Václav Tryhuk

The most general transformation of homogeneous retarded linear differential equations of the n-th order

Mathematica Slovaca, Vol. 33 (1983), No. 1, 15--21

Persistent URL: http://dml.cz/dmlcz/136313

Math. Slovaca 33, 1983, No. 1, 15—21

THE MOST GENERAL TRANSFORMATION OF HOMOGENEOUS RETARDED LINEAR DIFFERENTIAL EQUATIONS OF THE n-TH ORDER

VÁCLAV TRYHUK

On an interval $I=(a, b)$, $a\geqq -\infty$, $b\leqq\infty$ we shall consider an equation of the form

$$y^{(n)}(x)+\sum_{i=0}^{n-1}\left[a_i(x)y^{(i)}(x)+\sum_{j=1}^{m}b_{ij}(x)y^{(i)}(\xi_j(x))\right]=0 \tag{1}$$

($y^{(i)}(s)=\mathrm{d}^iy(s)/\mathrm{d}s^i$) with bounded or unbounded delays

$$\mu_j(x)=x-\xi_j(x)>0$$

on I, $1\leqq j\leqq m$, $m\geqq 1$ and $n\geqq 2$ being integers.

Let us suppose that a_i, b_{ij}, $\xi_j\in C^0(I)$, $\xi_j(x)\to b_-$ as $x\to b_-$, $\xi_k\neq\xi_j$ if $k\neq j$ on I, $b_{rs}\not\equiv 0$ on I for some r and s, $0\leqq i$, $r\leqq n-1$; $1\leqq k$, j, $s\leqq m$.

A continuous function y is said to be a solution of (1) if there exists $c\in I$ such that y satisfies (1) for all $x\in[c, b)$. In this case we say that y is a solution of (1) on $[c, b)$.

Let $c\in I$, $A=[c, b)$, $A_j=\{\xi_j(x)\colon\ \xi_j(x)<c,\ x\in A\}$ and $d=\inf\cup A_j$, $j=1, 2, \ldots, m$. Then we put $A_c=[d, c]$ if $d>-\infty$. Otherwise let $A_c=(-\infty, c]$.

For given functions $\sigma_0, \sigma_1, \ldots, \sigma_{n-1}\in C^0(A_c)$ we say that is a solution of (1) with the initial values $\{\sigma_k\}_0^{n-1}$ at c or simply a solution of (1) through (c; $\sigma_0, \sigma_1, \ldots, \sigma_{n-1}$) if y is a solution of (1) on A and

$$y^{(k)}(s)=\sigma_k(s)\quad\text{for all}\quad s\in A_c,$$

$0\leqq k\leqq n-1$.

If a_i, b_{ij}, ξ_j, σ_k ($0\leqq i$, $k\leqq n-1$; $1\leqq j\leqq m$) are continuous on I, there exists a unique solution of (1) through (c; $\sigma_0, \sigma_1, \ldots, \sigma_{n-1}$) (see [1], p. 34).

In this paper we derive the most general transformation, which converts any linear equation (1) into another equation of the same form

$$u^{(n)}(t)+\sum_{i=0}^{n-1}\left[A_i(t)u^{(i)}(t)+\sum_{j=1}^{m}B_{ij}(t)u^{(i)}(\eta_j(t))\right]=0, \tag{2}$$

where A_i, B_{ij}, $\eta_j \in C^0(J)$, $\eta_j \neq \eta_k$ if $j \neq k$ on J, $\operatorname{sgn}(t-\eta_j(t)) = \operatorname{sgn}(t-\eta_k(t)) \neq 0$ on J, $B_{pq} \not\equiv 0$ on J for some p and q, $0 \leqq i,\ p \leqq n-1$; $1 \leqq k,\ j,\ q \leqq m$.

Although the equation (1) was studied by many authors, they did not pay special attention to the question of transformations. EL'SGOL'C [1] and NORKIN [2] considered the transformation

$$x = f(t), \quad y = g(t)u,$$

that converts any equation (1) into another of the same form and order. This transformation was used by MELVIN [5] and others for a functional generally nonhomogeneous differential equations.

