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QUASI-CONTINUOUS MULTIVALUED MAPPINGS

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Let X be a topological space. The closure, the interior and the boundary of a set A will be denoted by $\bar{A}$, $\operatorname{Int} A$, $\operatorname{Fr} A$, respectively.

Definition 1. [4, 2]. *A set $A \subset X$ is called semi-open if there is an open set $U \subset X$ such that $U \subset A \subset \bar{U}$. A set A is semi-closed if its complementary $X \setminus A$ is semi-open.*

The following propositions are an immediate consequence of the definition.

Proposition 2. 1) *A set A is semi-open if and only if $\bar{A} = \overline{\operatorname{Int} A}$.*

2) *A set A is semi-open if and only if $A = (\operatorname{Int} A) \cup B$, where $B \subset \operatorname{Fr} A$.*

Proposition 3. [4, 2]. 1) *The union of semi-open sets and the intersection of an open and a semi-open set are semi-open.*

2) *If A is a semi-open (semi-closed) set, then all of the sets:* $\operatorname{Int} A$, $\bar{A}$ *are semi-open (semi-closed).*

The reader can easily verify the following:

Lemma 4. 1) *If A is a semi-open set, then* $\operatorname{Fr} A = \operatorname{Fr}(\operatorname{Int} A)$.

2) *If A is a semi-open (semi-closed) set, then* $\operatorname{Fr} A$ *is a border set.*

Any semi-open set U such that $x \in U$ will be called a semineighbourhood of a point x (briefly s-neighbourhood). Let X, Y be topological spaces and let $\mathscr{S}(Y)$, $\mathscr{C}(Y)$, $\mathscr{K}(Y)$ be classes of all non-empty, non-empty-closed and non-empty compact subsets of Y, respectively. For a multivalued map $F\colon X \to \mathscr{S}(Y)$ we will denote

$$F(A) = \bigcup_{x \in A} F(x), \quad F^-(B) = \{x \in X\colon F(x) \cap B \neq \emptyset\}, \quad F^+(B) = \{x \in X\colon F(x) \subset B\}$$

for any sets $A \subset X$, $B \subset Y$.

Definition 5. [8]. *A multivalued map $F\colon X \to \mathscr{S}(Y)$ is said to be l-quasi-continuous (u-quasi-continous) at a point $x_0 \in X$ if for any open set $W \subset Y$ such that $F(x_0) \cap W \neq \emptyset$ ($F(x_0) \subset W$) there is an s-neighbourhood U of the point x_0 such that $F(x) \cap W \neq \emptyset$ ($F(x) \subset W$) for every $x \in U$. A multivalued map F is*

l-quasi-continuous (u-quasi-continuous) in X if it is l-quasi-continuous (u-quasi--continuous) at every point of X.

Definition 6. *A multivalued map F is said to be:*
— *injective if for any $x_1, x_2 \in X$, $x_1 \neq x_2$ we have $F(x_1) \cap F(x_2) - \emptyset$* ([1], p. 22);
— *pre-semi-open if for any semi-open set $A \subset X$ the set $F(A)$ is semi-open.*

Theorem 7. *Let Y be a regular topological space and let F: $X \to \mathscr{S}(Y)$ be a pre-semi-open, u-quasi-continuous multivalued map. If one of the two conditions is satisfied:*

1) $\operatorname{Int} F(x) = \emptyset$ *for every $x \in X$; or*

2) *F is injective and l-quasi continuous, then F is lower semi-continuous.*

(For the definition of a lower and upper semi-continuity see [1].)

Proof: Suppose that F is not lower semi-continuous. There is an open set $G \subset Y$ such that $F^-(G)$ is not open; $F^-(G) \neq X$. Thus there is a point $x \in F^-(G)$ such that $x \in \operatorname{Fr} F^-(G)$.

