\mu$, is a $\hat{V} \cup \{\infty\}$-valued outer measure and $M_{\mu^*}$ is a $\sigma$-ring containing $\mathcal{S}(\mathcal{R})$. Further, $\mu^*$ is a complete $\hat{V} \cup \{\infty\}$-valued measure on $M_{\mu^*}$ and the restriction $\bar{\mu}$ of $\mu^*$ to $\mathcal{S}(\mathcal{R})$ is a $\hat{V} \cup \{\infty\}$-valued measure extending $\mu$ to $\mathcal{S}(\mathcal{R})$. If $\mu$ is further $\sigma$-finite on $\mathcal{R}$, so is $\mu^*$ on $\mathcal{H}(\mathcal{R})$ and $\bar{\mu} = \mu^*|\mathcal{S}(\mathcal{R})$ is a $\sigma$-finite $\hat{V} \cup \{\infty\}$-valued measure extending uniquely $\mu$ to $\mathcal{S}(\mathcal{R})$. If $\mu$ is a $V$-valued measure bounded by $h$ on $\mathcal{R}$, then $\bar{\mu} = \mu^*|\mathcal{S}(\mathcal{R})$ is a $V$-valued measure extending uniquely $\mu$ to $\mathcal{S}(\mathcal{R})$ and is also bounded by $h$.*

Remark. Since **R** is a weakly $(\sigma, \infty)$-distributive Stone algebra $C(S)$, where $S$ is a singleton with discrete topology and since any extended real valued measure is

**R**$\cup$\{strict.$\infty$\}-valued, the above theorem includes the classical Carathéodory extension theorem of numerical measures as a pasticular case.

## 5. Completion and outer regularity of vector lattice-valued measures

Throughout this section $V$ will denote a weakly $(\sigma, \infty)$-distributive vector lattice.

Let $\mu$ be a $V\cup\{\infty\}$-valued measure on a $\sigma$-ring $\mathcal{S}$. If $\tilde{\mathcal{S}}=\{E\cup N: E\in\mathcal{S}, N$ a subset of a set in $\mathcal{S}$ of $\mu$-measure zero$\}$ then $\tilde{\mathcal{S}}$ is a $\sigma$-ring. If $\tilde{\mu}$ is defined on $\tilde{\mathcal{S}}$ by $\tilde{\mu}(E\cup N)=\mu(E)$, then $\tilde{\mu}$ is a complete $V\cup\{\infty\}$-valued measure on $\tilde{\mathcal{S}}$. $\tilde{\mu}$ is called the completion of $\mu$ and $\tilde{\mathcal{S}}$ is called the completion of $\mathcal{S}$.

In this section we obtain a sufficient condition to obtain $M_{\mu^*}$ as $\mathcal{S}(\mathcal{R})$, where $\mu$ is a $\sigma$-finite $V\cup\{$strict.$\infty\}$-valued measure on a ring $\mathcal{R}$ of subsets of a set $X$. This result can be compared with the numerical analogue.

**Definition 5.1.** *Let $\mu$ be a $V\cup\{$strict.$\infty\}$-valued measure on a ring $\mathcal{R}$ of subsets of a set $X$, where $V$ is a weakly $(\sigma, \infty)$-distributive vector lattice and let $\mu^*$ be the outer measure on $\mathcal{H}(\mathcal{R})$ induced by $\mu$. Then $\mu$ is said to be outer regular if for each set $E$ in $\mathcal{H}(\mathcal{R})$, there is a set $F$ in $\mathcal{S}(\mathcal{R})$ such that*

(i) $E\subseteq F$;

(ii) *if* $G\in\mathcal{S}(\mathcal{R})$ *with* $G\subseteq F\backslash E$, *then* $\bar{\mu}(G)=0$ *and*

(iii) $\mu^*(E)=\bar{\mu}(F)$,

*where* $\bar{\mu}=\mu^*|\mathcal{S}(\mathcal{R})$.

A set $F$ in $\mathcal{S}(\mathcal{R})$ satisfying conditions (i) and (ii) above is called a measurable cover of $E$.

