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ABOUT σ-ADDITIVE AND σ-MAXITIVE MEASURES

ZDENA RIEČANOVÁ

The σ-additive and the σ-maxitive measures have some common properties. With the help of the $\oplus$-measure (Definition 2) we can study some problems of σ-additive and σ-maxitive measures simultaneously. In the presented paper we study the problem of extension (Theorems 1, 2).

1. Definitions and examples

N. Shilkret in [1] defined the σ-maxitive measure in the following way:

Definition 1. *Let $\mathcal{R}$ be a ring of subsets of a nonempty set X. A set function m: $\mathcal{R} \to \langle 0, \infty\rangle$ is called a σ-maxitive measure if $m(\emptyset)=0$ and $m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sup_i m(E_i)$ for each sequence $\{E_i\}_{i=1}^{\infty}$ of mutually disjoint sets in $\mathcal{R}$ such that $\bigcup_{i=1}^{\infty} E_i \in \mathcal{R}$.*

It is interesting that the σ-maxitive measures and the σ-additive measures have many common properties. One of their common generalizations may be the set function from the following definition.

Definition 2. *Let $\mathcal{R}$ be a ring of subsets of a nonempty set X. Let $\oplus$ be a binary operation on $\langle 0, \infty\rangle$, which is commutative, associative and $a \oplus 0 = a$ for all $a \in \langle 0, \infty\rangle$. A set function m: $\mathcal{R} \to \langle 0, \infty\rangle$ is called a $\oplus$-measure if $m(\emptyset)=0$ and $m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sup_n \{m(E_1) \oplus m(E_2) \oplus \ldots \oplus m(E_n)\}$ for each sequence $\{E_i\}_{i=1}^{\infty}$ of mutually disjoint sets from $\mathcal{R}$ such that $\bigcup_{i=1}^{\infty} E_i \in \mathcal{R}$.*

If $a \oplus b = a + b$ fro all $a, b \in \langle 0, \infty\rangle$, then the $\oplus$-measure on the ring $\mathcal{R}$ is a σ-additive measure on $\mathcal{R}$. If $a \oplus b = \max\{a, b\}$ for all $a, b \in \langle 0, \infty\rangle$, then the $\oplus$-measure on the ring $\mathcal{R}$ is a σ-maxitive measure on $\mathcal{R}$. The following is an example of a $\oplus$-measure which is neither additive nor maxitive.

Example 1. Let $\mathscr{R}$ be a ring of subsets of a nonempty set X and let m: $\mathscr{R} \to \langle 0, \infty)$ be a σ-additive measure on $\mathscr{R}$. Let $\bar{m}(A) = e^{m(A)}$ for all sets $A \in \mathscr{R}$, $A \neq \emptyset$ and $\bar{m}(\emptyset) = 0$. Then $\bar{m}$ is a set function on $\mathscr{R}$ which is neither additive nor maxitive but $\bar{m}$ is a $\oplus$-measure if we define $a \oplus b = ab$ for all $a, b \in (0, \infty)$ and $a \oplus 0 = a$, $a \oplus \infty = \infty$ for all $a \in \langle 0, \infty \rangle$.

Observe that if m is a $\oplus$-measure on a ring $\mathscr{R}$, then m is monotone and $m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sup_n m\left(\bigcup_{i=1}^{n} E_i\right)$ for each sequence of mutually disjoint sets in $\mathscr{R}$ such that $\bigcup_{n=1}^{\infty} E_n \in \mathscr{R}$. This follows from the relation

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sup_n \{m(E_1) \oplus m(E_2) \oplus \ldots \oplus m(E_n)\} \leqq$$

$$\leqq \sup_n m\left(\bigcup_{i=1}^{n} E_i\right) \leqq m\left(\bigcup_{i=1}^{\infty} E_i\right).$$

Definition 3. *Let $\mathscr{R}$ be a ring of subsets of a nonempty set X. A set function m: $\mathscr{R} \to \langle 0, \infty \rangle$ is called a supremeasure on $\mathscr{R}$ if $m(\emptyset) = 0$ and $m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sup_n m\left(\bigcup_{i=1}^{n} E_i\right)$ for each sequence of mutually disjoint sets in $\mathscr{R}$ such that $\bigcup_{n=1}^{\infty} E_n \in \mathscr{R}$.*

Examples of supremeasures are the σ-additive measures, the σ-maxitive measures and the $\oplus$-measures on $\mathscr{R}$. The relationship among these set functions is the following:

m is a σ-additive (or σ-maxitive) measure on $\mathscr{R} \Rightarrow$
m is a $\oplus$-measure on $\mathscr{R} \Rightarrow m$ is a supremeasure on $\mathscr{R}$.

