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ENTROPY FOR NONINVARIANT MEASURES

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Measure theoretic entropy, which is also called the Kolmogorov—Sinai invariant, was defined for measure preserving transformations of probability measure spaces.

In the presented paper we consider an extension of this invariant for nonsingular measurable transformations. This class of transformations has been studied in many papers. The following property (of a probability measure m with respect to a measurable transformation T) has been a matter of a great interest:

(P) there exists a probability measure μ equivalent to m and invariant under T.

A great number of necessary and sufficient conditions has been established. A list of them with an extensive survey of bibliography is in [5].

One of the generalizations of the Kolmogorov—Sinai invariant is the sequantial entropy introduced by Kushnirenko in [7] and extensively studied in [8] and [6]. It was defined for measure preserving transformations but the definition does not require the considered measure to be invariant. As a special case of this extended sequential entropy we get an extension of the measure theoretic entropy for noninvariant measures.

One of the most important results connected with measure theoretic entropy is Goodwyn's theorem comparing the latter with topological entropy. We show that this theorem is valid for our extension of the measure theoretic entropy if the considered regular measure has the property (P).

I. Extension of entropy

Recall that for a given subsequence $A=\{t_1, \dots, t_n, \dots\}$ of the sequence Z_+ of nonnegative integers the sequential entropy of a measure preserving transformation T (of a probability measure space $(X, \mathscr{B}, m)$) is defined by (cf. [7]):

$$h_{A,m}(T)=\sup\{\bar{H}_{A,m}(T, \xi): \xi\in\mathscr{P}\}, \tag{1.a}$$

where

$$\bar{H}_{A,m}(T, \xi)=\overline{\lim}\ 1/n\cdot H_m\left(\bigvee_{i=1}^{n} T^{-t_i}(\xi)\right). \tag{1.b}$$

$\mathcal{P}$ is the system of all finite measurable partitions of X
and H_m is the Shannon entropy of a finite measurable partition.

Note that this definition does not require T to be measure preserving. Hence we can apply it for any measurable transformation. If we put $A = Z_+$, we get the measure theoretic entropy $h_m(T)$ for noninvariant measures which is an extension of the Kolmogorov—Sinai invariant.

Let $(X, \mathcal{B}, m)$ be a probability measure space. Recall that a bijective point transformation T of the space X is called nonsingular (cf. [5]) if T and T^{-1} are measurable and preserve null sets. This means that for any $B \in \mathcal{B}$ $m(B) = 0$ implies $m(T^{-1}(B)) = m(T(B)) = 0$.

The terminology of the following definition follows the one of [9] p. 26.

Definition. *Suppose that for $i = 1, 2$, T_i is a nonsingular transformation of a probability measure space $(X_i, \mathcal{B}_i, m_i)$. Suppose further that there exists a measurable transformation φ, defined on almost all of X_1, which maps onto almost all of X_2. If φ is such that*

$$m_2(B) = m_1 \cdot \varphi^{-1}(B) \quad \text{for} \quad B \in \mathcal{B}_2 \tag{2.a}$$

and

$$\varphi \cdot T_1 = T_2 \cdot \varphi \quad \text{a.e. on} \quad X_1, \tag{2.b}$$

then we say that T_2 is a homomorphic image of T_1. If φ is invertible so that T_1 is a homomorphic image of T_2 under the homomorphism φ^{-1}, we say that T_1 and T_2 are isomorphic.

Proposition 1. *Suppose that T_2 is a homomorphic image of T_1 in the sense of the previous definition. Then for any sequence $A \subset Z_+$ we have*

$$h_{A,m_2}(T_2) \leq h_{A,m_1}(T_1). \tag{3}$$

Hence sequential entropy is an invariant of nonsingular transformations.

Proof. Let for $i = 1, 2$, $\mathcal{P}_i$ be the system of finite measurable partitions of X_i. Let $\xi \in \mathcal{P}_2$. Using (2.a) and (2.b) we get

$$H_{m_2}\left(\bigvee_{i=1}^{n} T_2^{-t_i}(\xi)\right) = H_{m_1}\left(\bigvee_{i=1}^{n} T_1^{-t_i}(\varphi^{-1}(\xi))\right) \tag{4.a}$$

for any positive integer n.

Therefore

$$\bar{H}_{A,m_1}(T_1, \varphi^{-1}(\xi)) = \bar{H}_{A,m_2}(T_2, \xi) \tag{4.b}$$

and

$$\begin{aligned} h_{A,m_2}(T_2) &= \sup\{\bar{H}_{A,m_2}(T_2, \xi) : \xi \in \mathcal{P}_2\} = \\ &= \sup\{\bar{H}_{A,m_1}(T_1, \varphi^{-1}(\xi)) : \xi \in \mathcal{P}_2\} = \\ &\leq \sup\{\bar{H}_{A,m_1}(T_1, \eta) : \eta \in \mathcal{P}_1\} = \\ &= h_{A,m_1}(T_1). \end{aligned} \tag{4.c}$$

Now we return to the property (P) mentioned above. The following result was established in [2].

Lemma. *Let T be a nonsingular transformation of a probability measure space $(X, \mathscr{B}, m)$. Then the measure m has the property (P) if and only if the following condition is satisfied:*

(C) For any $B \in \mathscr{B}$ there exists a limit

$$\bar{m}(B) = \lim_{n\to\infty} 1/n \cdot \sum_{i=0}^{n-1} m(T^{-i}(B)). \qquad (5)$$

The set function $\bar{m}$ is a probability measure invariant under the transformation T and equivalent to m.