**Theorem 5.2.** *Let $V$ be a weakly $(\sigma, \infty)$-distributive vector lattice satisfying the countable chain condition. If $\mu$ is a $\sigma$-finite $V\cup\{$strict.$\infty\}$-valued measure on a ring $\mathcal{R}$ of subsets of a set $X$, then $\mu$ is outer regular.*

Proof. By hypothesis there is an $h\in V$ such that $\mu(E)\in V[h]$ if $E\in\mathcal{R}$ and $\mu(E)<\infty$ and $\mu$ is $\hat{V}[h]\cup($strict.$\infty)$-valued. Let $A\in\mathcal{H}(\mathcal{R})$.

Case 1. Let $\mu^*(A)<\infty$. Then clearly from the definition of $\mu^*$ it follows that $\mu^*(A)\in\hat{V}[h]=V[h]$ since $\hat{V}=V$ as $V$ satisfies the countable chain condition. Also the finiteness of $\mu^*(A)$ implies that there is a set $B_0$ in $\mathcal{R}_\sigma$ with

$$A\subseteq B_0, \quad \mu_*(B_0)\in V[h].$$

Then as in the derivation of (6′) we have

$$\mu^*(A)=\bigwedge_{V[h]}\{\mu_*(B): A\subseteq B\in\mathcal{R}_\sigma, B\subseteq B_0\}.$$

As $V$ satisfies the countable chain condition, by Theorem V. 1.2.1 of Vulikh [13], there exists a sequence $\{B_n\}_n$ of sets such that $A\subseteq B_n\subseteq B_0$, $B_n\in\mathcal{R}_\sigma$ and

$$\mu^*(A)=\bigwedge_{n-1}^{\infty}\mu^*(B_n). \tag{12}$$

Let $F_n=\bigcap_{i=1}^{n}B_i$. Then $F_n\supseteq A$, $F_n\in\mathscr{R}_\sigma$, $\{F_n\}_n$ is a decreasing sequence and $\mu^*(F_n)=\mu_*(F_n)\leq\mu_*(B_0)<\infty$ for each $n$. Let $F=\bigcap_1^{\infty}F_n$. Then $F\in\mathscr{S}(\mathscr{R})$ and $F\supseteq A$. By Lemma 3.1 and by the monotoneity of $\mu^*$,

$$\mu^*(A)\leq\mu^*(F)=\bar{\mu}(F)=\bigwedge_1^{\infty}\bar{\mu}(F_n)<\bigwedge_1^{\infty}\mu_*(B_n)=\mu^*(A).$$

Thus

$$\mu^*(A)=\bar{\mu}(F),\quad A\subseteq F\in\mathscr{S}(\mathscr{R}).$$

Let $G\in\mathscr{S}(\mathscr{R})$ with $G\subseteq F\backslash A$. Then

$$\bar{\mu}(F)=\mu^*(A)\leq\mu^*(F\backslash G)=\bar{\mu}(F\backslash G)=\bar{\mu}(F)-\bar{\mu}(G),$$

and hence $\mu(G)=0$. Thus $F$ is a measurable cover of $A$.

Case 2. Let $\mu^*(A)=\infty$. Since $\mu$ is $\sigma$-finite on $\mathscr{R}$, by the remark under Lemma 4.12 $\mu^*$ is $\sigma$-finite on $\mathscr{H}(\mathscr{R})$. Hence there exists a sequence $\{A_i\}_i$ of sets in $\mathscr{H}(\mathscr{R})$ with

$$A\subseteq\bigcup_1^{\infty}A_i,\quad \mu^*(A_i)<\infty\quad\text{for}\quad i=1,2,\dots.$$

Therefore by case 1, there exists a measurable cover $F_i$ in $\mathscr{S}(\mathscr{R})$ for each $A_i$. Let $F=\bigcup_{i=1}^{\infty}F_i$. Then $F\in\mathscr{S}(\mathscr{R})$ and $\mu^*(F)=\infty$. If $G\in\mathscr{S}(\mathscr{R})$ with $G\subseteq F\backslash A$, then

$$G\cap F_i\in\mathscr{S}(\mathscr{R})\quad\text{and}\quad G\cap F_i\subseteq F_i\backslash A\subseteq F_i\backslash A_i$$

so that $\bar{\mu}(G\cap F_i)=0$. Then $\bar{\mu}(G)-\bar{\mu}\left(\bigcup_1^{\infty}G\cap F_i\right)\leq\bigvee_{n-1}^{\infty}\sum_{i=1}^{n}\bar{\mu}(G\cap F_i)=0$. Thus $F$ is a measurable cover of $A$ and $\bar{\mu}(F)=\mu^*(A)=\infty$.