Let $y \in F(x) \cap G$. Since Y is regular, there exists an open set V such that $y \in V \subset \bar{V} \subset G$. Hence we have $x \in F^-(\bar{V}) \subset F^-(G)$ and $x \in \operatorname{Fr} F\ (\bar{V})$. The set $X \backslash F^-(\bar{V})$ is non-empty and semi-open, and moreover $x \in \operatorname{Fr}[X \backslash F^-(\bar{V})]$. Therefore $U = \{x\} \cup [X \backslash F^-(\bar{V})] = \{x\} \cup F^+(Y \backslash \bar{V})$ is a semi-open set. Thus the sets $F(U)$ and $F(U) \cap V$ are semi-open. But $F(U) \cap V = [F(x) \cup F(F^+(Y \backslash \bar{V}))] \cap V = F(x) \cap V$. If 1) is satisfied, then $\operatorname{Int}[F(U) \cap V] = \operatorname{Int}[F(x) \cap V] = \emptyset$. If 2) holds then the set $F^-[F(U) \cap V] = \{x\}$ has the non-empty interior. On the other hand $\{x\} \subset \operatorname{Fr}[F^-(\bar{V})]$ and $\operatorname{Int} \operatorname{Fr}[F^-(\bar{V})] = \emptyset$ (lemma 4), therefore the proof is completed.

Remark 8. Theorem 7 remains true if we suppose instead of regularity that the space Y has a basis composed of open-closed sets. Simple examples show that such a space need not be regular.

Definition 9. *A multivalued map F: $X \to \mathscr{S}(Y)$ is said to be l-irresolute (u-irresolute) at a point $x_0 \in X$ if for any semi-open set $W \subset Y$ such that $F(x_0) \cap W \neq \emptyset$ ($F(x_0) \subset W$) there exists an s-neighbourhood U of the point x_0 such that $F(x) \cap W \neq \emptyset$ ($F(x) \subset W$) for every $x \in U$. A multivalued map is l-irresolute (u-irresolute) in X if it is l-irresolute (u-irresolute) at every point of X.*

A set A is said to be the open domain if $A = \operatorname{Int} \bar{A}$.

Theorem 10. *Let Y be a topological space, which has a basis composed of open domains and let F: $X \to \mathscr{S}(Y)$ be a pre-semiopen, u-irresolute multivalued map. If one of the conditions holds:* 1) $\operatorname{Int} F(x) = \emptyset$ *for every $x \in X$; or* 2) *F is injective, l-quasi-continuous; then F is lower semi-continuous.*

The proof is similar to that of theorem 7.

Theorem 11. *Let Y be a regular topological space or a space which has a basis composed of open-closed sets. If F: $X \to \mathscr{K}(Y)$ is a pre-semi-open, l-quasi-cont-*

inuous, u-irresolute and injective multivalued map, then it is upper semi-continuous.

Proof: Suppose that F is not upper semi-continuous and let $G \subset Y$ be an open set such that $F^+(G)$ is not open. Then $\emptyset \neq F^+(G) \neq X$. There exists a point $x \in F^+(G)$ such that $x \in \operatorname{Fr} F^+(G)$. Let Y be regular. Since $F(x)$ is compact, there exists an open set V such that $F(x) \subset V \subset \bar{V} \subset G$. So we have $x \in F^+(\bar{V}) \subset F^+(G)$ and $x \in \operatorname{Fr} F^+(\bar{V})$. Since the set $F^-(Y \setminus \bar{V})$ is semi-open and $x \in \operatorname{Fr} F^-(Y \setminus \bar{V})$, it follows that $U = \{x\} \cup F^-(Y \setminus \bar{V})$ is semi-open. Hence the sets $F(U) \cap V$ and $F^+[F(U) \cap V]$ are semi-open. On the other hand, $F^+[F(U) \cap V] = F^+[(F(x) \cup F(F^-(Y \setminus \bar{V}))) \cap V] = \{x\} \cup F^+[F(F^-(Y \setminus \bar{V})) \cap V]$. We will show that $f^+[F(F^-(Y \setminus \bar{V})) \cap V] = \emptyset$. On the contrary, assume that $x_0 \in F^+[F(F^-(Y \setminus \bar{V})) \cap V]$. Then $F(x_0) \subset F(F^-(Y \setminus \bar{V})) \cap V$ and by the injectivity of F we have $x_0 \in F^-(Y \setminus \bar{V})$; this is a contradiction. Hence $F^+[F(U) \cap V] = \{x\} \subset \operatorname{Fr} F^+(\bar{V})$; by lemma 4 $\operatorname{Int} F^+[F(U) \cap V] = \emptyset$ and the proof is completed. If we assume that Y has a basis composed of open-closed sets, the proof is exactly the same.