But no implication in the reverse direction holds, which is evident from Example 1 and from the following example.

Example 2. Let $X = (-\infty, \infty)$, $\mathscr{R} = 2^X$. Define

$$m(A) = \sup \{|x - y| : x, y \in A\} \quad \text{for all} \quad A \subset X, \ A \neq \emptyset$$

and $m(\emptyset) = 0$. Then m is a supremeasure on $\mathscr{R}$. Suppose that m is a $\oplus$-measure on $\mathscr{R}$. Put

$$A = \left(0, \frac{1}{2}\right\rangle \cup \bigcup_{n=2}^{\infty} \left(\frac{n+2}{n+1}, \frac{n+1}{n}\right\rangle.$$

Then

$$\frac{3}{2} = m(A) = \sup_n \left\{\frac{1}{2}, \frac{1}{2} \oplus \frac{1}{6}, \ldots, \frac{1}{2} \oplus \frac{1}{6} \oplus \ldots \oplus \frac{1}{n(n+1)}\right\}$$

and because

$$(0, 1\rangle = \bigcup_{n=1}^{\infty} \left(\frac{1}{n+1}, \frac{1}{n}\right\rangle,$$

we have

$$1 = m((0, 1\rangle) = \sup_n \left\{\frac{1}{2}, \frac{1}{2}\oplus\frac{1}{6}, \ldots, \frac{1}{2}\oplus\frac{1}{6}\oplus\cdots\oplus\frac{1}{n(n+1)}\right\},$$

which is a contradiction.

Example 3. Let X be a metric (or more generally pseudometric) space with a metric ϱ. Let $\mathcal{R} = 2^X$. Define $m(A) = \sup \{\varrho(x, y): x, y \in A\}$, (the diameter of A) for all $A \subset X$, $A \neq \emptyset$ and $m(\emptyset) = 0$. Then m is a supremeasure on $\mathcal{R}$, which is not a $\oplus$-measure and consequently m is neither σ-additive nor σ-maxitive.

Example 4. Let m be a σ-additive measure on a ring $\mathcal{R}$ of subsets of a nonempty set X. Define $\bar{m}(A) = \min \{m(A), 1\}$ for all $A \in \mathcal{R}$. Then

a) $\bar{m}$ is a supremeasure on $\mathcal{R}$

b) $\bar{m}$ is strongly subaditive on $\mathcal{R}$ (i.e. $\bar{m}(A \cup B) + \bar{m}(A \cap B) \leqq \bar{m}(A) + \bar{m}(B)$ for all $A, B \in \mathcal{R}$)

c) $\bar{m}$ is neither additive nor maxitive on $\mathcal{R}$.

Observe the following: Let m be a supremeasure on $\mathcal{R}$. Then:

(a) m is a σ-additive measure on $\mathcal{R}$ iff $m(A \cup B) = m(A) + m(B)$ for all A, $B \in \mathcal{R}$, $A \cap B = \emptyset$.

(b) m is a σ-maxitive measure on $\mathcal{R}$ iff $m(A \cup B) = \max \{m(A), m(B)\}$ for all A; $B \in \mathcal{R}$, $A \cap B = \emptyset$.

(c) If $\oplus$ is a binary operation on $\langle 0, \infty\rangle$, which is commutative, associative and $a \oplus 0 = a$ fro all $a \in \langle 0, \infty\rangle$ and if $a \leqq a \oplus b$ for all $a \in \langle 0, \infty\rangle$, then m is a $\oplus$-measure on $\mathcal{R}$ iff $m(A \cup B) = m(A) \oplus m(B)$ for all $A, B \in \mathcal{R}$, $A \cap B = \emptyset$.

2. An extension of a supremeasure

Let $\mathcal{R}$ be a ring of subsets of a nonempty set X and $\mathcal{H}(\mathcal{R})$ be the hereditary σ-ring generated by $\mathcal{R}$. Let m: $\mathcal{R} \to \langle 0, \infty\rangle$ be a supremeasure on $\mathcal{R}$. Denote

$$\mathcal{K} = \left\{\bigcup_{i=1}^{\infty} E_i: E_i \in \mathcal{R}, i = 1, 2, \ldots\right\}$$

and define $m_0: \mathcal{K} \to \langle 0, \infty\rangle$ and $m_1: \mathcal{H}(\mathcal{R}) \to \langle 0, \infty\rangle$ by the formulas

$$m_0\left(\bigcup_{i=1}^{\infty} E_i\right) = \sup_n m\left(\bigcup_{i=1}^{n} E_i\right) \quad \text{for all sets} \quad \bigcup_{i=1}^{\infty} E_i \in \mathcal{K}$$

$$m_1(A) = \inf \{m_0(E): A \subset E \in \mathcal{K}\} \quad \text{for all sets} \quad A \in \mathcal{H}(\mathcal{R}).$$