This completes the proof of the theorem.

**Proposition 5.3.** *Let $V$ be a weakly $(\sigma,\infty)$-distributive vector lattice. If $\mu$ is a $V\cup\{\text{strict.}\,\infty\}$-valued measure on $\mathscr{R}$ with $\mu^*$ its induced outer measure on $\mathscr{H}(\mathscr{R})$, then the following hold:*

(i) *If $E\in\mathscr{H}(\mathscr{R})$ with $F_1$ and $F_2$ as measurable covers, then*

$$\bar{\mu}(F_1\triangle F_2)=0.$$

(ii) *If $\mu$ is outer regular, then $\mu^*(E)=\bar{\mu}(F)$ for every measurable cover $F$ of $E$.*

(iii) *Further if $V$ satisfies the countable chain condition and $\mu$ is $\sigma$-finite, then*

$$\mu^*(E)=\bar{\mu}(F)\ \textit{for every measurable cover } F \textit{ of } E.$$

Proof. (i) follows by an argument similar to the numerical analogue in Halmos [5]. (ii) follows from (i) and (iii) follows from Theorem 5.4 and (ii) of the present proposition.

We state and prove the following main theorem of this section.

**Theorem 5.4.** *Let $V$ be a weakly $(\sigma, \infty)$-distributive vector lattice and $\mu$ a $\sigma$-finite $V \cup \{\text{strict.}\,\infty\}$-valued measure on a ring $\mathcal{R}$ of subsets of a set $X$. If $\mu^*$ is the outer measure induced by $\mu$ and if $\mu$ is outer regular, then $M_{\mu^*} = \mathcal{S}(\mathcal{R})$ and $\mu^*$ on $M_{\mu^*}$ is the completion of $\bar{\mu}$ on $\mathcal{S}(\mathcal{R})$, where $\bar{\mu} = \mu^* | \mathcal{S}(\mathcal{R})$.*

Proof. Clearly $\mathcal{S}(\mathcal{R}) \subseteq M_{\mu^*}$ since $\mu^*$ is complete on $M_{\mu^*}$. It is easy to check that $\mu^* | \mathcal{S}(\mathcal{R}) = \tilde{\mu}$, where $\tilde{\mu}$ is the completion of $\bar{\mu}$ on $\mathcal{S}(\mathcal{R})$. Thus it suffices to show that $M_{\mu^*} \subseteq \mathcal{S}(\mathcal{R})$. Since by hypothesis $\mu$ is outer regular, and $\mu^*$ is $\sigma$-finite, the proof of this is similar to the numerical analogue in Halmos [4] and hence we omit details.

**Corollary 5.6.** *If $\mu$ is a $\sigma$-finite $V \cup \{\text{strict.}\,\infty\}$-valued measure on a ring $\mathcal{R}$ of subsets of a set $X$ and if $V$ is a weakly $(\sigma, \infty)$-distributive vector lattice satisfying the countable chain condition, then $\mu$ is outer regular, $M_{\mu^*} = \mathcal{S}(\mathcal{R})$ and $\mu^*$ on $M_{\mu^*}$ is the completion of $\bar{\mu}$ on $\mathcal{S}(\mathcal{R})$, where $\bar{\mu} = \mu^* | \mathcal{S}(\mathcal{R})$.*

Proof. By Theorem 5.2 $\mu$ is outer regular. Now the corollary follows from the above theorem.

Remark. The corresponding analogue of the above corollary for a $\sigma$-finite extended real valued measure $\mu$ on $\mathcal{R}$ is a consequence of the fact that $\mathbf{R}$ is a weakly $(\sigma, \infty)$-distributive vector lattice satisfying the countable chain condition and that $\mu$ is $\mathbf{R} \cup \{\text{strict.}\,\infty\}$-valued.

