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1. Introduction

Let n and k be non-negative integers. The aim of this paper is to characterize graphs of order n, all of whose induced subgraphs of order k (possibly with some exceptions) have the same size, or are even mutually isomorphic.

We admit graphs with loops or multiple edges. Some of our results are generalizations of results of other authors concerning only graphs without loops or multiple edges [1, 12]. The results of the paper have been presented at the Sixth Hungarian colloquium on combinatorics held in Eger, 1981 [2].

2. Terminology and notation

All *graphs* considered in the paper are finite and undirected. An edge joining two different [identical] vertices is called a *link* [a *loop,* respectively]. A graph is said to be *loopless* [*simple*] if it has no loops [multiple edges]. A graph that is not loopless [simple] is said to be a *pseudograph* [a *multigraph,* respectively]. A loopless simple graph is called *ordinary.* The number of vertices [edges] of a graph G is called the *order* [the *size*] of G.

Let N be the set of non-negative integers. For $n \in N$ the symbol K_n denotes the complete graph of order n. For $m,\ n \in N-\{0\}$ let $K_{m,n}$ denote the complete bipartite graph whose first [second] part has m [n] vertices. The symbol K_0 denotes the *empty graph* (the graph without vertices or edges), and $\bar{G}$ is the complementary graph to a graph G. The graph $\bar{K}_n$ is called the *null graph* of order n. For $n \in N$, $n \geq 2$, the symbol $K_n - e$ denotes the graph arising from K_n by deleting an edge. Given $n,\ x,\ y \in N$, by a *quasicomplete graph* $Q(n,\ x,\ y)$ we mean the graph of order n with every two different vertices joined by the same number x of edges and having at every vertex the same number y of loops. For $x=1$, $y=0$ we get the complete graph K_n, for $x=y=0$ the null graph $\bar{K}_n$.

Vertices u and v of a graph G are said to be *similar* if there exists an automorphism α of G with $\alpha(u)=v$. Vertices u and v are said to be *pseudosimilar* if they are not similar but the vertex-deleted graphs $G-u$ and $G-v$ are isomorphic [3, 4, 6, 9, 11, 13, 14].

Given $k \in N$ and a graph G, by a *k-subgraph* (called a *k-section* in [8]) of G we mean an induced subgraph of order k of G. The number of edges incident with a vertex v of G (the loops being counted twice) is called the *degree* of v in G and is denoted by $\deg_G v$. A graph G whose vertices all have the same degree S ($S \in N$) is called a *regular graph of degree* $S = \deg G$.

3. *k*-subgraphs with the same size

Let $n, a, b, c, d \in N$, $n \geq 4$. Denote by $D_n(a, b, c, d)$ the graph of order n containing vertices u and v ($u \neq v$) joined by a edges such that each other vertex is joined to u (as well as to v) by b edges and to every other vertex by c edges; moreover, every vertex has d loops. (For $a=b=c$ we get the quasicomplete graph $Q(n, a, d)$.)

Lemma 1. *Let $k, n \in N$, $3 \leq k \leq n-2$, and let G be a loopless graph of order n. Then the following assertions are equivalent:*

1. *In G there exist two different vertices such that all k-subgraphs of G containing at least one of them have the same size.*
2. *G is isomorphic to a graph $D_n(a, b, c, 0)$ such that*

$$a+(k-3)b=(k-2)c. \tag{1}$$

P r o o f. If the second assertion holds, choose in G two vertices joined by a edges. It is easy to check that for these two vertices the first assertion holds.

Conversely, let the first assertion hold for the vertices u and v. From the suppositions it follows that $k \geq 3$ and $n \geq 5$. Choose k other vertices $v_1, v_2, \ldots, v_k$ and put $v_{k+1}=u$, $v_{k+2}=v$, $V=\{v_1, v_2, \ldots, v_{k+2}\}$. Let H be the subgraph of G induced by V. Each of the graphs $H_i = H - v_i$, $i=1, 2, \ldots, k$ is regular because if we delete from H_i any vertex v_j ($j \neq i$, $j \in \{1, 2, \ldots, k+2\}$), we get a k-subgraph $H_{i,j} = H_i - v_j$ of G with the size independent of v_j.