Lemma 1. *If E_i, $F_i \in \mathcal{R}$ $(i=1, 2, \ldots)$ and $\bigcup_{i=1}^{\infty} E \subset \bigcup_{i}^{\infty} F$, then $\sup_n m\left(\bigcup_{i}^{n} E_i\right) \leqq \sup_n m\left(\bigcup_{i=1}^{n} F_i\right)$.*

Proof. Let $A_n = \bigcup_{i=1}^{n} E_i$ $(n=1, 2, \ldots)$. We have $m(A_n) = m\left(\bigcup_{i=1}^{n} E_i\right) = m\left[\bigcup_{i=1}^{\infty}(F_i \cap A_n)\right] = \sup_k m\left[\bigcup_{i=1}^{k}(F_i \cap A_n)\right] \leqq \sup_k m\left(\bigcup_{i=1}^{k} F_i\right)$ for each n and thus our assertion is evident.

Corollary. (1) *m_0 is monotone on $\mathcal{K}$.*

(2) *If $A_i \in \mathcal{K}$ $(i=1, 2, \ldots)$, then $m_0\left(\bigcup_{i=1}^{\infty} A_i\right) = \sup_n m_0\left(\bigcup_{i=1}^{n} A_i\right)$.*

(3) *$m_0(E) = \sup \{m(F): E \supset F \in \mathcal{R}\}$ for all sets $E \in \mathcal{K}$.*

(4) *If m is strongly subadditive on $\mathcal{R}$ (i.e. $m(A \cup B) + m(A \cap B) \leqq m(A) + m(B)$ for all A, $B \in \mathcal{R}$), then m_0 is strongly subadditive on $\mathcal{K}$.*

The following lemma is a modification of Lemma 3.1 from [3].

Lemma 2. *If m is a strongly subadditive supremeasure on $\mathcal{R}$, then for each increasing sequence of sets A_n $(n=1, 2, \ldots)$ in $\mathcal{H}(\mathcal{R})$ and for each $\varepsilon > 0$ there holds:*

If $B_i \in \mathcal{K}$, $B_i \supset A_i$, $m_1(B_i) < m_1(A_i) + \frac{\varepsilon}{2^{i+1}}$ for all $i=1, 2, \ldots$, then

$$m_1\left(\bigcup_{i=1}^{n} B_i\right) < m_1(A_n) + \sum_{i=1}^{n} \frac{\varepsilon}{2^{i+1}}$$

for each n.

Proof. (By induction.) For $n=1$ the assertion holds. Suppose for some n the assertion holds. Then

$$m_1\left(\bigcup_{i=1}^{n+1} B_i\right) \leqq m_1\left(\bigcup_{i=1}^{n} B_i\right) + m_1(B_{n+1}) - m_1\left[\left(\bigcup_{i=1}^{n} B\right) \cap B_{n+1}\right] <$$

$$< m_1(A_n) + \sum_{i=1}^{n} \frac{\varepsilon}{2^{i+1}} + m_1(A_{n+1}) + \frac{\varepsilon}{2^{n+2}} - m_1\left[\left(\bigcup_{i=1}^{n} A_i\right) \cap A_{n+1}\right] =$$

$$= m_1(A_n) + \sum_{i=1}^{n} \frac{\varepsilon}{2^{i+1}} + m_1(A_{n+1}) + \frac{\varepsilon}{2^{n+2}} - m_1(A_n) = m_1(A_{n+1}) + \sum_{i=1}^{n+1} \frac{\varepsilon}{2^{i+1}}.$$

Theorem 1. *Let m be a strongly subadditive supremeasure on $\mathcal{R}$. Let*

$$m_1(A) = \inf\left\{\sup_n m\left(\bigcup_{i=1}^{n} E_i\right) : A \subset \bigcup_{i=1}^{\infty} E_i,\ E_i \in \mathcal{R}\ (i=1, 2, \ldots)\right\}$$

for all sets A in $\mathcal{H}(\mathcal{R})$. Then m_1 is a strongly subadditive supremeasure on $\mathcal{H}(\mathcal{R})$.