## 6. Some applications to positive operator valued measures in banach spaces and some characterizations of extendable spectral measures

The notion of positive operator valued measures in Banach spaces has been introduced by us in [13]. In this section we give Carathéodory extension of bounded positive operator valued measures in Banach spaces as a particular case of Theorem 4.14 and consequently, the Carathéodory extension of spectral measures, defined on a ring of sets, is obtained as a corollary, the latter generalizing Theorem 4 of Panchapagesan [12]. We also obtain two characterizations of extendable spectral measures in Banach spaces, which extend Theorems 8 and 9 of Panchapagesan [12] to the case of a ring of sets. Then it is proved that if $X$ is a Banach space containing no subspace isomorphic to $c_0$, then a spectral measure $E(\cdot)$ on $X$, defined on a ring of sets $\mathcal{R}$ is extendable uniquely to a spectral measure $\bar{E}(\cdot)$ on the $\sigma$-ring $\mathcal{S}(\mathcal{R})$, generated by $\mathcal{R}$, if and only if the range of $E(\cdot)$ is bounded. This generalizes a known result on the extension of vector measures to spectral measures on Banach spaces. (See 'Theorem on Exten ion p. 178 of Kluvánek [8]). Lastly, Theorem 6 of Panachapages n [12] is extended to positive operator valued measures in separable Banach spaces.

Before dealing with the applications, we give some definitions and results from [13] to make this section self-contained.

**Definition 6.1.** *Let $\mathscr{R}$ be a ring of subsets of a set $\mathfrak{S}$. Let $P(\cdot)$ be a map: $\mathscr{R} \to W$, where $W$ is a $W^*(\|\cdot\|)$-algebra of operators on a complex Banach space $X$. (See [11] for definition of $W^*(\|\cdot\|)$ -algebras). Then $P(\cdot)$ is called a positive operator valued measure in W (abbreviated as PO-measure (in W)) on $\mathscr{R}$ ıf the range of $P(\cdot)$ is contained in $H(W)^+$ (the set of all real scalar type operators in W wıth their spectra contained in the set of all non-negative reals), and ıf $P(\ )$ is countably additive in the strong operator topology $\tau_s$ of W.*

*Further, the PO-measure (in W) $P(\cdot)$, is said to be a spectral measure (in W) if the range of $P(\cdot)$ is contained in the set of all projections in W.*

*A PO-measure (in W), $P(\cdot)$, defined on $\mathscr{R}$, is said to be bounded if there exists a $T \in H(W)^+$ such that $P(\sigma) \leqslant T$ for all $\sigma \in \mathscr{R}$.*

Throughout this section $P(\ )$ is a PO measure (ın $W$), where $W$ ıs a $W^*(\|\cdot\|)$ algebra of operators on $X$.

**Proposition 6.2.** *A PO-measure (in W), $P(\cdot)$, on $\mathscr{R}$ ıs a spectral measure ıf and only if $P(\cdot)$ is multiplicative, i.e. $P(\sigma \cap \delta) - P(\sigma)p(\delta)$ for $\sigma$, $\delta$ in $\mathscr{R}$.*

**Proposition 6.3.** *$H(W)$ is a boundedly complete vector lattice and $P(\cdot)$ is a PO-measure (in W) on $\mathscr{R}$ if and only if $P(\cdot)$ is a $H(W)$-valued measure in the sense of Definition 1.1. Further, $H(W)$ is hyperstonian and hence is a weakly $(\sigma, \infty)$-distributive Stone algebra.*

**Proposition 6.4.** *Let $\mathscr{P}$ be a $\sigma$-complete Boolean algebra (abbreviated as B.A.) of projections on X, in the sence of Bade [1]. If W is the algebra generated by $\mathscr{P}$ in the weak operator topology of $B(X)$, then W is a $W^*(\|\cdot\|)$-algebra under a suitable equivalent norm $\|\cdot\|$ on X. Further, W is the algebra generated by the complete B.A.$\bar{\mathscr{P}}^s$ of projections on X in the uniform operator topology of $B(X)$, where $\bar{\mathscr{P}}^s$ denotes the closure of $\mathscr{P}$ in the strong operator topology of $B(X)$.*

Now we study the applications of results in earlier sections to PO-measures in Banach spaces.