STÄCKEL [3] and WILCZYNSKI [4] have proved that the most general point-transformation

$$T\colon x = f(t, u), \quad y = g(t, u),$$

converting every linear differential equation of the n-th order ($n \geqq 2$) of the form

$$y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_n(x)y = 0 \tag{3}$$

with continuous coefficients into another equation of the same form and order, is

$$x = f(t), \quad y = g(t)u,$$

where f and g are arbitrary functions satisfying some additional assumptions.

Using the same arguments as in [3] and [4] we obtain the form of the most general transformation for (1). We wish that the transformation $T = (f, g)$ be independent of coefficients of (1) for the same reason as in [4] (see [4], p. 8).

The case $n = 1$ is solved in [9].

The most general transformation of the equation (1).

Theorem. *For each $n \geqq 2$, the most general transformation converting any equation (1) into (2) is*

$$x = f(t), \quad y = g(t)u,$$

where $f, g \in C^n(J)$, $\dot{f}(t)g(t) \neq 0$ for all $t \in J$.

Furthermore

$$\xi_j \circ f = f \circ \eta_j$$

on J for $j = 1, 2, \ldots, m$.

Proof: If y is a solution of (1), there exists $c \in I$ such that y is defined on an interval $A \cup A_c$. A mapping $\Psi\colon A \cup A_c \to R^2$ defined by $\Psi(x) = (x, y(x))$, $x \in A \cup A_c$, is a one-to-one homeomorphism of $A \cup A_c$ into a graph of the given solution y. Conversely, to any point $(x_0, y_0) \in R^2$, $x_0 \in A$, there is a solution y of (1) such that a graph of y contains the point (x_0, y_0). For example, the interval

$A=[x_0, b)$ and some continuous functions $\sigma_0, \sigma_1, \ldots, s_{n-1}$; $\sigma_0(x_0)=y_0$ on A_{x_0} will do. Due to continuity of a_i, b_{ij}, ξ_j, σ_k $(1\leqq i,\ k\leqq n-1;\ 1\leqq j\leqq m)$ there exists a solution of (1) through $(x_0;\ \sigma_0, \sigma_1, \ldots, \sigma_{n-1})$.

If $G:=I\times R\subseteq R^2$ then G is open and $\Psi(x)=(x, y(x))\in G$ for any solution y of (1) and each $x\in I$ where y is defined. Consider a one-to-one homeomorphism Φ taking G into $U\subset R^2$ with properties $\Phi\in C^n(G)$ and Jacobian $|\Phi'(p)|\neq 0$ for all $p\in G$, i.e. Φ is a diffeomorphism. Then U is open and there is $\Phi^{-1}=(f, g)$ such that $|\Phi^{-1\prime}(q)|\neq 0$ for all $q\in U$ (see [6], p. 223; [8], p. 58—59).

The mapping Φ^{-1} is a point-transformation. Consequently, for any nontrivial solution y of (1) and an arbitrary fixed $x\in I$ where y is defined there is a unique set of mutually disjoint points in U

$$(t, u)=\Phi(\Psi(x)),\quad (t_j, u_j)=\Phi(\Psi(\xi_j(x))), \tag{4}$$

$1\leqq j\leqq m$. Using

$$(x, y(x))=\Psi(x)=\Phi^{-1}(t, u)=(f(t, u), g(t, u))$$

we get

$$y^{(k)}(x)=\frac{H}{\sigma^{k+1}}u^{(k)}(t)+D_k(t)\quad (2\leqq k\leqq n), \tag{5}$$

where $H=|\Phi^{-1\prime}(t, u)|=f_1g_2-g_1f_2\neq 0$ on U, $\sigma:=f_1+f_2\dot{u}(t)$ (f_i, g_i denotes the partial derivatives of f, g with respect to the i-th variable, $i=1, 2$) and $D_k(t)$ obtains terms of order lower than k (see also [3], [4]). Then