The regular graphs $H_1, H_2, \ldots, H_k$ have the same degree. Since, if $S = \deg H_s > \deg H_t = T$ for $s, t \in \{1, 2, \ldots, k\}$, then for every vertex $w \in V - \{v_s, v_t\}$ we have: if w is joined with v_t $[v_s]$ by A $[B$, respectively$]$ edges, then $A - B = S - T$ as $S = \deg_{H_s} v_t = \deg_{H_s} w$, $T = \deg_{H_t} v_s = \deg_{H_t} w$. Hence $A-B$ does not depend on w. Thus $A - B = S - T = k(A-B)$. As $k \neq 1$, it follows that $A = B$ so that $S = T$.

As $H_1, H_2, \ldots, H_k$ are regular graphs of the same degree, they have the same size. Therefore the vertices $v_1, v_2, \ldots, v_k$ have in H the same degree.

Suppose u and v are joined by a edges. We prove that each two the vertices v_1, $v_2, \ldots, v_k$ are joined by the same number, say c, of edges. Otherwise there are vertices v_p, v_q and v_r ($p, q, r \in \{1, 2, \ldots, k\}$) such that v_p is joined with v_q by c'

edges and with v_r by c'' edges, where $c'' \neq c'$. As $\deg_H v_q = \deg_H v_r$, we have $\deg_{H_p} v_q = \deg_H v_q - c' \neq \deg_H v_r - c'' = \deg_{H_p} v_r$. However, $\deg_{H_p} v_q = \deg_{H_p} v_r$, since H_p is a regular graph. This contradiction shows that $c' = c'' = c$.

For $i = 1, 2, \ldots, k$ denote by b_i $[b'_i]$ the number of edges joining u $[v$, respectively$]$ with v_i and put $b = b_1$. As the subgraphs $H_{1,k+1}, H_{2,k+1}, \ldots, H_{k,k+1}, H_{1,k+2}, H_{2,k+2}, \ldots, H_{k,k+2}$ have the same size, we have

$$\sum_{i\neq 1} b_i + \binom{k-1}{2} c = \sum_{i\neq 2} b_i + \binom{k-1}{2} c = \ldots = \sum_{i\neq k} b_i + \binom{k-1}{2} c =$$

$$= \sum_{i\neq 1} b'_i + \binom{k-1}{2} c = \sum_{i\neq 2} b'_i + \binom{k-1}{2} c = \ldots = \sum_{i\neq k} b'_i + \binom{k-1}{2} c$$

and it easily follows that $b = b_1 = b_2 = \ldots = b_k = b'_1 = b'_2 = \ldots = b'_k$ and the size of the graphs is

$$(k-1)b + \binom{k-1}{2} c.$$

However, $H_{1,2}$ has the same size so that

$$a + 2(k-2)b + \binom{k-2}{2} c = (k-1)b + \binom{k-1}{2} c.$$

It follows that (1) holds. Q.E.D.

Theorem 1. *Let $k, n \in N$, $2 \leqslant k \leqslant n-2$ and let G be a loopless graph of order n. Then the following three assertions are equivalent:*

I. *All k-subgraphs of G are isomorphic.*

II. *All k-subgraphs of G have the same size.*

III. *G is a quasicomplete graph.*

Proof. Obviously III $\Rightarrow$ I $\Rightarrow$ II. Thus we need to prove the implication II $\Rightarrow$ III only. Let II hold. If $k = 2$, then evidently III holds. Therefore we may suppose $k \geqslant 3$. According to Lemma 1 G is isomorphic to $D_n(a, b, c, 0)$, where (1) holds. It is sufficient to prove that $a = b = c$. In G there exist vertices u and v joined by a edges. A k-subgraph of G not containing u and containing [not containing] v has the size

$$(k-1)b + \binom{k-1}{2} c,$$

$$\left[\binom{k}{2} c, \text{ respectively}\right].$$

From the equality of these expressions it follows that $b = c$. Then (1) implies that $a = b$. Q.E.D.