**Theorem 6.5.** (Carathéodory extension Theorem for bounded PO-measures (in W)) *Let $P(\cdot)$ be a bounded PO measure (in W), defined on a rıng $\mathscr{R}$ of subsets of a set $\mathfrak{S}$.*

(i) *Then there is a unique bounded PO-measure (in W) $P(\cdot)$ defined on $\mathscr{S}(\mathscr{R})$, the $\sigma$-ring of sets generated by $\mathscr{R}$, such that $\bar{P}(\cdot)|\mathscr{R} - P(\cdot)$. Further, $\bar{P}(\cdot)$ arises through the Carathéodory exten ion procedure (of §4).*

(ii) *$\bar{P}(\cdot)$ is a spectral measure (in W) if and only if $P(\cdot)$ is so.*

*Consequently, every spectral measure $E(\cdot)$ on $\mathscr{R}$ (See definition 6.6) with its range contained in a $\sigma$- complete B.A.$\mathscr{P}$ of projections on X is extendable uniquely to a spectral measure $\bar{E}(\cdot)$ on $\mathscr{S}(\mathscr{R})$, the $\sigma$-ring generated by $\mathscr{R}$, by the Carathéodory extension procedure and the range of $\bar{E}(\cdot)$ is contained in $\bar{\mathscr{P}}^s$.*

Proof. (i) The hypotheses of Theorem 4.14 are satisfied by $P(\cdot)$ due to Proposition 6.3 and hence by Theorem 4.14 there is a unique bounded $H(W)$-valued measure $\bar{P}(\cdot)$ on $\mathscr{S}(\mathscr{R})$, extending $P(\cdot)$. Further, this extension arises by the Carathéodory extension procedure of § 4. Again, as $\bar{P}(\cdot)$ is a bounded $H(W)$-valued measure on $\mathscr{S}(\mathscr{R})$, $\bar{P}(\cdot)$ is a bounded PO-measure (in $W$) on $\mathscr{S}(\mathscr{R})$ with range in $H(W)$, by the first part of Proposition 6.3.

(ii) It suffices to prove that $\bar{P}(\cdot)$ is spectral (in $W$) if $P(\cdot)$ is so. Let $P(\cdot)$ be a spectral measure (in $W$). Then $P(\sigma)$ is a projection in $W$ for each $\sigma \in \mathscr{R}$. Let $\mathscr{B}_1 = \{P(\sigma): \sigma \in \mathscr{R}\} \cup \{I - P(\sigma): \sigma \in \mathscr{R}\}$. Then $\mathscr{B}_1$ is a B.A. of projections and $\mathscr{B}_1 \subseteq \mathscr{B}$ the B.A. of all projections in $W$. If $\mathscr{B}_0$ is the $\sigma$-complete B.A. of projections which generates $W$ in the weak operator topology (See definition of $W^*(\|\cdot\|)$-algebras in [11]), then $\bar{\mathscr{B}}_0^s = \mathscr{B}$ and hence $\mathscr{B}$ is a complete B.A. of projections by Theorem 2.7 of Bade [1]. From the definition of $P_*(\cdot)$ and $P^*(\cdot)$ (corresponding to $\mu_*$ and $\mu^*$ respectively in § 4) it is clear that the ranges of $P_*(\cdot)$ and $P^*(\cdot)$ are contained in $\mathscr{B}$, as $\mathscr{B}$ is complete. Thus $P^*(\cdot)$ and hence $\bar{P}(\cdot)$ are projection valued in $W$., i.e. $\bar{P}(\cdot)$ is a spectral measure (in $W$) on $\mathscr{S}(\mathscr{R})$.