$$a_{k+1}(x)y^{(k+1)}(x)+a_k(x)y^{(k)}(x)=$$

$$=\frac{H}{\sigma^{k+1}}\left(\frac{a_{k+1}(f)}{\sigma}u^{(k+1)}(t)+a_k(f)u^{(k)}(t)\right)+a_{k+1}(f)D_{k+1}(t)+a_k(f)D_k(t)$$

and the linearity of $\frac{a_{k+1}(f)}{\sigma}u^{(k+1)}(t)+a_k(f)u^{(k)}(t)$ in $\dot{u}(t), \ldots, u^{(k)}(t), u^{(k+1)}(t)$ implies $f_2\equiv 0$ since $\sigma=f_1+f_2\dot{u}$ and a transformation

$$(x, y(x))=(f(t, u(t)), g(t, u(t))),\quad (t, u(t))\in U, \tag{T}$$

is independent of coefficients $a_k(x)$ of (1), $2\leqq k\leqq n-1$, $a_n(x)\equiv 1$. Hence $H=f_1g_2\neq 0$ on U and

$$x=f(t) \tag{6}$$

is one-to-one mapping and $f^{-1}(I)=J\subset R$. From (4), (6) we have

$$\begin{aligned}(x, y(x))&=(f(t), g(t, u(t))),\\ (\xi_j(x), y(\xi_j(x)))&=(f(t_j), g(t_j, u(t_j))),\end{aligned} \tag{7}$$

thus

$$\xi_j(f(t))=f(t_j) \tag{8}$$

Denote $t_j = \eta_j(t)$. Then $\eta_j\colon J \to R$ and

$$\xi_j(f(t)) = f(\eta_j(t)), \quad 1 \leqq j \leqq m, \tag{9}$$

since (8) converts deviating arguments of (1) into deviating arguments of (2). Hence

$$\begin{gathered}(x, y(x)) = (f(t), g(t, u(t))),\\ (\xi_j(x), y(\xi_j(x))) = (f(\eta_j(t)), g(\eta_j(t), u(\eta_j(t)))),\end{gathered} \tag{10}$$

by means of (7), (9). We have $\eta_j(t) \neq \eta_k(t)$ as $j \neq k$ and $\eta_j(t) \neq t$ on J since $\xi_j(x) \neq \xi_k(x)$ as $j \neq k$ and $\xi_j(x) \neq x$ on I and the function f is monotonic on J and (9) holds, $1 \leqq k,\ j \leqq m$.

Thus

$$y'(x) = [g_1(t, u(t)) + g_2(t, u(t))\dot{u}(t)]/\dot{f}(t),$$

$$y^{(k)}(x) = \frac{H}{\dot{f}^{k+1}(t)}\, u^{(k)}(t) + D_k(t) = \frac{g_2(t, u(t))}{\dot{f}^k(t)}\, u^{(k)}(t) + D_k(t), \tag{11}$$

$$2 \leqq k \leqq n,$$

using (5), and the equation (1) becomes

$$\begin{aligned}
u^{(n)}(t) + \sum_{i=2}^{n-1} \Big[a_i(f)\dot{f}^{n-i} u^{(i)}(t) + \sum_{j=1}^{m} b_{ij}(f) \frac{\dot{f}^n}{\dot{f}^i(\eta_j)} \times \\
\times \frac{g_2(\eta_j, u(\eta_j))}{g_2}\, u^{(i)}(\eta_j) \Big] + a_1(f)\dot{f}^{n-1} \left(\frac{g_1}{g_2} + \dot{u} \right) + \\
+ a_0(f)\dot{f}^n \frac{g}{g_2} + \sum_{j=1}^{m} \Big[b_{1j}(f) \frac{\dot{f}^n}{\dot{f}(\eta_j)} \Big(\frac{g_1(\eta_j, u(\eta_j))}{g_2} + \\
+ \frac{g_2(\eta_j, u(\eta_j))}{g_2}\, \dot{u}(\eta_j) \Big) + b_{0j}(f)\dot{f}^n \frac{g(\eta_j, u(\eta_j))}{g_2} \Big] + \\
+ \frac{\dot{f}^n}{g_2} \Big[D_n(t) + \sum_{i=2}^{n-1} \Big(D_i(t) + \sum_{j=1}^{m} D_i(\eta_j) \Big) \Big] = 0,
\end{aligned} \tag{12}$$

where $g_2 \neq 0$ for all $t \in J$.