Proposition 2. *Let T, m and $\bar{m}$ be as in the previous lemma. Then we have*

$$h_m(T) \leq H_{\bar{m}}(T). \qquad (6)$$

Proof. We utilise the technique similar to the one used in the proof of proposition (18.12) in [1].

For any $B \in \mathscr{B}$ we have

$$\bar{m}(B) = \lim_{n\to\infty} m_n(B) \qquad (7.a)$$

where

$$m_n(B) = 1/n \cdot \sum_{i=0}^{n-1} m(T^{-i}(B)), \quad n = 1, 2, \ldots \qquad (7.b)$$

are probability measures.

Consider a partition $\xi \in \mathscr{P}$. For any positive integer d the partition $\xi^d = \bigvee_{i=0}^{d-1} T^{-i}(\xi) \in \mathscr{P}$. For any n, d such that $1 \leq d < n$ we have

$$n = q \cdot d + r \quad \text{and} \quad 0 \leq r < d. \qquad (8)$$

We calculate

$$\sum_{i=0}^{n-1} H_m(T^{-i}(\xi^d)) \geq \sum_{j=0}^{q-1} \sum_{i=0}^{d-1} H_m(T^{-j\cdot d-i}(\xi^d)). \qquad (9)$$

For any fixed $i \in \{0, \ldots, d-1\}$ we get from the known properties of the Shannon entropy (cf. [9]) that

$$\sum_{j=0}^{q-1} H_m(T^{-j\cdot d-i}(\xi^d)) \geq H_m\left(T^{-i}\left(\bigvee_{j=0}^{q-1} \bigvee_{k=0}^{d-1} T^{-j\cdot d-k}(\xi)\right)\right) = \qquad (10)$$

$$= H_m\left(\bigvee_{k=i}^{q\cdot d+i-1} T^{-k}(\xi)\right) \geq H_m(\xi^n) - d \cdot \ln p,$$

where $p = \mathrm{card}(\xi)$.

From (9) and (10) we get

$$\sum_{i=0}^{n-1} H_m(T^{-i}(\xi^d)) \geq d \cdot H_m(\xi^n) - d^2 \cdot \ln p. \tag{11}$$

The function

$$\eta(t) = \begin{cases} 0 & \text{for} \quad t=0 \\ -t \cdot \ln t & \text{for} \quad t \in (0, 1\rangle \end{cases} \tag{12}$$

is continuous and concave. Using these properties we get

$$\begin{aligned} H_{m_n}(\xi^d) &= \sum_{B \in \xi^d} \eta(m_n(B)) = \sum_{B \in \xi^d} \eta\left(1/n \cdot \sum_{i=0}^{n-1} m \cdot T^{-i}(B)\right) \geq \\ &\geq \sum_{B \in \xi^d} 1/n \cdot \sum_{i=0}^{n-1} \eta(m \cdot T^{-i}(B)) = \\ &= 1/n \cdot \sum_{i=0}^{n-1} \sum_{B \in \xi^d} \eta(m \cdot T^{-i}(B)) = \\ &= 1/n \cdot \sum_{i=0}^{n-1} H_m(T^{-i}(\xi^d)) \geq \\ &\geq d/n \cdot H_m(\xi^n) - d^2/n \cdot \ln p. \end{aligned} \tag{13}$$

Hence

$$\begin{aligned} H_{\bar{m}}(\xi^d) &= \sum_{B \in \xi^d} \eta(\bar{m}(B)) = \sum_{B \in \xi^d} \eta(\lim_{n \to \infty} m_n(B)) = \\ &= \lim_{n \to \infty} H_{m_n}(B) \geq d \cdot \overline{\lim_{n \to \infty}}\, 1/n \cdot H_m(\xi^n) = \\ &= d \cdot \bar{H}_m(T, \xi). \end{aligned} \tag{14}$$

Dividing by d and taking the limit with respect to d we get

$$\bar{H}_{\bar{m}}(T, \xi) \geq \bar{H}_m(T, \xi). \tag{15}$$

Therefore

$$h_{\bar{m}}(T) = \sup \{\bar{H}_{\bar{m}}(T, \xi) : \xi \in \mathcal{P}\} \geq h_m(T). \tag{16}$$

II. Comparing with topological entropy

Now we are able to prove the following comparison theorem.

Theorem. *Let T be an authomorphism of a Hausdorff compact topological space X. Let $\mathcal{B}(X)$ be the system of Borel subsets of X and m be a regular probability measure on $\mathcal{B}(X)$. Suppose further that T is nonsingular and that m has the* *property (P). Let $h(T)$ be the topological entropy of T. Then the following inequality holds*

$$h_m(T) \leq h(T). \tag{17}$$

Proof. The probability measure $\bar{m}$ given by (5) is regular because it is equivalent to m. According to Goodwyn's theorem the inequality

$$h_{\bar{m}}(T) \leq h(T) \tag{18}$$

holds. Combining (18) with (6) we get (17).

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ЭНТРОПИЯ ДЛЯ НЕИНВАРИАНТНЫХ МЕР

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Резюме

Вероятностная энтропия обобщается на случай неинвариантных вероятностей. Доказывается, что эта энтропия для регулярных вероятностей на метрических компактах не больше топологической энтропии.