By 2^Y we denote the set $\mathscr{C}(Y)$ with the Vietoris topology ([3], p. 162). As an immediate consequence of theorems 7 and 11 we have:

Corollary 12. *Let Y be a regular topological space or a space which has a basis composed of open-closed sets, and let $F\colon X \to \mathscr{K}(Y)$ be a pre-semi-open, injective, l-quasi-continuous map.* 1) *If F is u-quasi-continuous map, then the ordinary map $F\colon X \to 2^Y$ is quasi-continuous (for a definition of a quasicontinuity see* [5]). 2) *If F is u-irresolute, then the map $F\colon X \to 2^Y$ is continuous.*

An ordinary map $f\colon X \to Y$ may be interpreted as a multivalued map, which assigns to every point $x \in X$ the set $\{f(x)\}$. Moreover, we have $f^-(A) = f^+(A) = f^{-1}(A)$, where $f^{-1}(A)$ denotes the inverse image of the set $A \subset Y$. In this case l-quasicontinuity and u-quasi-continuity mean quasi-continuity of map f (called sometimes semi-continuity of f; cf. [2, 6]). By theorem 7 and remark 8 we have:

Corollary 13. *Let Y be a regular topological space or a space which has a basis composed of open-closed sets, and let*
$f\colon X \to Y$ be a pre-semi-open and quasi-continuous map. If one of two conditions is satisfied: 1) *the space Y is dense in itself* ([6], *theorem* 7); 2) *f is one-to-one; them f is continuous.*

Similarly, theorem 10 implies

Corollary 14. *Let Y be a topological space which has a basis composed of open domains. If $f\colon X \to Y$ is a pre-semi-open, irresolute map and one of the conditions is satisfied:* 1) *the space Y is dense in itself;* 2) *f is one-to-one; then f is continuous.*

From 14 (1) we have the theorem of Piotrowski ([6], theorem 8).

Theorem 15. *Let Y be a second countable topological space. If F: $X \to \mathcal{K}(Y)$ is a u-quasi-continuous multivalued map, then the set A of all points at which F is not upper semicontinuous is the first category in the sense of Baire.*

Proof: Let $\{V_n\}_{n=1}^{\infty}$ be a basis of Y. Let us denote by $\mathcal{A}$ the set of all finite, one-to-one sequences of natural numbers. Then we have $\mathcal{A}=\{\alpha_k\}_{k=1}^{\infty}$, where $\alpha_k=(n_{k,1}, n_{k,2}, \dots, n_{k,j(k)})$. Let us put $W_k=\bigcup_{i=1}^{j(k)} V_{n_{k,i}}$. If $x_0 \in A$, then there exists an open set $U \subset Y$ such that $x_0 \in F^+(U)$ and $x_0 \in \operatorname{Fr} F^+(U)$. By the compactivity of $F(x_0)$ there is a natural number k such that $F(x_0) \subset W_k \subset U$, hence $x_0 \in \operatorname{Fr} F^+(W_k)$. Thus we have $A \subset \bigcup_{k=1}^{\infty} \operatorname{Fr} F^+(W_k)$. As the sets $F^+(W_k)$ are semi-open, by lemma 4 $\operatorname{Fr} F^+(W_k)$, $k=1, 2, \dots$ are nowhere dense and the proof is completed.

Theorem 16. *Let Y be a second countable topological space. If F: $X \to \mathcal{S}(Y)$ is a l-quasi-continuous multivalued map, then the set A of all points at which F is not lower semicontinuous is the first category.*

Proof: It follows from the inclusion: $A \subset \bigcup_{n=1}^{\infty} \operatorname{Fr} F^-(V_n)$, where $\{V_n\}_{n=1}^{\infty}$ is a basis of a space Y.