Remarks. 1. Very recently T. Zaslavsky [15] has found a quite different proof of the implication II $\Rightarrow$ III. He has proved the following general result: Let r, k, $n \in N$, $2 \leq r \leq k \leq n-r$. Let all k-subgraphs of a loopless graph G have the same positive number of subgraphs isomorphic to K_r. Then G is a quasicomplete graph. For $r=2$ we obtain the implication II $\Rightarrow$ III.

2. In the case $k=n-1$ the condition I holds iff all vertices of G are mutually similar (i.e. G is vertex-symmetric), as was proved for ordinary graphs in [3]. Much later this was given as an open problem in [10]. This assertion as well as the equivalence I $\Leftrightarrow$ III (for $2 \leq k \leq n-2$) are (for ordinary graphs) completely proved in [1]; previously in [8, Theorem 5] both the assertions were given without any proof as "immediate". Note that if the (first) assertion is proved for ordinary graphs, then it can be easily extended to multigraphs and pseudographs: Since in such a graph G every vertex has the same number of loops, the reduction to loopless graphs is straightforward. The rest of the proof is the same as in [3].

3. The condition II for $k=n-1$ is evidently equivalent to the assertion that G is regular. In the cases $k=0$, $k=1$ and $k=n$ the conditions I and II are obviously fulfilled for every loopless graph G of order n.

Corollary 1 (Širáň [12]). *If k, $n \in N$, $2 \leq k \leq n-2$ and G is an ordinary graph of order n in which all k-subgraphs have the same size, then G is either complete or null.*

Proof. This is in fact the implication II $\Rightarrow$ III of Theorem 1 for the case that G is ordinary. Q.E.D.

Corollary 2 (for ordinary graphs see [1]). *Let k, $n \in N$, $2 \leq k \leq n-2$ and let G be a graph of order n. Then the assertions* I *and* III *of Theorem 1 are equivalent.*

Proof. Evidently III implies I. Conversely, if I holds for G, then I holds as well for the graph G^0 obtained from G by deleting all loops. Therefore according to Theorem 1 G^0 is quasicomplete. If G is not quasicomplete, then G has two vertices with a different number of loops. However, it is easy to show that then G does not fulfil I (cf. the proof of Corollary 4 to Theorem 3). Q.E.D.

4. Combinatorial inequalities and loops in induced subgraphs

We shall need several combinatorial relations.

Lemma 2. *Let n, k, a, $x \in N$, $1 \leq a \leq n-1$, $x \leq k$. Then we have:*

$$\binom{a}{x}\binom{n-a}{k-x} \leq \max\left\{\binom{n-1}{k},\ 2\binom{n-2}{k-1},\ \binom{n-1}{k-1}\right\}.$$

Proof. It is sufficient to prove that the expression on the left-hand side of the inequality is always less than or equal to one of the numbers

$$\binom{n-1}{k},\ 2\binom{n-2}{k-1},\ \binom{n-1}{k-1}.$$

We distinguish six cases:

(i) $x > a/2$. Then

$$\binom{a}{x} \leqslant \binom{a}{x-1}$$

so that

$$\binom{a}{x}\binom{n-a}{k-x} = \binom{a}{x}\left[\binom{n-a-1}{k-x-1} + \binom{n-a-1}{k-x}\right] \leqslant$$

$$\leqslant \binom{a}{x}\binom{n-a-1}{k-x-1} + \binom{a}{x-1}\binom{n-a-1}{k-x} \leqslant$$

$$\leqslant \sum_{y=0}^{a}\binom{a}{y}\binom{n-1-a}{k-1-y} = \binom{n-1}{k-1}.$$

(ii) $x < a/2$. Then

$$\binom{a}{x} \leqslant \binom{a}{x+1}$$

so that

$$\binom{a}{x}\binom{n-a}{k-x} = \binom{a}{x}\left[\binom{n-a-1}{k-x-1} + \binom{n-a-1}{k-x}\right] \leqslant$$