For proving the last part of the theorem, let $W$ be the weakly closed algebra generated by $\mathscr{P}^s$, which is a complete B.A. of projections by Theorem 2.7 of Bade [1]. By Proposition 6.4, $W$ is a $W^*(\|\cdot\|)$-algebra under a suitable equivalent norm $\|\cdot\|$ on $X$ and $E(\cdot)$ is a spectral measure (in $W$) on $\mathscr{R}$, with its range contained in $H(W)$. Now from (i) and (ii) of the theorem and from the fact that $\bar{\mathscr{P}}^s$ is the collection of all Projections in $W$, the last part of the theorem follows.

Remark. The above theorem is clearly a generalization of Theorem 7 of Berberian [2] to Banach spaces when the operators in the range of the PO-measure there commute with each other.

Now let us proceed to obtain some characterizations of extendable spectral measures in Banach spaces.

**Definition 6.6.** *Let $X$ be a complex Banach space and $E(\cdot)$ a set functions defined on a ring of sets $\mathscr{R}$ with values in $B(X)$, the algebra of all operators on $X$. Then $E(\cdot)$ is called a spectral measure on $X$ if it satisfies the following conditions:*

(i) $E(\emptyset) = 0$

(ii) $E(\sigma \cap \delta) = E(\sigma)E(\delta)$ *for* $\sigma, \delta \in \mathscr{R}$ *and*

(iii) $E\left(\bigcup_1^\infty \sigma_i\right) x = \lim_n \sum_1^n E(\sigma_i)x$, *for each* $x \in X$, *where* $\{\sigma_i\}_1^\infty$ *is a disjoint sequence of sets in* $\mathscr{R}$ *with their union in* $\mathscr{R}$.

When $E(\ )$ is a spectral measure, defined on a ring of sets $\mathscr{R}$, it is clear that the range of $E(\cdot)$, viz. $E(\mathscr{R})$, is commutative and is contained in the B.A. of projections

$$\mathscr{B}_0=\{E(\sigma)\colon \sigma\in\mathscr{R}\}\cup\{I-E(\sigma)\colon \sigma\in\mathscr{R}\}.$$

Using the results on extension of vector measures (See [8]) and the results in § 5 of Panchapagesan [12], we obtain the following characterization theorems of extendable spectral measures, defined on rings of sets.

**Theorem 6.7.** *Let $E(\cdot)$ be a spectral measure on a ring $\mathscr{R}$ of subsets of a set $\mathfrak{S}$, with its range in $B(X)$ Then $E(\cdot)$ can be extended to a spectral measure $E(\cdot)$ on the $\sigma$-ring $\mathscr{S}(\mathscr{R})$, generated by $\mathscr{R}$, if and only if for $x\in X$ $E(\mathscr{R})x$ is relatively weakly compact in the Banach space $X$, where*

$$E(\mathscr{R})x=\{E(\sigma)x\colon \sigma\in\mathscr{R}\}.$$

*When the extension $E(\ )$ exists as a spectral measure on $\mathscr{S}(\mathscr{R})$, then $E(\cdot)$ is unique.*

Proof. Let $\mathscr{B}_0=\{E(\sigma)\colon \sigma\in\mathscr{R}\}\cup\{I-E(\sigma)\colon \sigma\in\mathscr{R}\}$. Then $\mathscr{B}_0$ is a B.A. of projections on $X$ and $E(\mathscr{R})\subseteq\mathscr{B}_0$. If $E(\mathscr{R})x$ is relatively weakly compact for each $x\in X$, then

$$N(x)=\{Px\colon p\in\mathscr{B}_0\}=E(\mathscr{R})x\cup\{x-E(\mathscr{R})x\}$$

is also relatively weakly compact and hence by Lemma 5 of Panchapagesan [12] $\mathscr{B}_0$ is contained in a $\sigma$-complete B.A. of projections on $X$. Thus, in particular, the range of $E(\cdot)$ is contained in a $\sigma$-complete B.A. of projections on $X$. Hence $E(\cdot)$ is extendable uniquely to a spectral measure $E(\cdot)$ on $\mathscr{S}(\mathscr{R})$ by the last part of Theorem 6.5.