Remark 17. From theorem 15 or 16 we obtain a theorem of Levine [4] for an ordinary map.

Let Y be a metric space. If $A \subset X$ is a set of all quasicontinuity points of an ordinary map f: $X \to Y$, then the set $(\operatorname{Int} \bar{A}) \setminus A$ is the first category in the sense of Baire [5, remark 3]. For multivalued maps — in general — this condition does not hold.

Example 18. The multivalued map F defined on the space of real numbers by the formula: $F(x)=[0, 2]$ if x is rational and $F(x)=[1, 2]$ in the other case, has the set A of all u-quasicontinuity points equal to the set of rational numbers. Thus $(\operatorname{Int} \bar{A}) \setminus A$ is equal to the set of irrational numbers; this is not the first category. Similarly the set A of l-quasi-continuity points of the map F_1 given by: $F_1(x)=[1, 2]$ if x is rational and $F_1(x)=[0, 2]$ in the other case does not satisfy the above condition.

Theorem 19. *Let Y be a second countable regular space and let F: $X \to \mathcal{K}(Y)$ be an l-irresolute map. If $A \subset X$ is a set of u-quasi-continuity points of F, then the set $(\operatorname{Int} \bar{A}) \setminus A$ is the first category.*

Proof: Let $\{V_n\}_{n=1}^{\infty}$ be a basis of a space Y. Let us denote by $\mathcal{A}$ the set of all finite one-to-one sequences of natural numbers. Then $\mathcal{A}=\{\alpha_k\}_{k=1}^{\infty}$, where $\alpha_k=(n_{k,1}, n_{k,2}, \dots, n_{k,j(k)})$. Let us put $W_k=\bigcup_{i=1}^{j(k)} V_{n_{k,i}}$. We will denote by G_k the set of all points $x \in X$ such that the following condition is satisfied: if $F(x) \subset \operatorname{Int} \bar{W}_k$, then there exists a neighbourhood U of the point x such that $F(U) \subset \operatorname{Int} \bar{W}_k$. Let $x \in G_k$.

If $F(x) \subset \operatorname{Int} \bar{W}_k$, then $x \in \operatorname{Int} G_k$. In the other case $x \in F^-(Y \setminus \operatorname{Int} \bar{W}_k)$. Because $Y \setminus \operatorname{Int} \bar{W}_k$ is the semi-open set and F is l-irresolute, $F^-(Y \setminus \operatorname{Int} \bar{W}_k)$ is semi-open. Moreover $F^-(Y \setminus \operatorname{Int} \bar{W}_k) \subset G_k$, hence $x \in \overline{\operatorname{Int} G_k}$. Hence we have shown that G_k is semi-open. Let $x \in \bigcap_{k=1}^{\infty} G_k$ and let $V \subset Y$ be an open set such that $F(x) \subset V$. By the regularity of the space Y we have $F(x) \subset W_k \subset \bar{W}_k \subset V$ for some W_k, and $F(x) \subset \operatorname{Int} \bar{W}_k \subset V$. Then there exists an open set U such that $x \in U$ and $F(U) \subset \operatorname{Int} \bar{W}_k$. Thus F is upper semicontinuous at the point x. On the other hand every point of the upper semi-continuity of F belongs to $\bigcap_{k=1}^{\infty} G_k$; thus $\bigcap_{k=1}^{\infty} G_k$ is the set of all points at which F is upper semi-continuous. Hence $\bigcap_{k=1}^{\infty} G_k \subset A$. Now, let $x \in A$ be a point such that $F(x) \subset \operatorname{Int} \bar{W}_k$, and let U be any neighbourhood of x. There exists an open set U', $\emptyset \neq U' \subset U$ such that $F(U') \subset \operatorname{Int} \bar{W}_k$. It implies $U' \subset G_k$, therefore $U \cap \operatorname{Int} G_k \neq \emptyset$ and $x \in \bar{G}_k$. If $x \in A$ and $F(x) \not\subset \operatorname{Int} \bar{W}_k$, then $x \in G_k$. Finally we obtain $A \subset \bar{G}_k$. From this it follows $\operatorname{Int} \bar{A} \subset \bar{G}_k$ and $(\operatorname{Int} \bar{A}) \setminus G_k \subset \bar{G}_k \setminus G_k = \operatorname{Fr} G_k$. Since $\operatorname{Fr} G_k$ is nowhere dense by lemma 4 neither $(\operatorname{Int} \bar{A}) \setminus G_k$ is nowhere dense. Hence $\bigcup_{k=1}^{\infty} [(\operatorname{Int} \bar{A}) \setminus G_k]$ is the first category set. But