$$\leqslant \binom{a}{x+1}\binom{n-a-1}{k-x-1} + \binom{a}{x}\binom{n-a-1}{k-x} \leqslant$$

$$\leqslant \sum_{y=0}^{a}\binom{a}{y}\binom{n-1-a}{k-y} = \binom{n-1}{k}.$$

(iii) $x = a/2$, $n > 2k$. Then

$$\binom{n-a}{k-x} \leqslant \binom{n-a}{k-x+1}$$

so that

$$\binom{a}{x}\binom{n-a}{k-x} = \left[\binom{a-1}{x-1} + \binom{a-1}{x}\right]\binom{n-a}{k-x} \leqslant$$

$$\leqslant \binom{a-1}{x-1}\binom{n-a}{k-x} + \binom{a-1}{x}\binom{n-a}{k-x-1} \leqslant$$

$$\leqslant \sum_{y=0}^{a-1}\binom{a-1}{y}\binom{n-a}{k-1-y} = \binom{n-1}{k-1}.$$

(iv) $x=a/2$, $n<2k$. Then

$$\binom{n-a}{k-x} \leqslant \binom{n-a}{k-x-1}$$

so that

$$\binom{a}{x}\binom{n-a}{k-x} = \left[\binom{a-1}{x-1} + \binom{a-1}{x}\right]\binom{n-a}{k-x} \leqslant$$

$$\leqslant \binom{a-1}{x-1}\binom{n-a}{k-x} + \binom{a-1}{x}\binom{n-a}{k-x-1} \leqslant$$

$$\leqslant \sum_{y=0}^{a-1} \binom{a-1}{y}\binom{n-a}{k-1-y} = \binom{n-1}{k-1}.$$

(v) $x=a/2$, $n=2k$, $a \geqslant k$. It is easy to prove that for $i \geqslant j \geqslant 1$ we have:

$$\binom{2i}{i}\binom{2j}{j} \leqslant \binom{2i+2}{i+1}\binom{2j-2}{j-1}.$$

Hence

$$\binom{a}{x}\binom{n-a}{k-x} = \binom{2x}{x}\binom{2k-2x}{k-x} \leqslant \binom{2x+2}{x+1}\binom{2k-2x-2}{k-x-1} \leqslant$$

$$\leqslant \binom{2x+4}{x+2}\binom{2k-2x-4}{k-x-2} \leqslant \ldots \leqslant \binom{2k-2}{k-1}\binom{2}{1} = 2\binom{n-2}{k-1}.$$

(vi) $x=a/2$, $n=2k$, $a<k$. Using the substitution $a'=2k-a$ we get the case (v). Q.E.D.

Lemma 3. *Let $n, k \in N$, $n \geqslant 2$, $k \leqslant n$. Then we have:*

$$\max\left\{\binom{n-1}{k}, 2\binom{n-2}{k-1}, \binom{n-1}{k-1}\right\} = \begin{cases} \binom{n-1}{k} & \text{if } n>2k, \\ 2\binom{n-2}{k-1} & \text{if } n=2k, \\ \binom{n-1}{k-1} & \text{if } n<2k. \end{cases}$$

Proof. If $k=0$, then the assertion is evident. Otherwise it suffices to distinguish the possibilities $n>2k$, $n=2k$ and $n<2k$. Q.E.D.

Lemma 4. *Let n, k, a, $x \in N$, $1 \leqslant a \leqslant n-1$, $x \leqslant k \leqslant n$. Then we have:*

$$\max_a \max_x \left\{\binom{a}{x}\binom{n-a}{k-x}\right\} = \begin{cases} \binom{n-1}{k} & \text{if } n>2k, \\ 2\binom{n-2}{k-1} & \text{if } n=2k\neq 2, \\ 1 & \text{if } n=2k=2, \\ \binom{n-1}{k-1} & \text{if } n<2k. \end{cases}$$

Proof. In the case $n=2$, $k=1$ the assertion is evident. Otherwise it suffices to take into consideration that

$$\binom{n-1}{k} = \binom{1}{0}\binom{n-1}{k-0},$$

$$2\binom{n-2}{k-1} = \binom{2}{1}\binom{n-2}{k-1},$$

$$\binom{n-1}{k-1} = \binom{1}{1}\binom{n-1}{k-1}$$

and to use Lemmas 2 and 3. Q.E.D.