Proof. It is clear that $m_1(\emptyset)=0$ and m_1 is monotone on $\mathcal{H}(\mathcal{R})$. Let $\{A_n\}_{n=1}^{\infty}$ be an increasing sequence of sets in $\mathcal{H}(\mathcal{R})$ and $m_1(A_n)<\infty$ for each n. Let $\varepsilon>0$. Then for each i ($i=1, 2, \dots$) there exists $B_i \in \mathcal{K}$, $B_i \supset A_i$ such that $m_1(A_i) + \frac{\varepsilon}{2^{i+1}} > m_1(B_i)$. It follows from Lemma 2 that

$$m_1(A_n)+\sum_{i=1}^{n} \frac{\varepsilon}{2^{i+1}} > m_1\left(\bigcup_{i=1}^{n} B_i\right)$$

for each n and hence

$$m_1\left(\bigcup_{i=1}^{\infty} A_i\right) \leqq m_1\left(\bigcup_{i=1}^{\infty} B_i\right) = \sup_n m_1\left(\bigcup_{i=1}^{n} B_i\right) \leqq$$

$$\leqq \sup_n \left\{m_1(A_n)+\sum_{i=1}^{n} \frac{\varepsilon}{2^{i+1}}\right\} = \sup_n m_1(A_n)+\varepsilon.$$

On the other hand it is clear that $\sup_n m_1(A_n) \leqq m_1\left(\bigcup_{i=1}^{\infty} A_i\right)$ and hence m_1 is a supremeasure on $\mathcal{H}(\mathcal{R})$. The strong subadditivity of m_1 on $\mathcal{H}(\mathcal{R})$ follows from the strong subadditivity of m_0 on $\mathcal{K}$ and from the definition of m_1.

Remark. If the supremeasure m from Theorem 1 is a σ-maxitive measure on $\mathcal{R}$, then also its extension m_1 is a σ-maxitive measure on $\mathcal{H}(\mathcal{R})$. It suffices to show that $m_1(A \cup B) = \max\{m_1(A), m_1(B)\}$ for all $A, B \in \mathcal{H}(\mathcal{R})$, $A \cap B = \emptyset$. If $A, B \in \mathcal{K}$, then this assertion follows from the relation $\sup_n m\left(\bigcup_{i=1}^{n} E_i\right) = \sup_n \max\{m(E_1), \dots, m(E_n)\} = \sup_n m(E_n)$ for each sequence $\{E_i\}_{i=1}^{\infty}$ in $\mathcal{R}$. If $A, B \in \mathcal{H}(\mathcal{R})$, then there are $E, F \in \mathcal{K}$ such that $A \subset E$, $B \subset F$ and $m_1(A)+\varepsilon > m_1(E)$, $m_1(B)+\varepsilon > m_1(F)$, thus $m_1(A \cup B) \leqq m_1(E \cup F) = \max\{m_1(E), m_1(F)\} < \max\{m_1(A), m_1(B)\}+\varepsilon$ and hence $m_1(A \cup B) \leqq \max\{m_1(A), m_1(B)\}$. The reverse inequality is clear.

3. An extension of a $\oplus$-measure

Let $\oplus$ be a binary operation on $\langle 0. \infty\rangle$ such that

(a) it is commutative

(b) it is associative

(c) $a \oplus 0 = a$ for all $a \in \langle 0, \infty\rangle$

(d) $a \leqq a \oplus b$ for all $a, b \in \langle 0, \infty\rangle$

(e) $a_n \uparrow a,\ b_n \uparrow b \Rightarrow a_n \oplus b_n \uparrow a \oplus b$

(f) $a_n \downarrow a,\ b_n \downarrow b \Rightarrow a_n \oplus b_n \downarrow a \oplus b$

If m is a supremeasure on a ring $\mathcal{R}$ of subsets of X, then m is a $\oplus$-measure iff $m(A\cup B) = m(A)\oplus m(B)$ for all A, $B\in\mathcal{R}$, $A\cap B=\emptyset$. The last condition is equivalent to the following condition:

$$m(A\cup B)\oplus m(A\cap B)=m(A)\oplus m(B) \quad \text{for all} \quad A, B\in\mathcal{R}.$$

If $\mathcal{A}$ is a class of subsets of X, the notations

$$\mathcal{A}^{\nearrow}=\{A\subset X\text{: there is }\{A_n\}_{n=1}^{\infty}\text{ in }\mathcal{A},\ A_n\uparrow A\}$$
$$\mathcal{A}^{\searrow}=\{A\subset X\text{: there is }\{A_n\}_{n=1}^{\infty}\text{ in }\mathcal{A},\ A_n\downarrow A\}$$

are used.

The following theorem will be proved by transfinite induction. A similar method for extending functionals was used in [4].