$$(\operatorname{Int} \bar{A}) \setminus A \subset (\operatorname{Int} \bar{A}) \setminus \bigcap_{k=1}^{\infty} G_k = \bigcup_{k=1}^{\infty} [(\operatorname{Int} \bar{A}) \setminus G_k]$$

and this is the inclusion finishing the proof.

Theorem 20. *Let Y be a second countable space which has an open-closed basis and let F: $X \to \mathcal{K}(Y)$ be an l-quasi-continuous map. If $A \subset X$ is a set of u-quasi-continuity points, then $(\operatorname{Int} \bar{A}) \setminus A$ is the first category set.*

Proof: Let W_k, $k = 1, 2, \ldots$ be such as in the proof of theorem 19. We assume that G_k is the set of all points $x \in X$ satisfying the next condition: if $F(x) \subset W_k$, then there exists a neighbourhood U of x such that $F(U) \subset W_k$. Then G_k is the semi-open set, $\bigcap_{k=1}^{\infty} G_k$ is the set of all points at which F is upper semi-continuous and the remaining part of the proof is exactly the same as in theorem 19.

Theorem 21. *Let Y be a second countable regular space and let F: $X \to \mathcal{S}(Y)$ be a u-irresolute map. If A is a set of l-quasi-continuity points, then $(\operatorname{Int} \bar{A}) \setminus A$ is the first category.*

Proof: Let $\{V_n\}_{n=1}^{\infty}$ be a basis of a space Y. By G_k we will denote the set of all points $x \in X$ satisfying the condition: if $F(x) \cap \operatorname{Int} \bar{V}_k \neq \emptyset$, then there exists a neighbourhood U of x such that $F(x') \cap \operatorname{Int} \bar{V}_k \neq \emptyset$ for every $x' \in U$. The rest of the proof is such as in the 19.

Similary to the above we have

Theorem 22. *Let a space Y have a countable open-closed basis, and let F: $X \to \mathscr{S}(Y)$ be a u-quasi-continuous map. If $A \subset X$ is a set of l-quasi-continuity points, then* $(\operatorname{Int} \bar{A}) \setminus A$ *is the first category.*

From the proofs of these theorems there immediately follows:

Remark 23. Let Y be a second countable space, and F: $X \to \mathscr{K}(Y)$ any map. If: 1) Y is regular and F l-irresolute; or 2) Y has an open-closed basis and F is l-quasi-continuous, then a set of upper semi-continuity points of F is the intersection of a countable family of semi-open sets.

Remark 24. Let a space Y have a countable open-closed basis. If F: $X \to \mathscr{K}(Y)$ is lower semi-continuous map, then a set of upper semi-continuity points is G_δ.

Remark 25. If in the assumptions of 23 and 24 the words "l-irresolute", l-quasi-continuous", "lower semi-continuous" are replaced by "u-irresolute", "u-quasi-continuous" and "upper semi-continuous" respectively, then we have the analogous properties of the set of lower semi-continuity points.

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КВАЗИ-НЕПРЕРЫВНЫЕ МНОГОЗНАЧНЫЕ ОТОБРАЖЕНИЯ

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Резюме

В этой работе сформулированы некоторые условия касающиеся полунепрерывности сверху (снизу) квази-непрерывных многозначных отображений. Кроме этого оговорены некоторые свойства множества точек полунепрерывности сверху (снизу) этих отображений.