If k, $n \in N$, $n \geqslant 2$, denote by $f(k, n)$ the greatest $s \in N$ such that there exists a graph of order n having at least s mutually isomorphic k-subgraphs and containing at least two vertices with a different number of loops. Evidently for any $n \in N$, $n \geqslant 2$ we have: $f(0, n) = f(n, n) = 1$, $f(1, n) = f(n-1, n) = n-1$ and $f(k, n) = 0$ for $k \in N$, $k > n$.

Theorem 2. *Let k, $n \in N$, $n \geqslant 2$, $k \leqslant n$. Then the maximal number of mutually isomorphic k-subgraphs of a graph of order n having two vertices with different numbers of loops is*

$$f(k, n) = \begin{cases} \binom{n-1}{k-1} & \text{if } n<2k, \\ 2\binom{n-2}{k-1} & \text{if } n=2k\neq 2, \\ 1 & \text{if } n=2k=2, \\ \binom{n-1}{k} & \text{if } n>2k. \end{cases}$$

Proof. Choose a and x in such a way that the expression

$$\binom{a}{x}\binom{n-a}{k-x}$$

is maximal under the conditions

$$1 \leq a \leq n-1,$$
$$0 \leq x \leq k.$$

Observe that the above expression gives the number of k-subgraphs of $K_n(a)$ isomorphic to $K_k(x)$. (Here $K_n(a)$ denotes the graph arising from K_n by adding one loop to a different vertices.) Hence

$$f(k, n) \geq \binom{a}{x}\binom{n-a}{k-x}.$$

Let G be a graph of order n with $f(k, n)$ mutually isomorphic k-subgraphs and with two vertices having different numbers of loops. Let A be the set of vertices of G possessing the greatest number of loops. Put $|A| = a'$. Obviously $1 \leq a' \leq n-1$. Isomorphic k-subgraphs of G must evidently have the same number x' of vertices of A, therefore there is an $x' \in N$, $x' \leq k$, such that

$$f(k, n) \leq \binom{a'}{x'}\binom{n-a'}{k-x'} \leq \binom{a}{x}\binom{n-a}{k-x}.$$

The rest of the proof follows from Lemma 4. Q.E.D.

Corollary. *Let k, $n \in N$, $2 \leq k \leq n$. Then we have:*

$$\binom{n-2}{k-1} \leq f(k, n) \leq \binom{n}{k} - \binom{n-2}{k-1}.$$

Proof. It is sufficient to check the validity of the inequalities in all four cases from Theorem 2. Q.E.D.

The following weaker result will be useful for us.

Lemma 5. *Let k, $n \in N$, $2 \leq k \leq n-1$. If a graph G of order n has at least*

$$\binom{n}{k} - \frac{n-2}{k}$$

mutually isomorphic k-subgraphs, then all the vertices of G have the same number of loops.

Proof. Obviously

$$\frac{n-2}{k} < \binom{n-2}{k-1}$$

so that by Corollary to Theorem 2 we have

$$f(k, n) \leqslant \binom{n}{k} - \binom{n-2}{k-1} < \binom{n}{k} - \frac{n-2}{k}$$

and the assertion follows. Q.E.D.

5. Pseudosimilar vertices and exceptional k-subgraphs

Theorem 3. *Let $k, n \in N$, $3 \leqslant k \leqslant n-2$. Let G be a loopless graph of order n. Then the following three assertions are equivalent:*

I. *In G there exist*

$$\binom{n}{k} - 1$$

k-subgraphs with the same size and one k-subgraph with a different size.

II. *In G there exist at least*

$$\binom{n}{k} - \frac{n-2}{k},$$

but at most

$$\binom{n}{k} - 1$$

k-subgraphs with the same size.