Theorem 2. *Let m be a finite $\oplus$-measure on an algebra $\mathcal{R}$ of subsets of a nonempty set X. Let the supremeasure m_1 be an extension of m on the σ-ring $\mathcal{S}(\mathcal{R})$ generated by $\mathcal{R}$ and let m_1 be continuous from above on $\mathcal{S}(\mathcal{R})$ (i.e. $E_n\downarrow E\Rightarrow m_1(E_n)\downarrow m_1(E)$). Then m_1 is a $\oplus$-measure on $\mathcal{S}(\mathcal{R})$.*

Proof. For each ordinal $\alpha<\Omega$ (Ω is the first uncountable ordinal) we define a class $\mathcal{R}_\alpha$ of subsets of X as follows:

1. $\mathcal{R}_1=\mathcal{R}$.
2. $\mathcal{R}_\alpha=\mathcal{R}_{\alpha-1}^{\nearrow}$ if α is an even non-limit ordinal.
3. $\mathcal{R}_\alpha=\mathcal{R}_{\alpha-1}^{\searrow}$ if α is an odd non-limit ordinal.
4. $\mathcal{R}_\alpha=\bigcup_{\beta<\alpha} R_\beta$ if α is a limit ordinal.

Let $\mathcal{R}_\Omega=\bigcup_{\alpha<\Omega}\mathcal{R}_\alpha$. Then $\mathcal{R}_\Omega$ is a monotone class, $\mathcal{R}_\Omega\supset\mathcal{R}$ and hence $\mathcal{R}_\Omega\supset\mathcal{S}(\mathcal{R})$. If A, $B\in\mathcal{S}(\mathcal{R})$, then there is an ordinal $\alpha<\Omega$ such that A, $B\in\mathcal{R}_\alpha$. Hence it suffices to prove that for each ordinal $\alpha<\Omega$ there holds:

If A, $B\in\mathcal{R}_\alpha$, then $m_1(A\cup B)\oplus m_1(A\cap B)=m_1(A)\oplus m_1(B)$. We use the transfinite induction.

If $\alpha=1$, the assertion holds. Let $\alpha<\Omega$ be any ordinal and let the assertion holds for all $\beta<\alpha$. Hence

(a) If α is a non-limit ordinal, then there are monotone sequences $\{A_n\}_{n=1}^{\infty}$, $\{B_n\}_{n=1}^{\infty}$ in $\mathcal{R}_{\alpha-1}$ (both increasing or both decreasing) such that

$$m_1(A)=\lim_{n\to\infty} m_1(A_n),\quad m_1(B)=\lim_{n\to\infty} m_1(B_n)$$

and hence

$$m_1(A)\oplus m_1(B)=\lim_{n\to\infty}[m_1(A_n)\oplus m_1(B_n)]=$$
$$=\lim_{n\to\infty}[m_1(A_n\cup B_n)\oplus m_1(A_n\cap B_n)]=m_1(A\cup B)\oplus m_1(A\cap B).$$

(b) If α is a limit ordinal, the proof is trivial.

Remark. The existence of such an extension m_1 which is continuous from above on $\mathcal{S}(\mathcal{R})$ in the case of $\mathcal{R}$ being an algebra and m being finite, subadditive, continuous from above and exhausting on $\mathcal{R}$ (i.e. $A_n \in \mathcal{R}$, $n = 1, 2, \ldots$ mutualy disjoint and $\lim_{n\to\infty} m\left(\bigcup_{i=1}^{n} A_i\right) < \infty \Rightarrow \lim_{n\to\infty} m(A_n) = 0$) follows from [2] p. 217.

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О σ-АДДИТИВНЫХ И σ-МАКСИТИВНЫХ МЕРАХ

Здена Риечанова

Резюме

В работе показано, что некоторые проблемы σ-аддитивных и σ-макситивных мер возможно изучать одновременно при помощи ⊕-меры. Действительная функция m множества определенная на некотором кольце $\mathcal{R}$ подмножеств данного множества X, называется ⊕-мерой, если она неотрицательна,

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sup_n \left\{m(E_1) \oplus m(E_2) \oplus \ldots \oplus m(E_n)\right\}$$

для всякой последовательности непересекающихся множеств

$$\{E_n\}_{n=1}^{\infty}$$

из $\mathcal{R}$, соединение которых также принадлежит $\mathcal{R}$, и $m(\emptyset) = 0$. Здесь символом ⊕ обозначается любая бинарная операция в множестве $\langle 0, \infty)$, обладающая следующим свойствами: 1) она коммутативна; 2) она подчиняется сочетательному закону; 3) $a \oplus 0 = a$ для любого $a \in \langle 0, \infty)$. В работе изучается проблема продолжения ⊕-меры.