Conversely, if $E(\cdot)$ is extendable to a spectral measure $\bar{E}(\cdot)$ on $\mathscr{S}(\mathscr{R})$, then for $x\in X$ $\bar{E}(\cdot)x$ is a vector measure on $\mathscr{S}(\mathscr{R})$ which extends the vector measure $E(\cdot)x$ on $\mathscr{R}$ and hence by the 'Theorem on Extension' on p. 178 of Kluvánek [8], the range $E(\mathscr{R})x$ is relatively weakly compact in $X$.

The uniqueness of $\bar{E}(\cdot)$ on $\mathscr{S}(\mathscr{R})$ follows from the sufficiency part of the theorem and the last part of Theorem 6.5.

**Theorem 6.8.** *Let $E(\cdot)$ be a spectral measure on a ring $\mathscr{R}$ of subsets of a set $\mathfrak{S}$, with its range in $B(X)$. Then $E(\cdot)$ can be extended to a spectral measure $\bar{E}(\cdot)$ on the $\sigma$-ring $\mathscr{S}(\mathscr{R})$, generated by $\mathscr{R}$, if and only if the range of $E(\cdot)$ is contained in a $\sigma$-complete B.A. of projections on $X$. Then the extension $E(\cdot)$ on $\mathscr{S}(\mathscr{R})$ is also unique.*

Proof. The sufficiency part of the theorem follows from the last part of Theorem 6.5. The condition is also necessary. For, from Theorem 6.6 it follows that $E(\mathscr{R})x$ is relatively weakly compact for each $x\in X$ and so by repeating the argument in the first part of the proof of Theorem 6.6 we obtain that the range of $E(\cdot)$ is contained in a $\sigma$-complete B.A. of projections on $X$.

The part concerning the uniqueness of $\bar{E}(\cdot)$ on $\mathscr{S}(\mathfrak{R})$ is a consequence of Theorem 6.7.

When $X$ satisfies some extra conditions, we can show that the boundedness of the range of $E(\cdot)$ itself would suffice for the validity of Theorem 6.7. To this end we need the following definition from [8].

**Definition 6.9.** *A Banach space $X$ is said to have the B-P property if, given $\{x_n\}_1^\infty$, a sequence of elements of $X$ such that $\sum_{n\in\pi} x_n \in X$ for every finite set $\pi$ of possible natural numbers, with $\sum_{n=1}^{\infty} |\langle x_n, x^*\rangle| < \infty$ for every $x^* \in X^*$, the Banach dual of $X$, then there exists an element $x \in X$ with $x = \sum_1^\infty x_n$.*

**Theorem 6.10.** *If $X$ is a Banach space containing no subspace isomorphic to $c_0$, then a necessary and sufficient condition for a spectral measure $E(\cdot)$ on $X$, defined on a ring $\mathfrak{R}$ of subsets of a set $\mathfrak{S}$, to be extended uniquely to a spectral measure $\bar{E}(\cdot)$ on $\mathscr{S}(\mathfrak{R})$, the $\sigma$-ring generated by $\mathfrak{R}$, is that the range of $E(\cdot)$ on $\mathfrak{R}$ is bounded in $B(X)$. Consequently, when $X$ is a weakly complete Banach space, the spectral measure $E(\cdot)$ on $\mathfrak{R}$ is extendable uniquely to a spectral measure $\bar{E}(\cdot)$ on $\mathscr{S}(\mathfrak{R})$ if and only if the range of $E(\cdot)$ is bounded.*

Proof. If $X$ is a Banach space containing no subspace isomorphic to $c_0$, then $X$ has the B-P property by Theorem 5 of Bessaga and Pelczyński [3].

Let $\sup\{\|E(\sigma)\| : \sigma \in \mathfrak{R}\} = M < \infty$. Then for $x \in X$ $E(\cdot)x$ is a vector measure on the ring $\mathfrak{R}$ with its range $E(\mathfrak{R})x$ bounded by $M\|x\|$ in $X$. Hence from the Theorem on Extension on p. 178 of Kluvánek [8] and from the fact that $X$ has the B-P property, it follows that $\{E(\sigma)x : \sigma \in \mathfrak{R}\}$ is relatively weakly compact in $X$. Therefore by Theorem 6.7 $E(\cdot)$ is uniquely extendable to a spectral measure $\bar{E}(\cdot)$ on $\mathscr{S}(\mathfrak{R})$.