III. *$k = n-2$ and G is isomorphic to a graph $D_n(a, b, c, 0)$, where (1) holds but $a = b = c$ does not hold.*

Proof. The implication I $\Rightarrow$ II is evident.

II $\Rightarrow$ III. In G there are at least

$$\binom{n}{k} - \frac{n-2}{k}$$

k-subgraphs with the same size so that at most $(n-2)/k$ k-subgraphs have another size. Hence there exist two vertices not belonging to any of these latter exceptional k-subgraphs and the suppositions of Lemma 1 are fulfilled. Thus G is isomorphic to some $D_n(a, b, c, 0)$ with (1) valid. If $a = b = c$, then all $\binom{n}{k}$ k-subgraphs have the same size, contrary to the supposition of II.

Let $k \leqslant n-3$. Then the k-subgraphs of G that contain one or both of the two vertices joined by a edges have size

$$(k-1)b + \binom{k-1}{2} c = a + 2(k-2)b + \binom{k-2}{2} c$$

and the remaining k-subgraphs of G have size

$$\binom{k}{2} c.$$

If

$$(k-1)b+\binom{k-1}{2} c=\binom{k}{2} c,$$

then all k-subgraphs of G have the same size, contrary to II. Therefore the equality does not hold. Then the number of k-subgraphs of G with the same size is at most

$$\max\left\{\binom{n-2}{k}, \binom{n}{k}-\binom{n-2}{k}\right\}<\binom{n}{k}-\frac{n-2}{k},$$

which contradicts II. Hence $k=n-2$ and III holds.

III $\Rightarrow$ I. Let u and v be the vertices joined in $D_n(a, b, c, 0)$ by a edges. Evidently all k-subgraphs of G containing u or v have

$$(k-1)b+\binom{k-1}{2} c=a+2(k-2)b+\binom{k-2}{2} c$$

edges. The number of such k-subgraphs of G is

$$\binom{n}{k}-1.$$

If the remaining k-subgraph of G has the same size, then

$$(k-1)b+\binom{k-1}{2} c=\binom{k}{2} c.$$

It follows, as before, that $b=c$ and $a=b$, contrary to III. Q.E D.

Corollary 1. *Let $k, n \in N$, $3 \leqslant k \leqslant n-2$. Let G be a graph of order n. Then the following three assertions are equivalent:*

I. *In G there exist*

$$\binom{n}{k}-1$$

mutually isomorphic k-subgraphs and the remaining k-subgraph is not isomorphic to them.

II. *In G there exist at least*

$$\binom{n}{k}-\frac{n-2}{k},$$

but at most

$$\binom{n}{k}-1$$

mutually isomorphic k-subgraphs.

III. $k=3$, $n=5$ *and* G *is isomorphic to a graph* $D_5(a, b, c, d)$, *where* $a=c$, $a\neq b$.

Proof. Obviously III implies I and I implies II. We prove the implication II $\Rightarrow$ III.

If II holds, then by Lemma 5 G has in every vertex the same number d of loops. After deleting them there arises a graph that is (by Theorem 3, implication II $\Rightarrow$ III) isomorphic to some $D_n(a, b, c, 0)$, where $k=n-2$, (1) is true, but $a=b=c$ does not hold. Therefore G is isomorphic to a graph $D_n(a, b, c, d)$. If $a=b$ or $b=c$, then from (1) it follows that $a=b=c$ so that all k-subgraphs of G are mutually isomorphic — a contradiction. Therefore $a\neq b$ and $b\neq c$. Distinguish two cases:

A. $k=3$ so that $n=5$. From (1) it follows that $a=c$ and III holds.

B. $k\geqslant 4$ so that $n\geqslant 6$. If $a=c$, from (1) it follows that $(k-3)b=(k-3)a$, which is impossible, as $k\neq 3$ and $a\neq b$. Therefore a, b and c are mutually different. However, then G has three mutually non-isomorphic k-subgraphs: The first containing both u and v, the second only one of u and v, the third neither u nor v (where u and v are vertices joined by a edges). But then the number of mutually isomorphic k-subgraphs of G is less than

$$\binom{n}{k}-1=\binom{n}{k}-\frac{n-2}{k},$$

which contradicts II. Q.E.D.