The following relations

$$\frac{g(t, u(t))}{g_2(t, u(t))} = \alpha(t)u(t), \tag{13}$$

$$\frac{g_1(t, u(t))}{g_2(t, u(t))} = \beta(t)u(t), \tag{14}$$

$$\frac{g(\eta_j(t), u(\eta_j(t)))}{g_2(t, u(t))} = \gamma_j(t)u(\eta_j(t)), \quad 1 \leqq j \leqq m, \tag{15}$$

must be valid for suitable functions α, β, γ_j on J to obtain the linearity and homogenity of

$$a_0(f)\dot{f}^n \frac{g}{g_2}, \quad a_1(f)\dot{f}^{n-1}\frac{g_1}{g_2}, \quad b_{0_j}(f)\dot{f}^n \frac{g(\eta_j, u(\eta_j))}{g_2}$$

in $u(\eta_j)$, u, $\dot{u}$.

From (13) we have $g(t, u(t)) = \alpha(t)u(t)g_2(t, u(t)) \not\equiv 0$ on any subinterval on J since $g(t, u) = y(x)$ is a nontrivial solution of (1) on any subinterval on I. Thus $g_2(t, u)/g(t, u) = \alpha_1(t)/u$, $\alpha_1 = 1/\alpha$, on some $J_1 \subsetneq J$ and through integration we get

$$\ln|g(t, u)| = \alpha_1(t)\ln|u| + \ln|\alpha_2(t)|,$$

i.e.

$$g(t, u) = \alpha_2(t)u^{\alpha_1(t)}.$$

Hence

$$0 \neq g_2(t, u(t)) = \alpha_2(t)\alpha_1(t)u(t)^{\alpha_1(t)-1}$$

for all $t \in J$ and

$$g(t, u(t)) = \alpha_2(t)u^{\alpha_1(t)}, \quad \alpha_1(t)\alpha_2(t) \neq 0 \tag{16}$$

on J. Using (16) and (14) we obtain

$$\dot{\alpha}_2 u^{\alpha_1} + \dot{\alpha}_1 u^{\alpha_1}\ln|u| = \beta\alpha_1\alpha_2 u^{\alpha_1 - 1}u,$$

i.e.

$$\dot{\alpha}_2 + \dot{\alpha}_1\ln|u| = \beta\alpha_1\alpha_2 \tag{17}$$

on J. It is clear that only $\alpha_1(t) \equiv \lambda = const.$ complies with (17).

Finally, $\lambda = 1$ for the sake of equations

$$\alpha_2(\eta_j(t))u^{\lambda}(\eta_j(t)) = \lambda\gamma_j(t)\alpha_2(t)u(\eta_j(t))u(t)^{\lambda-1} \tag{18}$$

$(1 \leqq j \leqq m)$ obtained from (15). Consequently

$$g(t, u(t)) = \alpha_2(t)u(t), \quad \alpha_2(t) \neq 0 \quad \text{for all} \quad t \in J. \tag{19}$$

It remains to show that the required transformation rewritten as

$$x = f(t), \quad y = g(t)u,$$

$f, g \in C^n(J)$, $\dot{f}g \neq 0$ on J, $\xi_j(f(t)) = f(\eta_j(t))$, $t \in J$, $1 \leqq j \leqq m$, converts (1) into (2).