Conversely, if $E(\cdot)$ is extendable to a spectral measure $\bar{E}(\cdot)$ on $\mathscr{S}(\mathfrak{R})$, then by Theorem 6.8 the range of $E(\cdot)$ is contained in a $\sigma$-complete B.A. of projections, which is bounded by Theorem 2.2 of Bade [1].

The last part of the theorem is due to the fact that a weakly complete Banach space does not contain any subspace isomorphic to $c_0$. (See Theorem 5 and Corollary 6.8 of Bessaga and Pelczyński [3].

Remark. From Theorem 6.7 it follows that there is a complete analogue of the 'Theorem on Extension' on p. 178 of Kluvánek [8] for spectral measures. For brevity, the details are omitted.

We shall conclude this section with an application of §5.

**Theorem 6.11.** *If $P(\cdot)$ is a bounded PO-measure in a separable Banach space $X$ defined on a ring $\mathfrak{R}$ of subsets of a set $\mathfrak{S}$ with its range contained in $H(W)$, then $P(\cdot)$ is outer regular in the sense of Definition 5.1. Further, $M_{p^*} = \mathcal{S}(\mathfrak{R})$ and $P^*(\cdot)$ on $M_{p^*}$ is the completion of $\bar{P}$ on $\mathcal{S}(\mathfrak{R})$, where $\bar{P}(\cdot) = P^*(\cdot)\ |\mathcal{S}(\mathfrak{R})$. ($P^*(\cdot)$ is the outer measure induced by $P(\cdot)$).*

Proof. By Proposition 6.3 $P(\cdot)$ is an $H(W)$-valued bounded measure (in the sense of Definition 1.1) and $H(W)$ is a weakly $(\sigma, \infty)$-distributive Stone algebra. Since the latter part of the theorem follows from the outer regularity of $P(\cdot)$ in view of Theorem 5.4, it suffices to prove that $P(\cdot)$ is outer regular.

Since $P(\cdot)$ is a bounded PO-measure, there exists $T \in H(W)^+$ such that $P(\sigma) \leqslant T$ for every $\sigma \in \mathfrak{R}$. From the definition of partial ordering in $H(W)$ it is clear that $\|P(\sigma)\| \leqslant \|T\|$, where $\|S\| = \sup\limits_{\|x\|=1} \|Sx\|$, $\|\cdot\|$ on $X$ being that occuring in the definition of the $W^*(\|\cdot\|)$-algebra $W$.

The outer regularity of $P(\cdot)$ can be proved exactly on the same lines of the proof of Theorem 5.2 if we can shown that for each decreasing net $\{T_\infty\}$ of operators in $H(W)^+$ which is norm bounded there exists a decreasing sequence $\{T_n\}$ such that $\{T_n\} \subseteq \{T_\infty\}$ and $\bigwedge\limits_n T_n = \bigwedge\limits_\infty T_\infty$ in $H(W)$. For this, because of Theorem 3 of [11], it suffices to show that a decreasing sequence $\{T_n\} \subseteq \{T_\infty\}$ exists such that $\lim\limits_n T_n x = \lim\limits_\infty T_\infty x$, $x \in X$. But, since $X$ is separable, by following an argument similar to the classical Hilbert space case it can be shown that on norm bounded sets of $H(W)$ the strong operator topology is metrizable. Consequently, as the ran e of $P(\cdot)$ is norm bounded in $H(W)$, the result follows.

Remark. The above theorem generalizes Theorem 6 of [12] to PO-measures in separable Banach spaces. We also remark that the proof of Theorem 6 in [12] is erroneous, as Theorem 5 of Lumer [9] does not apply there. We do not know whether Theorem 6 of [12] is still valid without the additional hypothesis of separability of the Banach space.

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# О МЕРАХ СО ЗНАЧЕНИЯМИ ВО ВЕКТОРНЫХ СТРУКТУРАХ – I

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Резюме

В работе исследованы внешние векторные меры со значениями в структуре и продолжение метода векторной меры со значениями в структуре и определенной на кольце множеств Показаны тоже применения этой теории для спектральных и ограниченных операторных мер в пространствах Банаха