It is easy to find an analogous result for $k=2$:

Corollary 2. *Let* s, $n\in N$, $n>4$ *and*

$$\binom{n-1}{2}<s\leqslant\binom{n}{2}.$$

Let G *be a graph of order* n. *Then the following assertions are equivalent:*

I. *In* G *there exist* s *mutually isomorphic* 2*-subgraphs.*

II. *Every vertex of* G *has the same number of loops and in* G *there exist* s *pairs of vertices joined by the same number of edges.*

Proof. Evidently II implies I. Conversely, if I holds, then II follows from Theorem 2 for $k=2$, $n>4$, as $s>f(2, n)$. Q.E.D.

Remarks. 1. For every $n\geqslant 3$ there exists a graph G of order n with

$$\binom{n-1}{2}$$

mutually isomorphic 2-subgraphs that has two vertices with different numbers of loops, e.g. the graph of order n and size 1 having a single loop.

2. In the case $k=n-1$ it is also possible that all k-subgraphs of G are, with exactly one exception, isomorphic. A sufficient condition for this is that all vertices of G, with exactly one exception, are mutually similar. This condition is also necessary if $F(n) \leq n-2$ for any $n \in N$, $n \geq 8$, where F is the function defined below in the following problem.

It is well known [3, 4, 6, 7] that for $n \in N$ there exists a graph of order n with pseudosimilar vertices iff $n \geq 8$. Such an example can be constructed from the graph of Fig. 1 by adding $n-8$ isolated vertices (if we wish to have a connected example for any $n \geq 8$, we may take the complementary graph). Evidently, the vertices u and v are pseudosimilar.

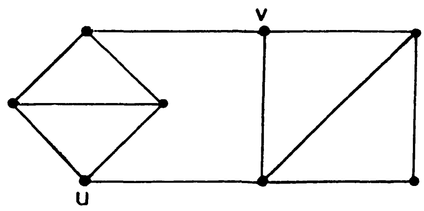

Fig. 1

Problem. *For $n \in N$, $n \geq 8$ determine the maximal cardinality $F(n) = |X|$ of a set X of vertices of a graph G of order n satisfying the following conditions:*

1. *Every two vertices of X are similar or pseudosimilar in G.*
2. *In X there exists a pair of pseudosimilar vertices of G.*

Remark. Let $n \in N$, $n \geq 8$. From Remark 2 after Theorem 1 it follows that $F(n) \leq n-1$. It is easy to prove that $F(n) \geq 2[n/8]$. An example of G of order n with $|X| = 2[n/8]$ can be constructed using $[n/8]$ disjoint copies of a graph of Fig. 1 and adding $n-8[n/8]$ isolated vertices. (The complementary graphs have the same property.)

Various constructions of graphs with pseudosimilar vertices are given in [5, 7].

Corollary 3. *Let $k, n \in N$, $2 \leq k \leq n-2$. Let G be an ordinary graph of order n. All the k-subgraphs of G, with at most one exception, are isomorphic if and only if one of the following cases occurs:*

1. *G is complete or null.*
2. *$k=2$ and G is isomorphic to $K_n - e$ or $\overline{K_n - e}$.*
3. *$k=3$, $n=5$ and G is isomorphic to $K_{2,3}$ or $\overline{K_{2,3}}$.*

Proof. This follows from Corollary 1 and from Theorem 1.

Corollary 4. *Let k, $n \in N$, $k \neq n-1$. Let G be a graph of order n. All k-subgraphs of G, with exactly one exception, are isomorphic if and only if one of the following cases occurs:*

1. $k=1$ *and all vertices of G, with just one exception, have the same number of loops.*

2. $k=2$, *all the vertices of G have the same number of loops and all pairs of (different) vertices of G, with just one exception, are joined by the same number of edges.*

3. $k=3$, $n=5$ *and G is isomorphic to a graph* $D_5(a, b, c, d)$, *where* $a=c$, $a \neq b$.