By succesive differentiation we find

$$y^{(k)}(x) = \frac{Y_k(t)}{\dot{f}^{2k-1}(t)}, \quad k = 1, 2, \dots, n, \tag{20}$$

where

$$Y_1(t) = \dot{g}(t)u(t) + g(t)\dot{u}(t)$$

and

$$Y_i(t) = \dot{Y}_{i-1}(t)\dot{f}(t) - (2i-3)\ddot{f}(t)\,Y_{i-1}(t), \tag{21}$$

$i=2, 3, \ldots, n$. Thus $Y_k(t)$ is a linear combination of $u, \dot{u}, \ldots, u^{(k)}$ $(1 \leqq k \leqq n)$ because Y_1 is linear in $u, \dot{u}$.

From $\xi_j \circ f = f \circ \eta_j$ we have

$$y^{(k)}(\xi_j(x)) = \frac{Y_k(\eta_j(t))}{\dot{f}^{2k-1}(\eta_j(t))}$$

and $Y_k(\eta_j(t))$ is a linear combination of $u(\eta_j), \dot{u}(\eta_j), \ldots, u^{(k)}(\eta_j)$ by means of (20) and (21), $k=1, 2, \ldots, n-1$.

Hence the Theorem is proved.

Remark. If the function f is strictly increasing (strictly decreasing) then $f(\eta_j(t)) = \xi_j(f(t)) = \xi_j(x) < x = f(t)$ implies $\eta_j(t) < t$ $(\eta_j(t) > t)$ for all $t \in J$ $(1 \leqq j \leqq m)$ and the transformation described in Theorem converts a retarded equation into a retarded (an advanced) equation.

REFERENCES

[1] EL'SGOL'C, L. E.: Vvedenije v Teoriju Differencial'nych Uravnenij s Otklonjajuščimsja Argumentom. Nauka, Moskva 1964.

[2] NORKIN, S. V.: Differencial'nyje Uravnenija vtorovo Porjadka s Zapazdyvajuščim Argumentom. Nauka, Moskva 1965.

[3] STÄCKEL, P.: Über Transformationen von Differentialgleichungen. J. Reine Agnew. Math. (Crelle Journal) 111, 1893, 290—302.

[4] WILCZYNSKI, E. J.: Projective differential geometry of curves and ruled surfaces. Teubner — Leipzig 1906.

[5] MELVIN, L. H.: A change of variables for functional differential equations. J. Diff. Equations 18, 1975, 1—10.

[6] SIKORSKI, R.: Diferenciální a integrální počet funkce více proměnných. Academia, Praha 1973.

[7] ŠEVELO, V. N.: Oscillacija Rešenij Differencial'nych Uravnenij s Otklonjajuščimsja Argumentom. Nauk. dumka, Kijev 1978.

[8] KURZWEIL, J.: Obyčejné diferenciální rovnice. SNTL, Praha 1978.

[9] TRYHUK, V.: The most general transformation of homogeneous linear differential retarded equations of the first order. Arch. Math. (Brno) 16, 1980, 225—230.

Received November 18, 1980

Katedra matematiky
VAAZ
600 00 Brno

САМОЕ ОБЩЕЕ ПРЕОБРАЗОВАНИЕ ОДНОРОДНОГО ЛИНЕЙНОГО УРАВНЕНИЯ n-ого ПОРЯДКА С ЗАПАЗДЫВАНИЕМ

Václav Tryhuk

Резюме

Штекел и Вилчински показали, что $x = f(t)$, $y = g(t)u$ — самое общее преобразование обыкновенного однородного дифференциального уравнения n-ого порядка, сохраняющее однородность, тип и порядок уравнения.

В статье доказывается, что $x = f(t)$, $y = g(t)u$ — самое общее преобразование для однородного линейного дифференциального уравнения с запаздыванием

$$y^{(n)}(x) + \sum_{i=0}^{n-1} \left[a_i(x) y^{(i)}(x) + \sum_{j=1}^{m} b_{ij}(x) y^{(i)}(\xi_j(x)) \right] = 0,$$

$$y^{(i)}(s) = \mathrm{d}^i y(s)/\mathrm{d}s^i, \quad n \geq 2.$$