Proof. If case 1, 2 or 3 occurs, then all k-subgraphs of G, with just one exception, are isomorphic.

Conversely, if the above assertion is valid, then obviously $1 \leq k \leq n-2$. If $k=1$, then case 1 occurs. Therefore suppose that $2 \leq k \leq n-2$ so that $n \geq 4$.

Let G have vertices v_0 and v_1 with different numbers of loops. In G there exist at least k other vertices. Choose among them $k-1$ vertices $v_2, v_3, \ldots, v_k$ such that the k-subgraph H_0 [H_1] induced by the set $\{v_0, v_2, v_3, \ldots, v_k\}$ [$\{v_1, v_2, v_3, \ldots, v_k\}$, respectively] is not exceptional. As H_0 and H_1 have different numbers of loops, they cannot be isomorphic. Therefore every vertex of G has the same number of loops.

If $k=2$, by Corollary 2 case 2 occurs. (The case $n=4$ can be easily completed from Theorem 2.) If $k \geq 3$, then according to Corollary 1 case 3 occurs. Q.E.D.

Remark. In Theorem 1 [Theorem 3] the condition that G is loopless cannot be deleted as shown by the example of Fig. 2 [Fig. 3] for $k=2$ [$k=3$, respectively], $n=5$.

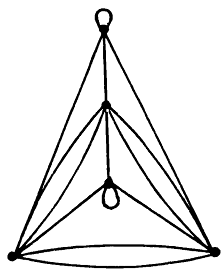

Fig. 2

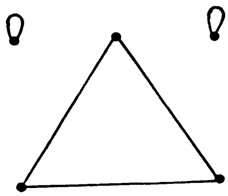

Fig. 3

The situation when loops are present has been partially solved in Corollary 2 to Theorem 1 and Corollaries 1, 2 and 4 to Theorem 3. Further results can be obtained in the following manner. Given $n, k \in N - \{0\}$ and a graph G of order n, denote by $G(k)$ the graph constructed from G in which every link of G is replaced by $k-1$ links (with the same end vertices) and every loop of G is replaced by $n-1$ links, joining the end vertex of the deleted loop with all the remaining vertices of G (one to each other vertex). Evidently $G(k)$ is a loopless graph in which any k-subgraph has a size $k-1$ times as great as the corresponding k-subgraph in G. Therefore we have:

Lemma 6. *Let k, $n \in N - \{0\}$ and G be a graph of order n. Two sets of k vertices induce in G k-subgraphs with the same size if and only if this is true in $G(k)$.*

Theorem 4. *Let k, n and s be positive integers and G be a graph of order n. All the k-subgraphs of G [all with exactly s exceptions] have the same size if and only if all the k-subgraphs of $G(k)$ [all with exactly s exceptions, respectively] have the same size.*

Proof. This follows from Lemma 6. Q.E.D.

Remarks. 1. Theorem 4 reduces the problem of the equality of the sizes to the corresponding problem for loopless graphs solved in Theorems 1 and 3.

2. Results of this paper will be used in further works concerning some invariants of graphs connected with the structure of induced subgraphs of a graph.

3. P. Erdös has suggested studying analogous problems for hypergraphs (oral communication, Eger, July 1981).

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ПОРОЖДЕННЫЕ ПОДГРАФЫ С ОДИНАКОВЫМ ПОРЯДКОМ И РАЗМЕРОМ

Juraj Bosák

Резюме

В статье дается характеризация конечных неориентированных графов, в которых все порожденные подграфы с заданным числом вершин (возможно, с некоторыми исключениями) имеют одинаковое число ребер, или даже изоморфны. Обобщаются результаты Акияма, Эксы и Харари [1] и Ширaня [12], касающиеся графов без петель и кратных ребер. При помощи комбинаторных неравенств находится максимальное число взаимно изоморфных порожденных подграфов графа, содержащего вершины с различным числом петель.